INTEGRAL POINTS ON VARIABLE SEPARATED CURVES

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Abstract. Let $f(x), g(x) \in \mathbb{Z}[x]$ be non-constant integer polynomials. We provide evidence, under the *abc*-conjecture, that there exists $\epsilon > 0$, depending on f and g, such that the equation $f(x) = bg(y)$ has only finitely many solutions (x_0, y_0, b_0) in positive integers such that $x_0 > 1$, $\frac{\log(y_0)}{\log(x_0)} > \frac{\deg(f)}{\deg(g)} - \epsilon$, and such that $f(x) - b_0 g(y)$ is geometrically irreducible and the genus of the smooth projective curve associated to the plane curve given by the equation $f(x) = b_0 g(y)$ is positive. Using the ABC-theorem for polynomials, we show in certain instances that there exists $\epsilon > 0$ such that, if $x(t)$, $y(t)$, and $b(t)$, are non-constant polynomials in $\mathbb{Q}[t]$ such that $f(x(t)) = b(t)g(y(t))$, then $\frac{\deg_t(y)}{\deg_t(x)} < \frac{\deg(f)}{\deg(g)} - \epsilon$.

1. Introduction

Let $f(x, y)$ and $g(x, y)$ be integer polynomials, and consider the diophantine equation $cf(x, y) = bg(x, y)$. We are interested in the solutions (x, y, b, c) in non-negative integers, and we will usually assume that $\gcd(b, c) = 1$. Siegel's Theorem on integer points on affine curves ([9], D.9.2.2; see also below) gives sufficient geometric conditions for the finiteness of the set of solutions in any infinite region in the (x, y, b, c)-space where the variables b and c are both bounded. One may wonder whether it is possible to find other infinite regions in the (x, y, b, c)-space where only one of b or c is bounded and where the set of positive integer solutions can be constrained. Clearly, any answer to such a question might depend strongly on the particular form of the given diophantine equation.

We will restrict our attention in this article to the case where $f(x, y) = f(x)$ and $g(x, y) = g(y)$ are non-constant polynomials in one variable. The diophantine properties of equations of the type $f(x) = g(y)$ is a subject with a long history, and we refer to [2] for some relevant literature. Our goal in this article is to provide evidence that Conjecture 1.1 below might hold.

Recall that a geometrically irreducible polynomial $h(x, y) \in \mathbb{Z}[x, y]$ defines a geometrically irreducible plane curve over $\mathbb{Q}$, and that given any geometrically irreducible plane curve $U/\mathbb{Q}$, its normalization $\tilde{U}/\mathbb{Q}$ is a smooth affine curve which is an open subscheme of a smooth projective geometrically irreducible curve $C/\mathbb{Q}$ of (geometric) genus $g \geq 0$. Given an equation $c_0 f(x) = b_0 g(y)$ with fixed $b_0, c_0 \in \mathbb{Z}$, Siegel's Theorem implies that if the genus of the curve $C/\mathbb{Q}$ associated with the plane curve defined by the equation $c_0 f(x) = b_0 g(y)$ is positive, then the equation $c_0 f(x) = b_0 g(y)$ has only finitely many solutions in integers. This 1928 theorem of Siegel was superseded when $g \geq 2$ in 1983 by Faltings' Theorem, which proves the finiteness of the set of rational solutions of this equation.

Conjecture 1.1. *Let $f(x), g(x) \in \mathbb{Z}[x]$ be non-constant integer polynomials.*

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(1) *There exists $\epsilon > 0$, depending on f and g, such that the equation $f(x) = bg(y)$ has only finitely many solutions (x_0, y_0, b_0) in positive integers such that $x_0 > 1$,*

$$\frac{\log(y_0)}{\log(x_0)} > \frac{\deg(f)}{\deg(g)} - \epsilon,$$

and such that $f(x) - b_0 g(y)$ is geometrically irreducible and the genus of the smooth projective curve associated to the plane curve given by the equation $f(x) = b_0 g(y)$ is positive.

A more optimistic version of the conjecture would be:

(2) *There exists $\epsilon > 0$, depending on f and g, such that for any fixed integer $c_0 > 0$, the equation $c_0 f(x) = bg(y)$ has only finitely many solutions (x_0, y_0, b_0) in positive integers such that $x_0 > 1$,*

$$\frac{\log(y_0)}{\log(x_0)} > \frac{\deg(f)}{\deg(g)} - \epsilon,$$

and such that $c_0 f(x) - b_0 g(y)$ is geometrically irreducible and the genus of the smooth projective curve associated to the plane curve given by the equation $c_0 f(x) = b_0 g(y)$ is positive.

An even more optimistic version of the conjecture would be to ask for ϵ in (1) to depend only on $\deg(f)$ and $\deg(g)$. Note that since b is allowed to vary in the statement of 1.1, this conjecture predicts the finiteness of the set of 'large solutions' on the union of *infinitely many* affine curves parameterized by b, where a 'large solution' is taken here to mean a solution (x, y) with $y > x^{\frac{\deg(f)}{\deg(g)} - \epsilon}$.

Proposition 2.1 shows that the statement of this conjecture is true when $\epsilon \leq 0$. We prove that special cases of Conjecture 1.1 hold in Proposition 2.2, under the assumption that the *abc*-Conjecture holds. We provide in section 4 extensive numerical evidence that Conjecture 1.1 might hold in the case of the family of curves of genus 3 given by the equation

$$cx(x+1)(x+2)(x+3) = by(y+1)(y+2).$$

Two precise conjectures for this equation are formulated in 4.2 and 5.7. A summary of earlier work on this type of equations when b and c are fixed is found in the introduction of Saradha and Shorey [20], where its study is motivated by a problem of Erdös. Our initial motivation for considering this equation came from a conjecture of the junior author in [29], 4.5, which we recall in 6.1. Conjecture 4.5 in [29] is motivated by the problem of the non-existence of non-trivial tight designs in algebraic combinatorics.

Assuming that Conjecture 1.1 holds for some $\epsilon > 0$, it is natural to wonder what is the largest possible such ϵ. Solutions $(x(t), y(t), b(t))$ to $f(x) = bg(y)$ with $x(t), y(t)$ and $b(t)$ in $\mathbb{Z}[t]$ provide some natural upper bounds for such ϵ, as we discuss in section 3. Using the ABC-theorem for polynomials in 3.12, we are able to show that in certain instances, there exists $\epsilon > 0$ such that, if $x(t), y(t), b(t) \in \mathbb{Q}[t]$ are such that $f(x) = bg(y)$, then $\frac{\deg_t(y)}{\deg_t(x)} < \frac{\deg(f)}{\deg(g)} - \epsilon$.

Assuming that Conjecture 1.1 holds for all ϵ in a range $[0, \epsilon_0]$, we can consider the counting function $\mathcal{N}(\epsilon)$, the number of solutions (x_0, y_0, b_0) in positive integers to $f(x) = bg(y)$ such that $\frac{\log(y_0)}{\log(x_0)} > \frac{\deg(f)}{\deg(g)} - \epsilon$. We provide some data in section 7 for the equation

$$x(x+1)(x+2)(x+3) = by(y+1)$$

which suggests that in this case, $\mathcal{N}(\epsilon)$ might be approximated by an exponential function of the form $\exp(\alpha\epsilon + \beta)$ for some positive constants α and β.

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2. Some motivations for the conjecture

Conjecture 1.1 is motivated by the following proposition.

Proposition 2.1. *Let $f(x), g(x) \in \mathbb{Q}[x]$ be of positive degree. Let $\delta > 0$.*

(1) *Then there exist only finitely many solutions (x, y, b) in positive integers to the equation $f(x) = bg(y)$ such that*
$$\frac{\log(y)}{\log(x)} \geq \frac{\deg(f)}{\deg(g)} + \delta.$$
(2) *Assume that the equation $f(x) - ag(y) = 0$ defines a smooth projective curve $X/\mathbb{Q}(a)$ having positive genus. Then there exist only finitely many solutions (x, y, b) in positive integers to the equation $f(x) = bg(y)$ such that*
$$\frac{\log(y)}{\log(x)} \geq \frac{\deg(f)}{\deg(g)}$$
and such that the associated smooth projective curve to $f(x) = bg(y)$ has positive genus.

Proof. Let $e := \deg(f)/\deg(g)$.

(1) Assume that there exists an infinite sequence $\{(x_n, y_n, b_n)\}$ of distinct solutions in positive integers to the equation $f(x) = bg(y)$ such that for all n,
$$y_n \geq x_n^{e+\delta}.$$
Clearly, $\lim_{n\to\infty} x_n = \infty$. We have for all n sufficiently large:
$$b_n = \frac{|f(x_n)|}{|g(y_n)|} \leq \frac{|f(x_n)|}{|g(x_n^{e+\delta})|}.$$
Since $\lim_{x\to\infty} f(x)/g(x^{e+\delta}) = 0$, we find a contradiction with the fact that $b_n \geq 1$ for all n.

(2) Assume that there exists an infinite sequence $\{(x_n, y_n, b_n)\}$ of solutions in positive integers to the equation $f(x) = bg(y)$ such that for all n, $y_n > x_n^e$. As before, $\lim_{n\to\infty} x_n = \infty$. We have for all n sufficiently large: $b_n = \frac{|f(x_n)|}{|g(y_n)|} \leq \frac{|f(x_n)|}{|g(x_n^e)|}$. Since $\lim_{x\to\infty} f(x)/g(x^e)$ exists, we find that the set $\{b_n, n \in \mathbb{N}\}$ is bounded. There are thus only finitely many curves of the form $f(x) = b_n g(y)$ to consider, and by hypothesis each has a smooth projective model with positive genus. Thus we obtain a contradiction by applying Siegel's Theorem to each such curve to obtain that the union of their integer points is finite. □

The following proposition proves that Conjecture 1.1 holds in some cases under the hypothesis that the *abc*-conjecture of Masser and Oesterlé holds ([12], p. 24, or [16], Conjecture 3). Recall that when F is any field and $g(t) \in F[t]$, then the radical $\mathrm{rad}(g)$ is defined to be the product of the distinct irreducible factors of g. When $m > 1$ is an integer, $\mathrm{rad}(m)$ denotes the product of the distinct primes which divide m.

Proposition 2.2. *Assume that the abc-Conjecture is true. Let $f(x) \in \mathbb{Z}[x]$ be a polynomial without multiple roots. Let $g(y) \in \mathbb{Z}[y]$ be a polynomial with at least one multiple root. Let*

$\delta > 0$. *Then the equation* $f(x) = bg(y)$ *in the variables* (x, y, b) *has only finitely many solutions* (x_0, y_0, b_0) *in positive integers with*

$$\frac{\log(y_0)}{\log(x_0)} > \frac{1}{\deg(g) - \deg(\mathrm{rad}(g))} + \delta.$$

In particular, if $\frac{\deg(f)}{\deg(g)} - \frac{1}{\deg(g)-\deg(\mathrm{rad}(g))} > 0$, *then Conjecture 1.1* (2) *holds for* f *and* g *with* $\epsilon < \frac{\deg(f)}{\deg(g)} - \frac{1}{\deg(g)-\deg(\mathrm{rad}(g))}$. *This is the case for instance when* $\deg(f) > \deg(g)$, *or when* $\deg(f) \geq 3$ *and* $g(y) = y^s$.

Proof. Let $\beta > 0$. Under our assumptions that the abc-Conjecture is true and that $f(x)$ does not have repeated roots, we find in [6], Corollary 1 to Theorem 5, that

$$\mathrm{rad}(f(x_0)) := \prod_{\text{primes } p|f(x_0)} p \gg |x_0|^{\deg(f)-1-\beta},$$

where the constant implied in the notation $\gg$ depends on β and f. Consider now a solution (x_0, y_0, b_0) in positive integers of the equation $f(x) = bg(y)$. Then

$$\mathrm{rad}(f(x_0)) = \mathrm{rad}(b_0 g(y_0)) \leq b_0 \mathrm{rad}(g(y_0)) = b_0 \mathrm{rad}(\mathrm{rad}(g)(y_0)) \leq \frac{f(x_0)}{g(y_0)} \mathrm{rad}(g)(y_0) = \frac{f(x_0)}{\frac{g}{\mathrm{rad}(g)}(y_0)}.$$

It follows that there exists a constant c_0 depending on β such that, for all x_0, y_0 sufficiently large,

$$x_0^{\deg(f)-1-\beta} \leq c_0 \frac{x_0^{\deg(f)}}{y_0^{\deg(g)-\deg(\mathrm{rad}(g))}}.$$

Let $d := \deg(g) - \deg(\mathrm{rad}(g))$ and recall that by hypothesis, $d > 0$. Thus, we can write:

$$\frac{\log(y_0)}{\log(x_0)} \leq \frac{1}{d} + \frac{\beta}{d} + \frac{\log(c_0)}{\log(x_0)d}.$$

It follows that if we choose $\delta > 0$, we can find β such that $\frac{\beta}{d} \leq \frac{1}{2}\delta$, and we obtain that for all x_0 large enough (depending of β), we have $\frac{\log(c_0)}{\log(x_0)d} \leq \frac{1}{2}\delta$. If follows that for all solutions (x_0, y_0, b_0) with x_0 and y_0 large enough, we have the desired inequality

$$\frac{\log(y_0)}{\log(x_0)} \leq \frac{1}{d} + \delta.$$

Clearly, for a fixed x_0, the equation $f(x_0) = bg(y)$ has only finitely many integer solutions (y_0, b_0). For a fixed y_0, $\frac{\log(y_0)}{\log(x_0)} \leq \frac{1}{d} + \delta$ as soon as x_0 is large enough. It follows that there can be only finitely many solutions (x_0, y_0, b_0) with $\frac{\log(y_0)}{\log(x_0)} > \frac{1}{d} + \delta$, as desired. □

Example 2.3 Consider a conic with infinitely many integral points, given by an equation $x^2 - dy^2 = -e$, with $d, e \in \mathbb{N}$. Let $f(x) \in \mathbb{Z}[x]$, with $\deg(f) \geq 3$. Assume that $f(x)$ is divisible by $(x^2 + e)$ in $\mathbb{Z}[x]$, and write $f(x) = (x^2 + e)h(x)$ with $\deg(h) > 0$. Then the equation $f(x) = by^2$ has infinitely many solutions (x, y, b) in positive integers with $y > x/\sqrt{d}$. Indeed, for each solution (x_0, y_0) of the equation $x^2 - dy^2 = -e$, we get the equality

$$f(x_0) = dh(x_0)y_0^2.$$

In this example, $\deg(f) \geq 3$ and $\deg(g) = 2$, and we find that for the statement of Conjecture 1.1 to hold for ϵ, we need

$$\frac{\deg(f)}{\deg(g)} - \epsilon > 1.$$

Proposition 2.2 shows that if the *abc*-Conjecture holds, then Conjecture 1.1 is true with $\epsilon < \frac{\deg(f)}{\deg(g)} - 1$. Thus the bound for ϵ provided in 2.2 when $g(y) = y^2$ is sharp.

Example 2.4 Let $a = 1, 3, 11, \ldots$ be an odd positive integer such that the hyperbola $an^2 - 2m^2 = 1$ has infinitely many integer solutions (n, m). Inspired by [27], 2.1, we find that $(x, y, b) = (2m^3 + 3m, n, a^2(m^2 + 2))$ is a solution to the equation $x^2 + 2 = by^4$. When $a = 1$, one easily checks that the solutions (x, y, b) in this parametric family are such that $y^3 > x$, and that $\lim_{y\to\infty} \frac{\log(y)}{\log(x)} = \frac{1}{3}$.

Proposition 2.2 shows that if the *abc*-Conjecture holds, then given any $\delta > 0$, there exist only finitely many positive solutions (x, y, b) to the equation $x^2 + 2 = by^4$ with $\frac{\log(y)}{\log(x)} > \frac{1}{3} + \delta$. Since there exist infinitely many positive integer solutions (m, n) to $n^2 - 2m^2 = 1$, we find that the statement obtained under the *abc*-conjecture when $\delta > 0$ does not hold for any $\delta \leq 0$.

The y-values of the first solutions (x, y, b) in the case of the infinite family with $a = 1$ are $3, 17, 99, 577, 3363, 19601, ...$, and are part of the sequence A001541 in [17]. The first solution found to the equation $x^2 + 2 = by^4$ with $\frac{\log(y)}{\log(x)} > \frac{1}{3}$ which does not belong to the infinite family with $a = 1$ is $(15379924712, 3777, 1162306)$ with $\frac{\log(y)}{\log(x)} = 0.3511498...$ We do not know of a solution with $\frac{\log(y)}{\log(x)} > 0.36$. The number of integer solutions to $x^2 + 2 = by^4$ for a given b is considered in [1] and [26].

We note that it follows from 3.12 (a) that if there exist non-constant polynomials $x(t), y(t), b(t)$ in $\mathbb{Q}[t]$ such that $x(t)^2 + 2 = b(t)y(t)^4$, then $\frac{\deg_t(y)}{\deg_t(x)} < \frac{1}{3}$. This can be easily shown directly by noting that if $x(t)^2 + 2 = b(t)y(t)^4$, then $y(t)^3$ divides $x'(t)$.

Remark 2.5 Fix $f(x) \in \mathbb{Z}[x]$ of positive degree. Given $g_0(x) = x^2$, it follows from 2.2 under the abc-conjecture that Conjecture 1.1 (1) holds for the pair $f(x)$ and $g_0(x)$ for $\epsilon < 1/2$. We note here that the same statement does not hold for all quadratic polynomials $g(x)$. For instance, in 3.4, we present an example where $g_1(x) = x(x + 1)$ and Conjecture 1.1 (1) can hold for the pair $f(x)$ and $g_1(x)$ only when $\epsilon < 1/6$.

Let now $g(x)$ be any non-constant integer polynomial. Fix a positive integer c_0, and assume that $c_0 f(x) = bg(y)$ defines a curve of positive genus over $\mathbb{Q}(b)$. Denote by $\mathcal{N}_{c_0}(\epsilon)$ the set of all solutions (x_0, y_0, b_0) in positive integers to the equation $c_0 f(x) = bg(y)$ with $\frac{\log(y_0)}{\log(x_0)} > \frac{\deg(f)}{\deg(g)} - \epsilon$ and such that $c_0 f(x) = b_0 g(y)$ defines a curve of positive genus over $\mathbb{Q}$. (Note that we do not impose here that $\gcd(c_0, b_0) = 1$.) It is clear that if $\mathcal{N}_1(\epsilon)$ is infinite, then $\mathcal{N}_{c_0}(\epsilon)$ is also infinite for any positive integer c_0. It is natural to wonder whether a converse could be true, namely, if there exists $c_0 > 0$ such that $\mathcal{N}_{c_0}(\epsilon)$ is infinite, does it automatically follow that $\mathcal{N}_1(\epsilon)$ is also infinite? It this question had a positive answer, then Conjecture 1.1 (2) would follow from Conjecture 1.1 (1).

Proposition 2.2 (under the abc-conjecture) shows that Conjecture 1.1 (1) holds for some pairs $f(x)$ and $g(x)$ with a constant ϵ that depends only on $\deg(f)$ and $\deg(g)$. One may wonder whether this stronger conclusion holds in general. If such were the case, then clearly 1.1 (2) would also hold.

Remark 2.6 Consider the surface $X/\mathbb{Q}$ in the (x,y,b)-affine space given by the equation $f(x)=bg(y)$. We note here that an additional assumption, such as one of the form $\frac{\log(y)}{\log(x)}>\delta$, seems essential in order to obtain a non-trivial finiteness statement for the number of positive integer solutions to $f(x)=bg(y)$. For instance, one might ask, in the spirit of the Lang-Vojta conjecture ([9], F.5.3.6), whether there exist finitely many irreducible algebraic curves on the surface X such that the complement of the union of these curves in X contains only finitely many integer points. This question is easily shown to have a negative answer when $f(x)=xh(x)$ since in that case, for each integer $e>0$, the parametric curve C_e given by $(x(y),y,b(y))$ with

$$\begin{aligned} x(y) &:= eg(y) \\ b(y) &:= eh(eg(y)) \end{aligned}$$

lies on the surface X, and contains infinitely many integer points. Note that when in addition $h(0)\neq 0$, then the curves C_e and $C_{e'}$ do not intersect on X when $e\neq e'$.

Conjecture 1.1 predicts the existence of $\epsilon>0$ such that there exists only finitely many positive solutions (x,y,b) with $\frac{\log(y)}{\log(x)}>\frac{\deg(f)}{\deg(g)}-\epsilon$. Assuming this conjecture, we can then ask a more general question:

What is the largest value of $\epsilon>0$ such that there exists only finitely many positive solutions (x,y,b) with $\frac{\log(y)}{\log(x)}>\frac{\deg(f)}{\deg(g)}-\epsilon$ lying outside of a finite set S of curves on the surface X.

In the example where $f(x)=xh(x)$, the existence of the infinitely many curves C_e each containing infinitely many positive solutions shows that the answer to the more general question can only produce in this case an ϵ with $\epsilon<\frac{\deg(f)}{\deg(g)}-\frac{1}{\deg(g)}$.

Let us now slightly modify Example 2.3 so that we can use an explicit example in [7]. Consider the surface $f(x)=by^2$ in the (x,y,b)-space with $f(x):=(x^2+x+1)h(x)$. Consider the curve C_k in the surface X obtained by intersecting X with the surface given by $x^2+x+1=ky^2$. It is known [7] that if $k=(q^2+3)/4$ for some integer q, then the conic C_k contains infinitely many integral points with positive coordinates. It is easy to check that the conics C_k and $C_{k'}$ have no integer solutions in common when $k\neq k'$.

For each point (x_0,y_0) on C_k having positive coordinates, we can write $f(x_0)=h(x_0)ky_0^2$, so $(x_0,y_0,b_0:=h(x_0)k)$ is an integer point on the surface X such that $\frac{\log(y)}{\log(x)}>1-\frac{\log(b_0)}{2\log(x_0)}$. Thus the existence of the infinitely many curves C_k each containing infinitely many positive solutions shows that the answer to the more general question can only produce in this case an ϵ with $\epsilon<\frac{\deg(f)}{\deg(g)}-\frac{1}{\deg(g)-1}$.

3. Upper bounds for ϵ

An immediate type of constraint on the possible ϵ's in Conjecture 1.1 comes from the existence of parametric solutions. Suppose that $f(x),g(x)\in\mathbb{Z}[x]$ have positive degree, and that we can find $x(t),y(t),b(t)\in\mathbb{Q}[t]$ such that the equation

$$f(x(t))=b(t)g(y(t))$$

holds in $\mathbb{Q}[t]$. Clearly when $x(t)$ is not constant,

$$\frac{\deg_t(y)}{\deg_t(x)}=\frac{\deg(f)}{\deg(g)}-\frac{\deg_t(b)}{\deg(g)\deg_t(x)}\leq\frac{\deg(f)}{\deg(g)}. \tag{3.1}$$

Let $\mathrm{Int}(\mathbb{Z})$ denote the ring of integer-valued polynomials:

$$\mathrm{Int}(\mathbb{Z}):=\{h(t)\in\mathbb{Q}[t]\mid h(\mathbb{Z})\subseteq\mathbb{Z}\}.$$

Lemma 3.2. *Let $f(x), g(x) \in \mathbb{Z}[x]$ be non-constant. Suppose that we can find a solution as above with $x(t), y(t), b(t) \in \mathrm{Int}(\mathbb{Z})$ and both $x(t)$ and $b(t)$ not constant. Then $\frac{\deg_t(y)}{\deg_t(x)} < \frac{\deg(f)}{\deg(g)}$. Assume that $f(x) - ag(y)$ is geometrically irreducible in $\mathbb{Q}(a)[x,y]$, and that its associated smooth projective curve over $\mathbb{Q}(a)$ has positive genus. If Conjecture 1.1 holds for some ϵ for the pair $f(x)$ and $g(x)$, then*

$$\frac{\deg_t(y)}{\deg_t(x)} < \frac{\deg(f)}{\deg(g)} - \epsilon.$$

Proof. The first inequality follows from (3.1). It is clear that the function $\frac{y(t)^{\deg_t(x)}}{x(t)^{\deg_t(y)}}$ has a finite limit when t tends to infinity. Thus, given any $\delta > 0$, there exists a positive integer t_0 such that for all $t_1 > t_0$,

$$\left| \frac{\log(|y(t_1)|)}{\log(|x(t_1)|)} - \frac{\deg_t(y)}{\deg_t(x)} \right| < \delta.$$

By hypothesis, the values of the polynomials $x(t), y(t)$, and $b(t)$, are integers when t is an integer. If Conjecture 1.1 holds for ϵ, we see that there exists an integer $t_2 > t_0$ such that

$$\frac{\deg_t(y)}{\deg_t(x)} - \delta < \frac{\log(|y(t_2)|)}{\log(|x(t_2)|)} \leq \frac{\deg(f)}{\deg(g)} - \epsilon.$$

Since this is true for any $\delta > 0$, the result follows. □

Remark 3.3 The hypothesis in the above lemma that the polynomials $x(t), y(t)$, and $b(t)$, belong to $\mathrm{Int}(\mathbb{Z})$ is needed in its proof. Indeed, consider $f(x) = x(x^2+x+1)(x^2+x+3)$ and $g(y) = y(y+1)$. Then $x(t) := t$, $y(t) := \frac{1}{2}(t^2+t+1)$, and $b(t) := 4t$, is a solution to $f(x) = bg(y)$ in $\mathbb{Q}[t]$, and $y(t)$ never takes integer values. The existence of this solution does not imply an upperbound on the ϵ for which Conjecture 1.1 holds for f and g. (A computation with Magma shows that there are no other solutions $(x(t), y(t), b(t))$ of $f(x) = bg(y)$ in $\mathbb{Q}[t]$ with $\deg_t(x) = 1$, $\deg_t(y) = 2$ and positive leading coefficients.)

Example 3.4 Consider the equation

$$x(x+1)(x+2) = by(y+1). \tag{3.5}$$

The sets of integer solutions of (3.5) when $b = 1$ and $b = 3$ are considered in [22], Theorem B34 and Theorem A23, and [15]. Let $P := (0,0)$ and $Q := (-2,0)$. It is quite possible that the Mordell-Weil group of the elliptic curve over $\mathbb{Q}(t)$ defined by (3.5) when $b = t$ is generated by P and Q. The integral solution $-3P - 3Q = (x(t), y(t))$, with

$$\begin{array}{rcl} x(t) & = & (t^2-t-1)(t+1) \\ y(t) & = & (t^2-t-1)(t^2+t-1), \end{array} \tag{3.6}$$

is of interest since $\frac{\deg_t(y)}{\deg_t(x)} = 4/3$, while $\frac{\deg(f)}{\deg(g)} = 3/2$. If Conjecture 1.1 holds for some $\epsilon > 0$, then Lemma 3.2 implies that $\epsilon \leq 3/2 - 4/3 = 1/6$.

Example 3.7 Consider the equation

$$x(x+1)(x+2)(x+3) = by(y+1), \tag{3.8}$$

which is preserved by the involutions $x \mapsto -x-3$ and $y \mapsto -y-1$. When $b = 4$, this equation defines a singular plane curve, parameterized by $x(t) = t$ and $y(t) = t(t+3)/2$.

The 20 integer solutions of (3.8) when $b = 12$ are described in [22], Theorem A24. Equation (3.8) has some parametric solutions of interest with $\frac{\deg_t(y)}{\deg_t(x)} = 3/2$:

$x(t)$	$y(t)$	$b(t)$
$(t+1)(2t-3)$	$t(2t^2-t-2)$	$(2t+1)(2t-3)$
$(t-1)(2t+3)$	$(t-1)(t+1)(2t+1)$	$(2t-1)(2t+3)$

Except for when $t = 1, 2$, these parameterizations produce solutions (x, y, b) with $\frac{\log(y)}{\log(x)} < 3/2$, and $\lim_{t\to\infty} \frac{\log(y)}{\log(x)} = 3/2$. If Conjecture 1.1 holds for Equation (3.8) for some $\epsilon > 0$, then Lemma 3.2 implies that $\epsilon < 1/2$. Note that the above solutions are linked by the involution $t \mapsto -t$ and $y \mapsto -y-1$. Note also that replacing t by $t-1$ in the second solution produces a second $\mathbb{Q}[t]$-rational point on the same genus 1 curve given by the equation $x(x+1)(x+2)(x+3) = (2t+1)(2t-3)y(y+1)$.

3.9 The importance of parametric solutions leads us to ask whether there exists an analogue to Conjecture 1.1 in the function field case. Motivated by Lemma 3.2, we consider now only the following more restricted question. Let F be a field. Let $f(x), g(x) \in F[x]$ be non-constant. Does there exist $\epsilon > 0$ such that if a solution $(x_0(t), y_0(t), b_0(t))$ to $f(x) = bg(y)$ is such that

(a) $x_0(t), y_0(t), b_0(t) \in F[t]$ and $x_0(t)$ and $b_0(t)$ are not both constant, and
(b) The equation $f(x) = b_0(t)g(y)$ has only finitely many integral solutions $(x(t), y(t))$ with $x(t), y(t) \in F[t]$ (for integral points on curves over function fields, see for instance [19] or [25]),

then
$$\frac{\deg_t(y_0)}{\deg_t(x_0)} < \frac{\deg_x(f)}{\deg_x(g)} - \epsilon.$$
Note that under our hypotheses, we have $\frac{\deg_t(y_0)}{\deg_t(x_0)} \leq \frac{\deg_x(f)}{\deg_x(g)}$. It is possible to obtain in some cases an upper bound for $\frac{\deg_t(y)}{\deg_t(x)}$ smaller than $\frac{\deg(f)}{\deg(g)}$ using the ABC-theorem for polynomials (see, e.g., [9], F.3.6, or [12], page 19). This theorem, also called the Mason-Stothers Theorem ([13], and [24], 1.1), states that:

Let $a(t)$, $b(t)$, and $c(t)$ be relatively prime polynomials over a field F such that $a + b = c$ and such that not all of them have vanishing derivative. Then
$$\max\{\deg(a), \deg(b), \deg(c)\} \leq \deg(\mathrm{rad}(abc)) - 1.$$
When $\mathrm{char}(F) = 0$, we find that if suffices to assume that $a(t)$, $b(t)$, and $c(t)$ are relatively prime polynomials and not all constant.

We use the ABC-theorem and a key lemma of Belyi to obtain the following theorem, whose proof uses some ideas of Granville in [6].

Theorem 3.10. *Let $F(x, y) \in \mathbb{Q}[x, y]$ be a homogeneous polynomial. Let $m(t), n(t) \in \mathbb{Q}[t]$ be two coprime polynomials, not both constant. Then*
$$\deg_t(\mathrm{rad}(F(m, n))) - 1 \geq (\deg(\mathrm{rad}(F)) - 2)\max(\deg_t(m), \deg_t(n)).$$

Proof. Belyi's Lemma in the form given in [6], Lemma 1, is used to show the existence of three coprime homogeneous polynomials $a(x, y), b(x, y)$, and $c(x, y)$ in $\mathbb{Z}[x, y]$ of degree $D > 0$ such that $a + b = c$, and such that $\mathrm{rad}(abc)$ has degree $D + 2$ and is divisible by rad(F). Since $m(t)$ and $n(t)$ are coprime, and so are $a(x,y)$ and $b(x,y)$, we find that $a(m,n)$ and $b(m,n)$ are coprime in $\mathbb{Q}[t]$. Since $m(t)$ and $n(t)$ are not both constant, we find that $a(m,n)$ and $b(m,n)$ cannot be both constant. The ABC-Theorem can then be applied to $a(m,n)+b(m,n)=c(m,n)$ to obtain the following inequality:

$$D\max(\deg_t(m),\deg_t(n))+1\leq\deg_t(\mathrm{rad}(a(m,n)b(m,n)c(m,n))).$$

Write $\mathrm{rad}(abc)=\mathrm{rad}(F)(x,y)h(x,y)$ for some $h(x,y)\in\mathbb{Q}[x,y]$. Then

$$\begin{aligned}\deg_t(\mathrm{rad}(a(m,n)b(m,n)c(m,n))) &= \deg_t(\mathrm{rad}(\mathrm{rad}(abc)(m,n)))\\ &= \deg_t(\mathrm{rad}(\mathrm{rad}(F)(m,n)h(m,n))))\\ &\leq \deg_t(\mathrm{rad}(F(m,n)))+\deg(h)\max(\deg_t(m),\deg_t(n))\end{aligned}$$

Putting these inequalities together, along with the expression $\deg(h)=(D+2)-\deg(\mathrm{rad}(F))$, concludes the proof. □

Corollary 3.11. *Let $f(x)\in\mathbb{Q}[x]$. Let $m(t)\in\mathbb{Q}[t]$ be of positive degree. Then*

$$\deg_t(\mathrm{rad}(f(m)))\geq 1+\deg_t(m)(\deg(\mathrm{rad}(f))-1).$$

Proof. Apply Theorem 3.10 to $F(x,y):=y^{\deg(f)+1}f(x/y)$, and to the polynomials $m(t)$ and $n(t)=1$. □

Note that the inequality in the corollary becomes an equality when 0 is a simple root of f and $m(t)=t^s$.

Theorem 3.12. *Let $f(x),g(x)\in\mathbb{Q}[x]$. Let $(x_0(t),y_0(t),b_0(t))$ be a solution of the equation $f(x)=bg(y)$ with $x_0(t),y_0(t),b_0(t)\in\mathbb{C}[t]$ and $x_0(t)$ not constant. Then*

$$(\deg(g)-\deg(\mathrm{rad}(g)))\deg_t(y_0)\leq(\deg(f)-\deg(\mathrm{rad}(f))+1)\deg_t(x_0)-1, \tag{3.13}$$

and

$$(\deg(\mathrm{rad}(g))-1)\deg_t(y_0)\leq\deg(\mathrm{rad}(f))\deg_t(x_0)-1. \tag{3.14}$$

In particular,

(a) *If g has at least one multiple root, then*

$$\frac{\deg_t(y_0)}{\deg_t(x_0)}<\frac{\deg(f)-\deg(\mathrm{rad}(f))+1}{\deg(g)-\deg(\mathrm{rad}(g))}.$$

If in addition $\deg(f)>\deg(g)$ and f has no multiple roots, then $\frac{\deg_t(y_0)}{\deg_t(x_0)}<\frac{1}{\deg(g)-\deg(\mathrm{rad}(g))}<\frac{\deg(f)}{\deg(g)}$.

(b) *If $\deg(\mathrm{rad}(g))>1$, then*

$$\frac{\deg_t(y_0)}{\deg_t(x_0)}<\frac{\deg(\mathrm{rad}(f))}{\deg(\mathrm{rad}(g))-1}.$$

If in addition f has at least one multiple root, g has no multiple root, and $\deg(f)<\deg(g)$, then $\frac{\deg_t(y_0)}{\deg_t(x_0)}<\frac{\deg(f)-1}{\deg(g)-1}<\frac{\deg(f)}{\deg(g)}$. Similarly, if in addition $f(x)=x^r$ and $\deg(f)>\deg(g)$, then $\frac{\deg_t(y_0)}{\deg_t(x_0)}<\frac{1}{\deg(\mathrm{rad}(g))-1}<\frac{\deg(f)}{\deg(g)}$.

Proof. We apply Corollary 3.11 to $f(x)$ and $x_0(t)$, to obtain the first inequality below:

$$\begin{aligned}1+\deg_t(x_0)(\deg(\mathrm{rad}(f))-1) &\leq \deg_t(\mathrm{rad}(f(x_0)))\\ &= \deg_t(\mathrm{rad}(b_0 g(y_0))\\ &\leq \deg_t(b_0)+\deg_t(\mathrm{rad}(g(y_0))\\ &= \deg_t(b_0)+\deg_t(\mathrm{rad}(\mathrm{rad}(g)(y_0))\\ &\leq \deg_t(b_0)+\deg(\mathrm{rad}(g))\deg_t(y_0)\\ &= \deg(f)\deg_t(x_0)-\deg(g)\deg_t(y_0)+\deg(\mathrm{rad}(g))\deg_t(y_0),\end{aligned}$$

and (3.13) follows. To prove (3.14), we apply Corollary 3.11 to $g(y)$ and $y_0(t)$ to obtain the first inequality below:

$$\begin{aligned}(\deg(\mathrm{rad}(g))-1)\deg_t(y_0) &\leq \deg_t(\mathrm{rad}(g(y_0)))-1\\ &\leq \deg_t(\mathrm{rad}(b_0 g(y_0))-1\\ &= \deg_t(\mathrm{rad}(f(x_0))-1\\ &\leq \deg(\mathrm{rad}(f))\deg_t(x_0)-1.\end{aligned}$$

□

Remark 3.15 Let F be any field. Let $f(x)$ and $g(x)$ be non-constant polynomials in $F[x]$. Then the following statements are equivalent:

(a) There exists $\epsilon > 0$ such that $\frac{\deg_t(y_0)}{\deg_t(x_0)} < \frac{\deg(f)}{\deg(g)} - \epsilon$ for any solution $(x_0(t), y_0(t), b_0(t))$ to the equation $f(x) = bg(y)$ in polynomials in $F[t]$ with $x_0(t)$ and $b_0(t)$ non-constant.

(b) There exists $C_1 > 0$ such that $\frac{\deg_t(x_0)}{\deg_t(b_0)} < C_1$ for any solution $(x_0(t), y_0(t), b_0(t))$ to the equation $f(x) = bg(y)$ in polynomials in $F[t]$ with $x_0(t)$ and $b_0(t)$ non-constant.

(c) There exists $C_2 > 0$ such that $\frac{\deg_t(y_0)}{\deg_t(b_0)} < C_2$ for any solution $(x_0(t), y_0(t), b_0(t))$ to the equation $f(x) = bg(y)$ in polynomials in $F[t]$ with $x_0(t)$ and $b_0(t)$ non-constant.

The equivalences follow from the equality

$$\deg(f)\deg_t(x_0) = \deg_t(b_0) + \deg(g)\deg_t(y_0).$$

The equivalence of (b) and (c) is immediate. To show that (c) implies (a), we note that

$$\frac{\deg_t(y_0)}{\deg_t(x_0)} = \frac{\deg(f)}{\deg(g)} - \frac{\deg_t(b_0)}{\deg(g)\deg_t(x_0)} < \frac{\deg(f)}{\deg(g)} - \frac{\deg_t(y_0)}{C_2\deg(g)\deg_t(x_0)},$$

and we can take

$$\epsilon = \frac{\deg(f)}{(C_2\deg(g)+1)\deg(g)}.$$

Assuming (b), we find that (a) holds with $\epsilon = \frac{1}{C_1\deg(g)}$. Assuming (a), we find that $\frac{\deg_t(x_0)}{\deg_t(b_0)} < \frac{1}{\epsilon\deg(g)}$.

We can slightly improve Theorem 3.12, in the special case where $x^r = bg(y)$ and $r \leq \deg(g)$, using the main result of [14].

Proposition 3.16. *Let F be a field of characteristic 0. Let $f(x) = x^r$, $r > 1$, and $g(y) \in F[y]$. Assume that the equation $x^r = g(y)$ defines a smooth projective geometrically connected curve over F of positive genus. Then,*

(a) *Given any solution* $(x_0(t), y_0(t), b_0(t))$ *to* $x^r = bg(y)$ *in polynomials in* $F[t]$ *with* $x_0(t)$ *and* $b_0(t)$ *not constant, we have* $\frac{\deg_t(y_0)}{\deg_t(x_0)} < \frac{\deg(f)}{\deg(g)} - \frac{\deg(f)}{(84\deg(g)+1)\deg(g)}$.

(b) *Given any solution* $(x_0(t), y_0(t), b_0(t))$ *to* $g(x) = by^r$ *in polynomials in* $F[t]$ *with* $x_0(t)$ *and* $b_0(t)$ *not constant, we have* $\frac{\deg_t(y_0)}{\deg_t(x_0)} < \frac{1}{r}\deg(g) - \frac{1}{84r}$.

Proof. (a) Given a solution $(x_0(t), y_0(t), b_0(t))$ to $x^r = bg(y)$, we let $\overline{g}(y) := b_0(t)g(y) \in F[t][y]$. We apply to the equation $x^r = \overline{g}(y)$ the main result of [14], Theorem on page 168, to obtain the following inequality:

$$\deg_t(y_0) \leq 78\deg_t(b_0) + 6 \leq 84\deg_t(b_0). \tag{3.17}$$

In the notation of [14], page 168, we apply the Theorem to the case where $K = F(t)$ with $g_K = 0$, and where S consists only in the place corresponding to the point at infinity, with local ring $F[1/t]_{(1/t)}$, so that $\mathcal{O}_S = F[t]$. Given two coprime polynomials $u(t)$ and $v(t)$ in $F[t]$, the reader will check that the height $H(u/v)$ introduced on page 168 of [14] reduces to the simple formula $H(u/v) = \max(\deg(u), \deg(v))$ when the field K is $F(t)$. The height of the polynomial $\overline{g}(y)$ is easily checked to be $\deg(b_0(t))$. To be able to apply the Theorem, we note that our hypothesis that the equation $x^r = g(y)$ defines a smooth projective geometrically connected curve over F of positive genus implies that the hypotheses on the factorization of the polynomial $\overline{g}(y) \in \overline{F(t)}[y]$ in the Theorem is satisfied (when the condition (†) in the Theorem is satisfied, the resulting curve has genus 0 and two points at infinity, or is the union of such curves). The genus g of the superelliptic curve given by an equation of the form $x^r = \alpha\prod_{i=1}^n (y-\alpha_i)^{r_i}$ when $\gcd(r, r_1, \ldots, r_n) = 1$ can be computed using the Riemann-Hurwitz formula, and depends only on r and $r_1, \ldots, r_n$. For instance, when r divides $\sum_i r_i$, we find that $2g-2 = r(n-2) - \sum_i \gcd(r, r_i)$. Thus our assumption on the genus of $x^r = g(y)$ implies that the same assumption holds for the curve $x^r = bg(y)$. Part (a) of the proposition then follows from (3.17) and from the fact that (c) implies (a) in Remark 3.15.

(b) Given a solution $(x_0(t), y_0(t), b_0(t))$ to $g(x) = by^r$, we let $\overline{g}(x) := \frac{1}{b_0(t)}g(x) \in F(t)[x]$. We apply to the equation $y^r = \overline{g}(x)$ the main result of [14], Theorem on page 168, to obtain the following inequality:

$$\deg_t(x_0) \leq 78\deg_t(b_0) + 6 \leq 84\deg_t(b_0). \tag{3.18}$$

Part (a) of the proposition then follows from (3.18) and from the fact that (b) implies (a) in Remark 3.15. □

Theorem 3.12 does not answer Question 3.9 for the equation $f(x) = bg(y)$ when both $f(x)$ and $g(x)$ do not have multiple roots. We can answer the question in the particular case where $\deg(g) = 2$ by again using [14].

Proposition 3.19. *Let* F *be a field of characteristic* 0. *Let* $f(x) \in F[x]$ *with* $\deg(f) \geq 3$, *and let* $g(y) \in F[y]$ *be separable of degree* 2. *Then*

(a) *Given any solution* $(x_0(t), y_0(t), b_0(t))$ *to* $f(x) = bg(y)$ *in polynomials in* $F[t]$ *with* $x_0(t)$ *and* $b_0(t)$ *not constant, we have* $\frac{\deg_t(y_0)}{\deg_t(x_0)} < \frac{1}{2}\deg(f) - \frac{1}{168}$.

(b) *Given any solution* $(x_0(t), y_0(t), b_0(t))$ *to* $bf(y) = g(x)$ *in polynomials in* $F[t]$ *with* $x_0(t)$ *and* $b_0(t)$ *not constant, we have* $\frac{\deg_t(y_0)}{\deg_t(x_0)} < \frac{2}{\deg(f)} - \frac{2}{(84\deg(f)+1)\deg(f)}$.

Proof. (a) It is easy to see that to prove Part (a), it is sufficient to prove Part (a) in the case where $g(y) = y^2 + d \in F[y]$ with $d \neq 0$ (the latter condition must hold since we assume that $g(y)$ is separable). Given a solution $(x_0(t), y_0(t), b_0(t))$ to $f(x) = b(y^2 + d)$, we let

$\overline{f}(x) := \frac{1}{b_0(t)}f(x) - d \in F(t)[x]$. We apply to the equation $\overline{f}(x) = y^2$ the main result of [14], Theorem on page 168, to obtain the following inequality:

$$\deg_t(x_0) \leq 78 \deg_t(b_0) + 6 \leq 84 \deg_t(b_0). \tag{3.20}$$

To be in a position to use the main result of [14], we need to verify that the hyperelliptic curve defined by $y^2 = \overline{f}(x)$ defines a curve over the field $F(t)$ of positive genus. For this, we need to verify that when $db_0(t)$ has positive degree, $f(x) - b_0(t)d$ has at least 3 distinct roots of odd multiplicities in the algebraic closure of $F(t)$. Since $\deg(f) \geq 3$, $f(x) - b_0(t)d$ at least 3 distinct roots unless it has a multiple root, and this latter condition is equivalent to a root of $f'(x)$ also being a root of $f(x) - b_0(t)d$. Thus any multiple root of $f(x) - b_0(t)d$ is in fact in $\overline{F}$, and we find that $f(x) - b_0(t)d$ has no multiple roots since it has no root in $\overline{F}$. Indeed, if there exists $\alpha \in \overline{F}$ such that $f(\alpha) - b_0(t)d = 0$, we would have $b_0(t)d$ constant. Part (a) of the proposition then follows from (3.20) and from the fact that (b) implies (a) in Remark 3.15.

(b) As in (a) it suffices to prove the result when $g(x) = x^2 + d \in F[x]$ with $d \neq 0$. Given a solution $(x_0(t), y_0(t), b_0(t))$ to $bf(y) = x^2 + d$, we let $\overline{f}(y) := b_0(t)f(y) - d \in F[t][y]$. We apply to the equation $\overline{f}(y) = x^2$ the main result of [14], Theorem on page 168, to obtain the following inequality:

$$\deg_t(y_0) \leq 78 \deg_t(b_0) + 6 \leq 84 \deg_t(b_0). \tag{3.21}$$

Part (b) of the proposition then follows from (3.21) and from the fact that (c) implies (a) in Remark 3.15. □

Example 3.22 Recall Equation (3.8):

$$x(x+1)(x+2)(x+3) = by(y+1).$$

Proposition 3.19 (a) shows if there exists a solution $(x(t), y(t), b(t))$ of Equation (3.8) with polynomials in $\mathbb{Q}[t]$ with $x(t)$ and $b(t)$ not constant, then $\frac{\deg_t(y)}{\deg_t(x)} < 2 - \frac{1}{168}$. In Example 3.7, we presented such a solution with $\frac{\deg_t(y)}{\deg_t(x)} = \frac{3}{2}$, and the data discussed in section 7 is consistent with the possibility that $\frac{\deg_t(y)}{\deg_t(x)} \leq \frac{3}{2}$ for any such solution. We present in this example solutions to Equation (3.8) with

$$\frac{\deg_t(y)}{\deg_t(x)} \in \{\frac{n+1}{n+2}, n \in \mathbb{N}, 1, \frac{4}{3}, \frac{3}{2}\}.$$

The data discussed in section 7 suggests that if $\frac{\deg_t(y)}{\deg_t(x)} \geq 1$, then $\frac{\deg_t(y)}{\deg_t(x)} \in \{1, \frac{4}{3}, \frac{3}{2}\}$.

First, note that using the solution $(x(t), y(t), t)$ in (3.6), we obtain two solutions of Equation (3.8) with $\frac{\deg_t(y)}{\deg_t(x)} = \frac{4}{3}$, namely:

$$(x(t), y(t), t(x(t)+3)) \quad \text{and} \quad (x(t)-1, y(t), t(x(t)-1)).$$

A much less obvious solution $(x(t), y(t), b(t))$ to Equation (3.8) with polynomials in $\mathbb{Q}[t]$ and with $\frac{\deg_t(y)}{\deg_t(x)} = \frac{4}{3}$ is as follows:

$$\begin{aligned} x &= \tfrac{1}{54}(t-6)(t^2+8t+21), \\ y &= \tfrac{1}{216}(t-6)(t+3)(t^2+5t-12), \\ b &= \tfrac{4}{729}(t^2-4t-3)(t^2+8t+21). \end{aligned}$$

This solution was obtained by carefully considering the solutions (x, y, b, c) in positive integers to $cx(x+1)(x+2)(x+3) = by(y+1)$ with $\frac{\log(y)}{\log(x)}$ around 4/3. Using the changes of variables $t = 36s-3$ or $t = 108s+6$, we obtain two solutions $(x(s), y(s), b(s))$ in integer polynomials to the equation $27x(x+1)(x+2)(x+3) = by(y+1)$. Composing further the change $t = 36s - 3$ with the change $s = 27r + 5$ produces the following solution $(x(r), y(r), b(r))$ in integer polynomials to the original Equation (3.8) $x(x+1)(x+2)(x+3) = by(y+1)$:

$$\begin{aligned} x &= (108r+19)(157464r^2+58644r+5461), \\ y &= 27(27r+5)(108r+19)(52488r^2+19386r+1789), \\ b &= 16(1944r^2+700r+63)(157464r^2+58644r+5461). \end{aligned}$$

It is easy to write down solutions to Equation (3.8) with $\deg_t(y) = \deg_t(x) = 1$, such as $y(t) = t$ and $x(t) \in \{t-2, t-1, t, 2t-1, 2t, 3t\}$. We note that there are also at least eight solutions with $\deg_t(y) = \deg_t(x) = 2$ which are not obtained by composition from solutions with $\deg_t(y) = \deg_t(x) = 1$; one such solution has $x(t) = \frac{1}{2}t(t+5)$ and $y(t) = \frac{1}{2}t(t+3)$.

There is a infinite supply of solutions with $\frac{\deg_t(y)}{\deg_t(x)} = \frac{2}{3}$. Indeed, the equation

$$cx(x+s) = \beta y(y+1)$$

has a family of parametric solutions $(x_c(c,s,t), y_c(c,s,t), \beta_c(c,s,t))$ obtained as follows:

$$\begin{aligned} x_c(t) &= t(tc+2)((s+t)c+1), \\ y_c(t) &= c(t(s+t)c+2t+s), \\ \beta_c(t) &= t(tc+2). \end{aligned} \tag{3.23}$$

Let $h(x) \in \mathbb{Z}[x]$ be a polynomial of positive degree, such as $h(x) = (x+2)(x+3)$. Fix $s_0 > 0$, such as $s_0 = 1$. For each positive integer c_0, we can use the above solution to produce a solution $(x_{c_0}, y_{c_0}, b_{c_0})$ to $x(x+s_0)h(x) = by(y+1)$ by setting $b_{c_0} := \beta(c_0, s_0, t)h(x_{c_0})/c_0 \in \mathbb{Q}[t]$. When $c_1 \neq c_0$, then the solutions $(x_{c_0}, y_{c_0}, b_{c_0})$ and $(x_{c_1}, y_{c_1}, b_{c_1})$ do not trace the same curve on the surface $x(x+s_0)h(x) = by(y+1)$.

Consider now the equation

$$x(x+1) = \beta y(y+1), \tag{3.24}$$

which corresponds to the equation $X^2 - \beta Y^2 = (1-\beta)/4$ after the change of variables $X := (x+1/2)$ and $Y := (y+1/2)$. Using the above general solution (3.23), we can produce a solution (x_0, y_0, β_0) to (3.24) with $\beta_0 = t(t+2/c)$ when $c \neq 0$, and with $\deg_t(x_0) = 3$ and $\deg_t(y_0) = 2$. Using this solution (x_0, y_0, β_0), we can produce polynomials solutions $(x_n(t), y_n(t), \beta_n(t))$ to (3.24) for each $n \geq 0$ with $\frac{\deg_t(y_n)}{\deg_t(x_n)} = \frac{n+2}{n+3}$ as follows. In general, when $\beta := t(t+e)$, the polynomial Pell equation $P^2 - \beta Q^2 = 1$ has a fundamental solution consisting of $P = (\frac{2}{e}t+1)$ and $Q = \frac{2}{e}$. Given a solution (X_0, Y_0, β_0) to equation $X^2 - \beta Y^2 = (1-\beta)/4$, identify (X_0, Y_0) with $X_0 + \sqrt{\beta_0}Y_0$ and (P, Q) with $P + \sqrt{\beta_0}Q$, and multiply out $(X_0 + \sqrt{\beta_0}Y_0)(P + \sqrt{\beta_0}Q)^n$ to obtain the new solution $(X_n + \sqrt{\beta_0}Y_n)$, with $\deg_t(X_n) = n + \deg_t X_0$ and $\deg_t(Y_n) = n + \deg_t Y_0$.

Let us return now to Equation (3.8), and note that given a solution $(x_n(t), y_n(t), \beta_n(t))$ to (3.24) as above, we obtain the solution $(x_n(t), y_n(t), b_n(t))$ to Equation (3.8) with $b_n(t) := \beta_n(t)(x_n(t)+2)(x_n(t)+3)$. For instance, when $n = 2$, and $c = 2$, we find a solution to Equation (3.8) with $\frac{\deg_t(y_2)}{\deg_t(x_2)} = \frac{4}{5}$ with $x_2(t) = 4t(t+1)(2t+1)(8t^2+16t+5)$ and $y_2(t) = 4(2t+1)(8t^3+20t^2+12t+1)$.

There is a infinite supply of solutions with $\frac{\deg_t(y)}{\deg_t(x)} = \frac{1}{2}$. Indeed, this was noted already in Remark 2.6 since x divides $f(x)$ in Equation (3.8).

4. Computations in the case of products of consecutive integers

Our goal in this section is to provide some numerical evidence that Conjecture 1.1 may hold in a specific example. Let $f(x) := x(x+1)(x+2)(x+3)$ and $g(x) := x(x+1)(x+2)$. Consider the following diophantine equation $cf(x) = bg(y)$ in the variables (x, y, b, c):

$$cx(x+1)(x+2)(x+3) = by(y+1)(y+2). \tag{4.1}$$

The involution $x \mapsto -x-3$ fixes the equation.

We find two obvious parametric solutions $(x(t), y(t), b(t), 1)$ with integer polynomials having positive leading coefficients, namely $(t-3, t-3, t, 1)$ and $(t, t+1, t, 1)$. Since $\frac{\deg(f)}{\deg(g)} = 4/3$ and we exhibited a parametric solution with $\deg_t(x) = \deg_t(y)$, Lemma 3.2 implies that Conjecture 1.1 (1) for the pair $f(x)$ and $g(x)$ can only hold when $\frac{\deg(f)}{\deg(g)} - \epsilon > \frac{\deg_t(y)}{\deg_t(x)}$, that is, for $\epsilon < 1/3$. We conjecture the following:

Conjecture 4.2. *Keep the above notation.*

(a) *Conjecture* 1.1 (2) *holds for $f(x)$ and $g(x)$ for any $\epsilon > 0$ such that $\epsilon < 1/3$. In other words, for any fixed integer $c > 0$, and any $\delta > 0$, Equation* (4.1) *has only finitely many solutions in positive integers (x, y, b) with $y \geq x^{1+\delta}$.*

(b) *Fix a positive integer $c \neq 1, 4$. Then Equation* (4.1) *has only finitely many solutions in positive integers (x, y, b) with $y \geq x+2$ and* $\gcd(b, c) = 1$.

The cases $c = 1$ and $c = 4$ admit a parametric family of solutions with $y \geq x+2$, and are treated in the next section. We computed the set of all solutions (x, y, b, c) of (4.1) with $c \in [1, 300]$, $y \geq x+2$, $x \leq 10^9$ and $\gcd(b, c) = 1$. We found 5050 such solutions. The data is available for download on the website of the senior author. We report now on this data, which supports the above conjecture.

Remark 4.3 The determination of all solutions to (4.1) for a given c is computationally very expensive already when the upper bound for x is 10^9. We used 150 CPU cores on the zcluster at the University of Georgia to produce our data. The computation of all solutions with $x \leq 10^9$ was most expensive in the case of the prime $c = 179$, and took 82.45 days. The case $c = 1$ took 80.45 days. The shortest case was $c = 210$, where the computation took 1.93 days. Our algorithm runs as follows.

(1) Fix c, and for each x in the chosen domain, list all divisors b of $f(x)$ such that $\gcd(b, c) = 1$.

(2) For each b found for a given x, check the existence of y such that $cf(x) = bg(y)$. This implies computing the $\deg(g)$-root of $cf(x)/b$ to some low precision, and then checking the value $g(y_0)$ at a few integers y_0 close to this root.

In step (1), instead of listing all divisors b of $f(x)$ and then checking that $\gcd(b, c) = 1$, we list all divisors of

$$\frac{x}{\gcd(x, c)} \cdot \frac{(x+1)}{\gcd(x+1, c)} \cdot \frac{(x+2)}{\gcd(x+2, c)} \cdot \frac{(x+3)}{\gcd(x+3, c)}.$$

Proceeding in this manner is much faster if c has many (small) prime factors, as in the case of $c = 210$ mentioned above.

In the table below, for each value of c, we exhibit the solution (x, y, b) to (4.1) with $y \geq x + 2$ which has the *largest value of* x among the *complete* list of all such solutions with $x \leq 10^9$ and $\gcd(b, c) = 1$. As we shall see, this largest x-value remains quite small compared to 10^9. We indicate also for a given c the number n of solutions to (4.1) with $y \geq x + 2$ and $0 < x \leq 10^9$.

c	x	y	b	n	c	x	y	b	n
2	284	1064	11	9	**22**	260	262	5655	12
3	713	1610	187	10	**23**	59943	61594	1270834	41
5	285	350	779	19	**24**	284	286	6745	5
6	68	70	391	4	**25**	351	648	1412	12
7	9590	59730	278	17	**26**	635	1014	4081	20
8	142	638	13	5	**27**	320	322	8560	6
9	104	106	910	6	**28**	3924	20383	785	11
10	207	350	437	7	**29**	16352	29783	78504	34
11	1918	4520	1616	33	**30**	356	358	10591	5
12	140	142	1645	2	**31**	894	1610	4768	23
13	358015	564718	1185928	32	**32**	380	382	12065	2
14	779	1064	4301	14	**33**	15040	33462	45080	19
15	176	178	2596	6	**34**	620	712	13995	19
16	713	895	5797	5	**35**	1935	2924	19668	29
17	10582	37960	3899	34	**36**	10945	17710	93041	3
18	272	350	2329	7	**37**	8225	20349	20108	32
19	4030	9918	5143	37	**38**	18422	29279	174405	14
20	7565	25024	4183	7	**39**	197583	260494	3362621	20
21	6498	7370	93575	18	**40**	476	478	18921	6

Consider the parametric solution (x, y, b, c) to (4.1) with[1]

$$x(c) := 12c - 4, \quad y(c) := 12c - 2, \quad b(c) := (12c - 7)c + 1. \tag{4.4}$$

Among the solutions (x, y, b, c) with $c \in [1, 300]$, $y \geq x + 2$, and $x \leq 10^9$, there are 83 values of c, including $c = 6, 9, 12, 15, 22, 24, 27, 30, 32, 40$ in the above table, where for such c the solution (x, y, b, c) with largest x-value is equal to the solution (4.4). In particular, for these c-values, since $x = 12c - 4$, the 'largest' solution found when c is fixed, has a very small x-value. There are an additional 73 values of c where exactly one solution (x, y, b, c) has $y > \max(x + 2, 12c - 4)$. It is natural to wonder whether there exist infinitely many values of c such that the solution (x, y, b, c) to (4.1) with largest x-value is equal to the solution (4.4) discussed above.

In the complete list of all 5050 solutions to (4.1) with $c \in [1, 300]$, $y \geq x + 2$, $x \leq 10^9$, and $\gcd(b, c) = 1$, only 55 solutions (x, y, b, c) have $x \in [10^5, 10^9]$, and only 21 solutions have

[1] A slighly more general parametrization is as follows: $x(c) := \beta c - 4$, $y(c) = \beta c - 2$, and $b(c) = (\beta c - 7)c + 12/\beta$, with β a positive integer dividing 12. When c is prime to 6, such parametrization produces a solution in positive integers with $\gcd(b, c) = 1$.

$x \in [10^6, 10^9]$. We list below the values c for which these 21 solutions with $x > 10^6$ occur:

$$1, 4, 44, 49, 65, 79, 89, 104, 139, 156, 161, 185, 223, 263, 298.$$

Five of these large solutions occur with $c = 1$ or 4, and as discussed in the next section, these solutions in fact belong to an infinite family. Of the 16 remaining solutions with $c \neq 1, 4$, all have $x \in [10^6, 10^7]$, except when $c = 79$ and 156, where $x \in [10^7, 10^8]$. The fact that we found only a very small number of solutions with $x \in [10^6, 10^9]$, and that most of these large x-values are close to 10^6, provides some evidence that Conjecture 4.2 may hold. We list below the two solutions with $x > 10^7$ found when $c \neq 1, 4$. In case of $c = 79$, we found two solutions with $x > 10^6$. This also occurred for $c = 89$ and $c = 263$.

c	x	y	b
79	1509512	36635458	8342
79	35125944	36635458	2445875082
156	75128030	122201728	2723331973

Remark 4.5 For the value $c = 187$, we found 69 solutions (x, y, b) in positive integers with $y > x + 1$ and $x \leq 10^9$. For the values $c = 32, 96, 108, 192, 200, 240, 252$, the only solutions to Equation (4.1) in positive integers with $y > x + 1$ and $x \leq 10^9$ have $y = x + 2$. It is natural to wonder whether there exist infinitely many values of c such that the only solutions (x, y, b, c) in positive integers to Equation (4.1) with $y > x + 1$ have $y = x + 2$. We leave it as an exercise to show that for a given c and for a given $e \geq 2$, there are only finitely many solutions in positive integers to (4.1) with $y = x + e$.

Remark 4.6 For $c = 1$ and a fixed integer b, the set of integer solutions to the equation (4.1) can be completely determined numerically when b is not too large. Indeed, the quotient by the involution $x \mapsto -x - 3$ is an elliptic curve given by the affine equation $X(X + 2) = by(y + 1)(y + 2)$, with $X := x(x + 3)$. Setting $v := b(X + 1)$ and $u := b(y + 1)$, we find an integral equation for the quotient curve of the form $v^2 = u^3 - b^2u + b^2$.

One can obtain the list of all integral points on this elliptic curve E using the function E.integral_points() in Sage [18]. It is then an easy matter to check what the integral solutions of (4.1) are. When $b = 1$, the set of integral solutions was completely determined in [4]. When $b = 4$, the set of integral solutions is considered in [22], Table T34.

Remark 4.7 Recall that if $\alpha/\beta \in \mathbb{Q}^*$ is a rational number with $\alpha, \beta \in \mathbb{Z}$ and $\gcd(\alpha, \beta) = 1$, the (logarithmic) height of α/β is $h(\alpha/\beta) := \log(\max(|\alpha|, |\beta|))$. Letting $a := b/c$, we can consider the equation $x(x+1)(x+2)(x+3) - ay(y+1)(y+2)$ as defining a plane curve X_a parametrized by the affine line $\operatorname{Spec} \mathbb{Q}[a]$. Given a solution (x, y, b, c) in positive integers to (4.1), we may compare the height of this solution, $\log(\max(x, y, b, c))$, with the height of the parameter a, namely $\log(\max(b, c))$.

We record below the only five solutions (x, y, b, c) in positive integers to (4.1) with $x \in [1, 10^9]$ and $c \in [1, 300]$ that we found where $\rho := \frac{\log(\max(x,y,b,c))}{\log(\max(b,c))} \geq 3$. There are only seventeen additional such solutions with $2 < \rho < 3$:

x	y	b	c	ρ
2	4	1	1	undefined
19	55	1	1	undefined
5	14	1	2	3.807
63	350	2	5	3.639
3	8	1	2	3.000

The ratio ρ is related to the Strong Mordell Conjecture for the family $X_a \to \operatorname{Spec}\mathbb{Q}[a]$ (see [9], F.4.3.2, and [5], Version 2 on page 357).

5. Product of consecutive integers: The cases $c = 1$ and $c = 4$.

Let us return to Equation (4.1). Letting $a := b/c$, we call $X_a/\mathbb{Q}$ the plane curve given by

$$x(x+1)(x+2)(x+3) = ay(y+1)(y+2).$$

This curve has 12 obvious points, namely:

$$(x_0, y_0) \text{ with } x_0 \in \{0, -1, -2, -3\} \text{ and } y_0 \in \{0, -1, -2\}.$$

We discuss in this section how a special geometric fact about the curve $X_a/\mathbb{Q}$ affects its arithmetic. Recall that given any five distinct points in the plane, no three on the same line, there exists a unique smooth conic which contains them, and that in general six points are not all contained on a single conic. It turns out that the twelve obvious points on the curve $X_a/\mathbb{Q}$ can be partitioned in two disjoint packets of six points, each lying on a smooth conic. More precisely,

$$(-3,0),\ (-2,-2),\ (-2,-1),\ (-1,-1),\ (-1,0), \text{ and } (0,-2),$$

lie on the conic

$$2x^2 + 2xy - y^2 + 8x + y + 6 = 0, \tag{5.1}$$

and the complement $(-3,-2)$, $(-3,-1)$, $(-2,0)$, $(-1,-2)$, $(0,-1)$, and $(0,0)$, lie on the conic $2x^2 - 2xy + 3y^2 + 4x + 3y = 0$. The real solutions of this latter conic form an ellipse in the plane, and will not be of interest in this discussion. The real solutions of the first conic form a hyperbola, and we find on it infinitely many integral points.

Fix a. Then the intersection of the curve X_a and the conic (5.1) contains the six obvious points listed above plus two more points (x, y) given by

$$2x = \frac{3(y-3)}{4} - a,$$

and

$$(y+1)^2 - 56(y+1)a + 16a^2 - 16 = 0. \tag{5.2}$$

These equations were obtained using Magma [3] by asking for the primary decomposition of the ideal (0) in the affine algebra $\mathbb{Q}[x, y, a]/I$, where I is generated by the equation of the curve X_a and the equation of the conic (5.1). Note that substituting $a = \frac{3(y-3)}{4} - 2x$ in (5.2) gives the relation $32(2x^2 + 2xy - y^2 + 8x + y + 6) = 0$.

Setting $a = \frac{3(y-3)}{4} - 2x$, we find that

$$x(x+1)(x+2)(x+3) - ay(y+1)(y+2) = \\ (-1/4)(2x^2 + 2xy - y^2 + 8x + y + 6)(2x^2 - 2xy + 3y^2 + 4x + 3y).$$

Therefore, any point (x, y) on the conic (5.1) lies on the curve X_a with $a = \frac{3(y-3)}{4} - 2x$.

Lemma 5.3. *If (x, y) is an integer point on the conic (5.1), then $a = \frac{3(y-3)}{4} - 2x$ is either an integer if y is odd, or its denominator is equal to 4 if y is even.*

Proof. Clearly, if x and y are integers, then $a = b/2^k$ for some $k = 0, 1$ or 2; and $k = 2$ when y is even. Suppose then that $y = 2t + 1$ for some integer t. Then it follows from (5.1) that

$$x(x + y) + 4x + 3 + y(1 - y)/2 = 0.$$

Since $x(x + y) + 4x$ is always even, we find that $(1 - y)/2$ must be odd. Hence, $y = 4s - 1$ for some integer s, and $(y - 3)/4$ is an integer. □

Let us write the conic (5.1) in standard form:

$$\begin{aligned} 2x^2 + 2xy - y^2 + 8x + y + 6 &= 3x^2 + 9x - (y - x)^2 + (y - x) + 6 \\ &= 3(x + 3/2)^2 - (y - x - 1/2)^2 - 1/2. \end{aligned}$$

Thus, setting $X = 2y - 2x - 1$, and $Y := 2x + 3$, we find that

$$X^2 - 3Y^2 = -2. \tag{5.4}$$

In particular, any integer point on the conic (5.1) produces an integer solution to $X^2-3Y^2 = -2$.

The integer solutions to (5.4) are well-understood. First, we have an obvious solution $(1, 1)$. The fundamental solutions (X, Y) of the Pell equation $X^2-3Y^2 = 1$ are $(\pm 2, \pm 1)$. Set $\epsilon := 2+\sqrt{3}$. All solutions of $X^2-3Y^2 = -2$ are of the form (X, Y) with $X+Y\sqrt{3} = (1+\sqrt{3})\epsilon^i$ for some i (see, e.g., [8], Theorem 3, A, and [21]).

Proposition 5.5. *When $c = 1$ or 4, Equation (4.1) has infinitely many solutions (x, y, b) in positive integers with $y \geq x + 2$. More precisely, for each integer $i \geq 2$, given $X_i + Y_i\sqrt{3} := (1 + \sqrt{3})\epsilon^i$, consider the point (x_i, y_i) on the conic (5.1) with $x_i := (Y_i - 3)/2$ and $y_i := x_i + (X_i + 1)/2$. If i is odd, then $(x_i, y_i, b_i := \frac{3(y_i-3)}{4} - 2x_i, c = 1)$ is a solution to (4.1) in positive integers. If i is even, then $(x_i, y_i, b_i := 3(y_i - 3) - 8x_i, c = 4)$ is a solution to (4.1) in positive integers.*

Proof. Let us note first that x_i and y_i are always integers. For this, it suffices to note that X_i and Y_i are always odd. This is indeed the case because if $X + Y\sqrt{3} = (m + n\sqrt{3})\epsilon$ with m and n odd, then X and Y are always odd.

Since the coefficients of ϵ are positive, it is clear that $X_i, Y_i > 0$. From the formula $y_i = x_i + (X_i + 1)/2$, we find that $y_i > x_i + 2$.

To conclude the proof, it remains to show, in view of Lemma 5.3, that i is odd if and only if y_i is odd. It is clear that y_i is odd if and only if $X_i + Y_i$ is divisible by 4. Computing now $X_i + Y_i\sqrt{3} = (m + n\sqrt{3})\epsilon^2$, we find that $X_i = 7m + 12n$, and $Y_i = 4m + 7n$. In particular, $X_i + Y_i \equiv m + n \mod 4$. When $i = 2$, and $m = n = 1$, we find that $m + n \equiv 2 \mod 4$. When $i = 3$, and $(m, n) = (5, 3)$, we find that $m + n \equiv 4 \mod 4$, as desired. □

5.6 Our initial small computer search for solutions (x, y, b) to (4.1) when $c = 1$ and $c = 4$ found the first four solutions in the infinite sequences given in 5.5 (listed in the table below), and produced the sequence $1, 14, 195, 2716$ for b when $c = 1$. This was recognized as Sequence A007655 in The Online Encyclopedia of Integer Sequences [17], and clearly indicated some possible structure in the set of solutions. The analogue sequence for $c = 4$ is A028230.

$c = 1$	$c = 4$
$(19, 55, 1)$	$(4, 14, 1)$
$(284, 779, 14)$	$(75, 208, 15)$
$(3975, 10863, 195)$	$(1064, 2910, 209)$
$(55384, 151315, 2716)$	$(14839, 40544, 2911)$

A much larger computer search found all solutions to (4.1) with $c = 1$ or $c = 4$ and $0 \leq x \leq 10^9$ and $y \geq x + 2$. When $c = 1$, only three such solutions, $(2, 4, 10)$, $(8, 10, 6)$, and $(152, 340, 14)$, do not belong to the infinite family in Proposition 5.5. When $c = 4$, we found six exceptional solutions not belonging to the infinite family in Proposition 5.5: $(12, 14, 39)$, $(39, 63, 41)$, $(44, 46, 165)$, $(74, 110, 95)$, $(130, 208, 131)$, and $(5642, 7903, 8217)$. These computations motivate the following conjecture.

Conjecture 5.7. *Let $c = 1$ or $c = 4$. Then the equation*

$$cx(x + 1)(x + 2)(x + 3) = by(y + 1)(y + 2)$$

has only finitely many solutions in positive integers (x, y, b) with $y \geq x + 2$ in addition to the infinite family of solutions described in Proposition 5.5.

This conjecture is consistent with Conjecture 4.2 (a). Indeed, the positive integer solutions (x, y, b, c) found in Proposition 5.5 when $c = 1$ or $c = 4$ all lie on a hyperbola. It follows that for such solutions $\lim_{x\to\infty} \frac{\log(y)}{\log(x)} = 1$. Thus, for any $\delta > 0$, Conjecture 5.7 predicts only finitely many positive integer solutions with $\frac{\log(y)}{\log(x)} \geq 1 + \delta$.

Remark 5.8 Let us return to the sequence introduced in 5.6 when $c = 4$, namely, $1, 15, 209, 2911, \ldots$ (A028230 in [17]). This sequence is recognized as a subsequence of a larger sequence $1, 4, 15, 56, 209, 780, 2911, \ldots$ (A001353 in [17]). In the entry for A001353 in [17], Jonathan Vos Post asks whether this sequence ever contains a prime. We note in this remark a factorization of the elements of this sequence that shows that the elements of A001353 are never prime.

Recall that $\epsilon := 2 + \sqrt{3}$. Let $\{s_n\}$ denote the sequence A001353, and let us define s_n to be the coefficient of $\sqrt{3}$ in the element ϵ^n. Define $X_n + Y_n\sqrt{3} := (1 + \sqrt{3})\epsilon^n$. We now consider the equality

$$\epsilon^{2n+1} = (X_n + Y_n\sqrt{3})^2\epsilon(1 + \sqrt{3})^{-2}$$

and compare the $\sqrt{3}$-terms. It is easy to check that $\epsilon(1 + \sqrt{3})^{-2} = 1/2$, so we obtain the equality $s_{2n+1} = X_nY_n$. Since $\{X_n\}$ and $\{Y_n\}$ are increasing sequences, we find that s_{2n+1} is never prime.

We now offer a factorization of s_n of the following form. Set $x_n := (Y_n - 3)/2$ and $y_n := (X_n + Y_n)/2 - 1$. This change of variables already appears in 5.5, and we used already in 5.5 the fact that x_n and y_n are integers. It is very easy to check that $s_{n+1} = y_n + 1$. The point (x_n, y_n) lies on the conic (5.1), which we rewrite in the following form:

$$2(x+1)(x+2) = (y+1)(y-2x-2).$$

It follows that

$$s_{n+1} = y_n + 1 = \frac{2(x_n+1)(x_n+2)}{y_n - 2x_n - 2}.$$

We leave it to the reader to check that $y_n - 2x_n - 2 < x_n$, so that the above factorization of s_{n+1} shows that indeed s_{n+1} is always composite.

Remark 5.9 Fix integers $0 < \alpha < \beta < \gamma$, and $0 < m < n$. Consider the following diophantine equation in the variables (x, y, b, c):

$$cx(x+\alpha)(x+\beta)(x+\gamma) = by(y+m)(y+n). \tag{5.10}$$

The equation has 12 obvious integer solutions (x, y, b, c), which occur for all values of b, c, namely:

$$(x, y, b, c) \text{ with } x \in \{0, -\alpha, -\beta, -\gamma\} \text{ and } y \in \{0, -m, -n\}.$$

Let us now impose a new constraint: that the following set of six points $(-\gamma, 0)$, $(-\beta, -n)$, $(-\beta, -m)$, $(-\alpha, -m)$, $(-\alpha, 0)$, $(0, -n)$ all lie on a conic. Note that no three of the last five points on this list can be colinear.

Lemma 5.11. *The six points above all lie on the same conic C if and only if*

$$\gamma = \beta + \alpha\frac{m}{n-m}.$$

Proof. We write down the conic C which contains the last five points in the list by computing the determinant of the 6×6-matrix whose first row is

$$(x^2, xy, y^2, x, y, 1)$$

and whose subsequent five rows are of the form $(a^2, ab, b^2, a, b, 1)$ for each of the five given points (a, b). It turns out that the first point $(-\gamma, 0)$ belongs to C if and only if $\gamma = \beta + \alpha\frac{m}{n-m}$. □

Note that the above condition on γ is compatible with the hypothesis that $\gamma > \beta$. It is possible to find examples of Equation (5.10) which do not have any obvious involution or parametric solutions with c constant (contrary to Equation (4.1)), but where we can find several values of c where Equation (5.10) has infinitely many solutions with $y > x + \gamma$. For instance, when $c \in \{1, 5, 25\}$, then there are infinitely many solutions (x, y, b) in positive integers with $y > x + 6$ to the equation

$$cx(x+3)(x+4)(x+6) = by(y+2)(y+5).$$

6. An application

Our initial motivation for considering the equation

$$cx(x+1)(x+2)(x+3) = by(y+1)(y+2)$$

came from a conjecture of the junior author in [29], 4.5. In [29], 1.2, Xiang proves the non-existence of non-trivial tight 8-designs in algebraic combinatorics. His method of proof led him to a conjecture in elementary number theory which, if true, could be used to give a new proof of the fact that there exist only finitely non-trivial tight $2e$-designs. His conjecture 4.5 in [29] can be stated as follows.

Conjecture 6.1. (Z. Xiang) *Let $n \geq 3$ be an integer. For every nonzero integer c, there are only finitely many pairs (x, y) of positive integers satisfying $y \geq x + 2$ and such that the n rational numbers*

$$\frac{x(x+1)}{y}, \frac{x(x+1)(x+2)}{y(y+1)}, \ldots, \frac{x(x+1)(x+2)\ldots(x+n)}{y(y+1)\ldots(y+n-1)}$$

all have denominators that divide c.

We now show that Conjecture 6.1 follows from our conjectures 4.2 and 5.7.

Proposition 6.2. *Conjectures* 4.2 *and* 5.7 *together imply Conjecture 6.1.*

Proof. Conjecture 6.1 follows if we can show that for every non-zero positive integer c, there are only finitely many pairs (x, y) of positive integers satisfying $y \geq x + 2$ and such that the denominators of both $\frac{x(x+1)}{y}$ and $\frac{x(x+1)(x+2)(x+3)}{y(y+1)(y+2)}$ divide c. Conjecture 4.2 in this article immediately implies that for every non-zero positive integer c with $c \neq 1, 4$, there are only finitely many pairs (x, y) of positive integers satisfying $y \geq x+2$ such that the denominator of $\frac{x(x+1)(x+2)(x+3)}{y(y+1)(y+2)}$ divides c.

Assume now that $c = 1$ or 4. Recall that the parametric solutions (x_n, y_n, b_n) of the equation

$$cx(x+1)(x+2)(x+3) = by(y+1)(y+2)$$

found in Proposition 5.5 lie on the conic (5.1), which we can rewrite as

$$2(x+1)(x+3) - y(y-2x-1) = 0.$$

Hence, setting $z_n := y_n - 2x_n - 1$, we have $y_n z_n = 2(x_n+1)(x_n+3)$. Using this identity, we get

$$\frac{2x_n(x_n+1)}{y_n} = z_n - \frac{6(x_n+1)}{y_n}.$$

For n positive, one checks that $0 < b_n < x_n$, so that

$$\frac{8}{3}(x_n+1) < y_n < 3(x_n+1).$$

Hence, $6(x_n+1)/y_n$ is not an integer and, therefore, $x_n(x_n+1)/y_n$ is not an integer. Conjecture 5.7 implies that there are only finitely many solutions (x, y) with $y \geq x+2$ in addition to the parametric solutions, as desired. □

7. The counting function

Let $f(x) := x(x+1)(x+2)(x+3)$ and $g(y) := y(y+1)$. We report in this section on some data on the solutions (x, y, b, c) in positive integers with $\gcd(b, c) = 1$ to the equation

$$cx(x+1)(x+2)(x+3) = by(y+1). \tag{7.1}$$

In 7.2, we discuss some evidence that Conjecture 1.1 (2) holds for the pair f and g with $\epsilon = 1/4$, i.e., that for a fixed c, there exist only finitely many positive integer solutions (x, y, b, c) to (7.1) with $\frac{\log(y)}{\log(x)} > 7/4$.

In 7.6, we consider all positive integer solutions (x, y, b, c) with $c \in [1, 300]$ and $x \in [1, 10^9]$, and remove from this set of solutions all solutions which can be explained 'geometrically', that is, solutions which (are known to) belong to a parametric family. We introduce then the following counting function for this set of 'not-geometrically explained' solutions:

$$N_{4/3,10^9,300}(B) \;:=\; \text{Number of solutions } (x,y,b,c) \text{ with } \gcd(b,c)=1,\ \tfrac{\log(y)}{\log(x)} \in [4/3,B], x\in[1,10^9], \text{ and } c\in[1,300].$$

The data shows a surprisingly good fit between $N_{4/3,10^9,300}(B)$ and a function of the form $\alpha - \beta e^{-\gamma B}$, where α, β, γ are positive constants.

7.2 Data when $\frac{\log(y)}{\log(x)} > \frac{7}{4}$.

A computer search found the complete list L of all 12730 solutions (x,y,b,c) to Equation (7.1) in positive integers with $c \in [1,300]$, $b/c \neq 4$, $x \leq 10^9$, $\gcd(b,c)=1$, and $\frac{\log(y)}{\log(x)} > 7/4$. The parametric solution

$$x(c) = 144c+2, \quad y(c) = 24c(144c+5), \quad b(c) = 36c+1, \tag{7.3}$$

has $\frac{\log(y)}{\log(x)} > 7/4$ when $c \geq 9$. For 156 values of c when $c \in [9,300]$, the solution (x,y,b,c) with the largest x-coordinate for the given c among the solutions in L is the solution given in the parametric family (7.3) above. We found only 232 solutions in L with x-coordinate larger than the x-coordinate of the corresponding solution in the parametric family (7.3).

7.4 Given any parametric solution $(x(c), y(c), b(c), c)$, we can use the involutions $x \mapsto -x-3$ and $y \mapsto -y-1$ if necessary to obtain a new solution where the leading coefficients of both $x(c)$ and $y(c)$ are positive. Consider then the set S of all parametric solutions $(x(c), y(c), b(c))$ with degrees $(1,2,1)$, and $x(c)$ and $y(c)$ having positive leading coefficients (such as (7.3)). Then the involution

$$I : (x(c), y(c), b(c)) \longmapsto (-x(-c)-3, y(-c), -b(-c))$$

preserves this set of solutions. Let $x(c) := a_1c+a_0$, $y(c) := c_2c^2+c_1c+c_0$, and $b(c) := b_1c+b_0$ be polynomials in $\mathbb{Q}[c]$ such that $(x(c), y(c), b(c), c)$ is a solution with $a_1, c_2 > 0$ and $b_0 \neq 0$. Since $\deg_c(x) = 1$, we can find a new solution in $\mathbb{Q}[c]$ such that $x(c)$ is monic. It turns out that the set of all such solutions can be explicitly computed, and is given by the following four solutions and their images under the involution I:

(7.5)

$x(c)$	$y(c)$	$b(c)$
$c+2$	$\frac{c(c+5)}{6}$	$36c+144$
$c+1$	$\frac{c(c+4)}{3}$	$9c+18$
$c+1$	$\frac{c(c+3)}{2}$	$4c+16$
$c-1$	c^2-1	$c+2$

Using Magma [3], one computes that the ideal in $A := \mathbb{Q}[a_0, b_0, b_1, c_0, c_1, c_2]$ generated by the coefficients of the polynomial $cx(x+1)(x+2)(x+3) - by(y+1)$ in $A[c]$ has exactly 18 distinct minimal prime ideals. The 16 prime ideals which do not contain b_0c_2 are maximal. Exactly half of these determine a coefficient c_2 which is non-negative.

To obtain solutions to Equation 7.1 in positive integers, we evaluate the above solutions at integer values c_0 such that $y(c_0) \in \mathbb{Z}$. For instance, using the first solution above, for each appropriate values of c_0 we obtain the integer solution (x,y,b,c) given as

$$x = c_0+2, y = \frac{c_0(c_0+5)}{6}, b = \frac{36c_0+144}{\gcd(c_0,144)}, c = \frac{c_0}{\gcd(c_0,144)}.$$

From this, we can write down explicit integer parametrizations of solutions. For instance, by setting $c_0 = 144d$ in the above expression we obtain (7.3), and by setting $c_0 = 6d+1$, we obtain:

$$x = 6d + 3, y = (6d + 1)(d + 1), b = 36(6d + 5), c = 6d + 1.$$

When $c = 90, 96, 144, 150, 162$, we found only eight solutions in positive integers to (7.1) with $x < 10^9$ and $\log(y)/\log(x) > 7/4$, and all solutions that we found are obtained from the above parametric solutions (7.5) and their images under I, specialized to the case where $c_0 = b(0)d$. One may wonder whether for these values of c, these eight solutions are the only solutions in positive integers with $\log(y)/\log(x) > 7/4$.

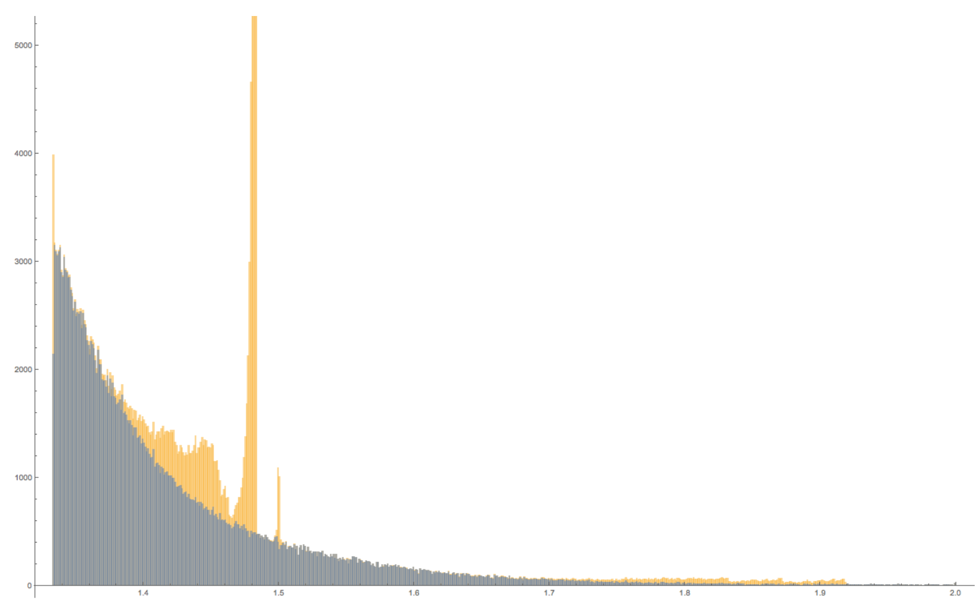
When $c = 247$, we found a much larger number of solutions, namely 125 in total, with $x < 10^9$ and $\log(y)/\log(x) > 7/4$. Five of these solutions have their x-coordinate larger than the x-coordinate $144c+2$ of the parametric solution (7.3). Only three of these solutions have $x > 10^6$.

We found a total of 36 solutions (x, y, b, c) in L with $x > 10^6$. We only found two values of c with solutions with $10^8 < x < 10^9$, when $c = 39$ and when $c = 281$:

x	y	b	c	$\log(y)/\log(x)$
169036273	2130028657415099	7018	39	1.86
554344548	2537086889346525	4122444	281	1.76

7.6 Data when $\frac{\log(y)}{\log(x)} > \frac{4}{3}$.

A second computer search found the complete list L' of all solutions (x, y, b, c) to Equation (7.1) in positive integers with $c \in [1, 300]$, $b/c \neq 4$, $x \leq 10^9$, $\gcd(b, c) = 1$, and $\frac{\log(y)}{\log(x)} > 4/3$. We present a histogram below of this data produced using Mathematica [28]. We partitioned the interval $[1.33, 2]$ in intervals I_d of length 0.001 and for each d counted the number of solutions found with $\frac{\log(y)}{\log(x)} \in I_d := [d, d+0.001)$. Thus in the histogram below, the horizontal axis pertains to the quantity $\frac{\log(y)}{\log(x)}$ and the vertical axis corresponds to the number of solutions found.



The parametric solutions to Equation (7.1) with $c = 1$, found in 3.7, explain the very tall peak around $\frac{\log(y)}{\log(x)} = 1.47$. The histogram also shows a smaller sharp peak in the number of solutions around $\frac{\log(y)}{\log(x)} = 1.5$. This peak is explained geometrically by the existence of the six parametric solutions in 7.7. The 'bulge' in the number of solutions above the interval $[1.40, 1.46]$ is also explained by the existence of many other parametric solutions, as we explain in 7.8. It turns out that much of the 'irregularity' in the histogram, shown in orange, is explained by the geometry of the hypersurface $cf(x) = bg(y)$.

7.7 In the tables of solutions below, we let $z(300) := \frac{\log(y(300))}{\log(x(300))}$. The following are solutions $(x(c), y(c), b(c))$ to Equation (7.1), with degrees $(2, 3, 3)$.

$x(c)$	$y(c)$	$b(c)$	$z(300)$
c^2+c-3	$(c+1)(c^2+c-1)$	$(c-1)(c^2+c-3)$	1.50015
c^2+c-2	$(c+1)(c^2+c-1)$	$(c-1)(c^2+c+1)$	1.50015
$9c^2+3c-3$	$(3c+1)(9c^2+3c-1)$	$(3c-1)(3c^2+c-1)$	1.50004
c^2-c-3	$c^2(c-2)$	$(c+1)(c^2-c-3)$	1.49986
c^2-c-2	$c^2(c-2)$	$(c+1)(c^2-c+1)$	1.49986
$9c^2-3c-3$	$9c^2(3c-2)$	$(3c+1)(3c^2-c-1)$	1.49996

Consider the set of polynomial solutions $(x(c), y(c), b(c))$ to Equation (7.1) with degrees $(2, 3, 3)$, and $x(c)$ and $y(c)$ having positive leading coefficients. The involution

$$I : (x(c), y(c), b(c)) \longmapsto (x(-c), -y(-c) - 1, -b(-c))$$

preserves this set of solutions, and the last three solutions above are obtained from the first three by this involution.

We note that the first solution above is such that $b(0) = 3$. It follows that for integers c_0 divisible by 3, $\gcd(b(c_0), c_0) \neq 1$. On the other hand, we may consider the new solution $(x(3c), y(3c), b(3c)/3)$, still with integer polynomials, which has the property that $\gcd(b(c_0)/3, c_0) = 1$ for all integer values of c_0. The third solution above is obtained from the first solution by this process.

7.8 The first two solutions in 7.7 are of the form $(x-1, y, b(x-1))$ and $(x, y, b(x+3))$ where $(x, y, b) := ((c-2)(c+1), c^2(c-2), c+1)$ is a solution to the equation $cx(x+1)(x+2) = by(y+1)$. We found three additional solutions to Equation (7.1) coming from this modified equation, listed below. Obviously, their images under the involution I are also solutions.

$x(c)$	$y(c)$	$b(c)$	$z(300)$
$2c^2+5c+1$	$c(c+2)(2c+3)$	$(2c+1)(x+3)$	1.47131
c^2+4c+2	$\frac{c(c+2)(c+3)}{2}$	$4(c+2)(x+3)$	1.43902
c^2+4c+1	$\frac{c(c+2)(c+3)}{2}$	$4(c+2)x$	1.43902

We list below solutions to Equation (7.1) obtained from solutions to the equation $cx(x+1)(x+3) = by(y+1)$.

$x(c)$	$y(c)$	$b(c)$	$z(300)$
$3c^2+7c+1$	$c(c+2)(3c+4)$	$(3c+1)(x+2)$	1.45606
c^2-4	$\frac{c(c-2)(c+1)}{2}$	$4(c+2)(x+1)$	1.43895
c^2+5c+3	$\frac{c(c+2)(c+4)}{3}$	$9(c+3)(x+2)$	1.40340
$3c^2+8c+2$	$\frac{c(c+2)(3c+5)}{2}$	$4(3c+2)(x+1)$	1.40064
$(c+3)(2c+1)$	$\frac{c(c+2)(2c+5)}{3}$	$9(2c+3)(x+1)$	1.38048

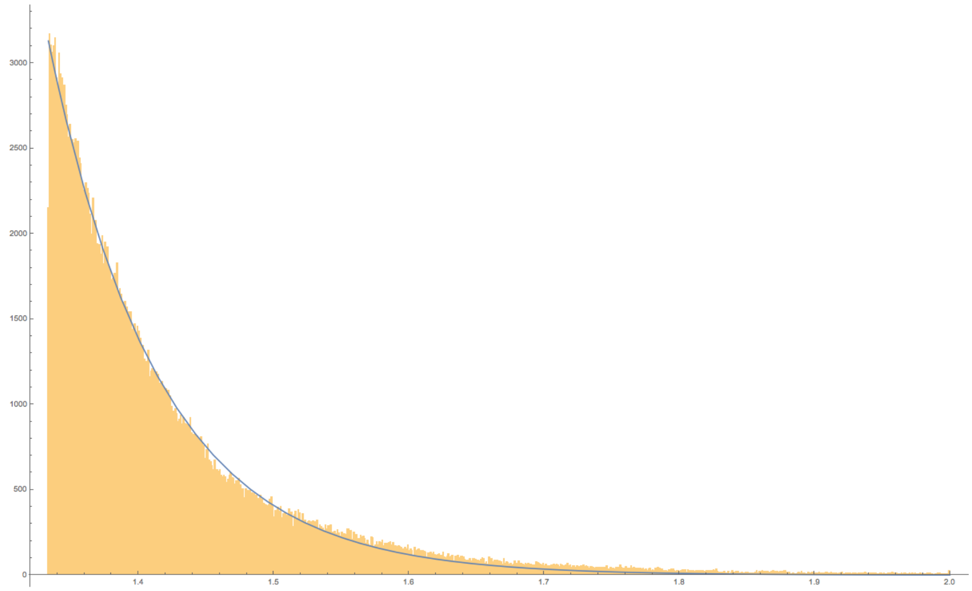
It is an easy matter for Magma to compute the parametric solutions (x, y, b) of degree $(2, 3, 1)$ to $cx(x+1)(x+2) = by(y+1)$ and to $cx(x+1)(x+3) = by(y+1)$, and each such solution produces solutions to Equation (7.1). We list below a set of integral parametric solutions to (7.1) which are not of that form. (We leave it to the reader to produce further such solutions using the involution I, or a substitution $c \mapsto \alpha c$ when appropriate.)

$x(c)$	$y(c)$	$b(c)$	$z(300)$
$4(24c^2+13c+1)$	$c(8c+3)(24c+7)$	$8(4c+1)(6c+1)(12c+5)$	1.40049
$(4c+3)(8c+1)$	$c(8c+3)(8c+5)$	$8(2c+1)(4c+1)(4c+3)$	1.43004
$6c^2-11c+3$	$(c-1)(2c-1)(3c-1)$	$(2c-3)(3c-4)(6c-5)$	1.43233
$2(12c^2+11c+1)$	$c(4c+3)(12c+5)$	$4(2c+1)(3c+2)(6c+1)$	1.43854

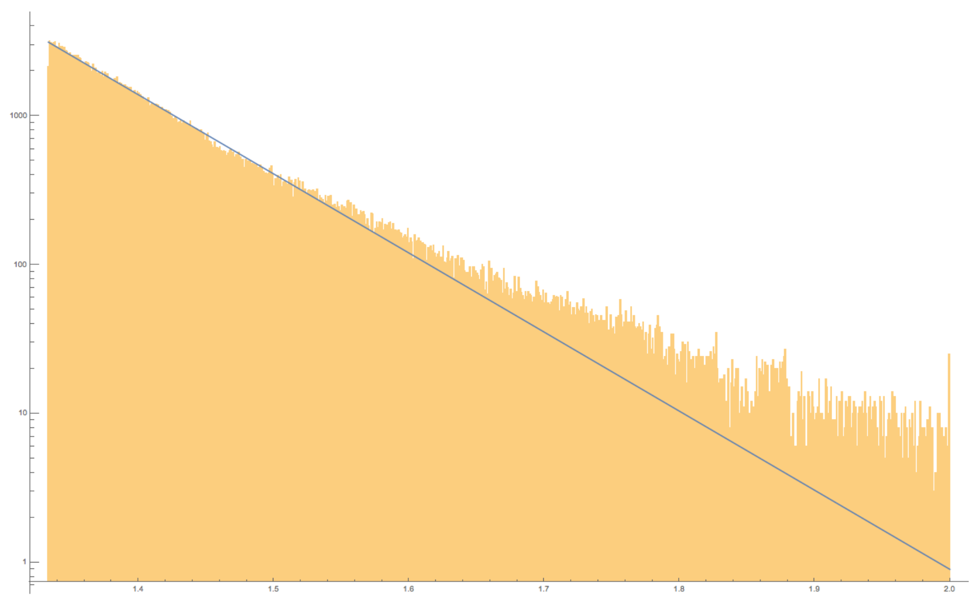
A priori, finding all polynomial solutions (x, y, b) of degrees $(2, 3, 3)$ is a problem with 7 variables (the coefficients of $x(c)$ and $y(c)$), and 6 relations (since the remainder of the division of $cf(x(c))$ by $y(c)(y(c)+1)$ has degree at most 5), and such a problem seem computationally hard at this time. As we now explain, to find the four solutions listed above, we considered instead a different problem where the number of variables and the degrees of the relations can be considerably decreased.

We may assume that c divides $y(y+1)$, otherwise we are reduced to an easier problem. In fact, we may assume that c divides y by using the involution $y \mapsto -y-1$ if necessary. Given a solution (x, y, b) to $cx(x+1)(x+2)(x+3) = by(y+1)$ with $\deg_c(x) = 2$, we claim that one of the polynomials x, $x+1$, $x+2$, and $x+3$, must be irreducible in $\mathbb{Q}[c]$. Indeed, otherwise we find that $\Delta := a_1^2 - 4a_2a_0$ is a square in $\mathbb{Z}$, and so are $\Delta - 4a_2$, $\Delta - 8a_2$, and $\Delta - 12a_2$. Since we are looking for a solution with $a_2 \neq 0$, it follows that we would then have a 4-term arithmetic progression in squares, and this was proved not to exist by Fermat [23]. Hence, we may assume that $x+i$ is irreducible, and that $x+i$ divides y or $y+1$ (if this were not the case, $x+i$ would divide b, and we would be reduced to an easier case). Thus, either $y(c) = tc(x(c)+i)$, and there are only 4 variables in total, a_0, a_1, a_2 and t, or $y(c)+1 = (x(c)+i)(tc+s)$, with $1 = (x(0)+i)s$. This determines s, and leaves only 4 variables in total again, a_0, a_1, a_2 and t.

7.9 Remove now from the set L' all possible positive integer solutions (x, y, b, c) which belong to one of the parametric solutions generated by the solutions discussed in this section. We present a histogram below of this data. We partitioned the interval $[1.33, 2]$ in intervals I_d of length 0.001 and for each d counted the number of solutions found with $\log(y)/\log(x) \in I_d := [d, d+0.001)$. The blue graph on this histogram is the graph of the function $h(B) := \exp(\alpha(B-2)+\beta)$, with $\alpha = -12.2368$ and $\beta = -0.1092$ obtained using a Mathematica command to produce the exponential function that fits best the data on this histogram when $t \in [1.5002, 2]$.



Below is the same data shown on a logarithmic scale:



Assuming that Conjecture 1.1 (2) holds for all ϵ in a range $[0, \epsilon_0]$, we can consider for each integer $c_0 > 0$ the counting function $\mathcal{N}_{c_0}(\epsilon)$, the number of solutions (x_0, y_0, b_0, c_0) in positive integers to $c_0 f(x) = bg(y)$ with $\gcd(c_0, b_0) = 1$ and such that $\frac{\log(y_0)}{\log(x_0)} > \frac{\deg(f)}{\deg(g)} - \epsilon$. In this section, we considered $\sum_{c_0 \in [1,300]} \mathcal{N}'_{c_0}(\epsilon)$, where $\mathcal{N}'_{c_0}(\epsilon)$ is a modified version of $\mathcal{N}_{c_0}(\epsilon)$, where we remove from the count in $\mathcal{N}_{c_0}(\epsilon)$ all the solutions known to lie in parametric families. The data presented in this section suggests that $\sum_{c_0 \in [1,300]} \mathcal{N}'_{c_0}(\epsilon)$ might be approximated by an exponential function. It is natural to wonder whether $\mathcal{N}_{c_0=1}(\epsilon)$ itself might also be approximated by an exponential function of the form $\exp(\alpha\epsilon + \beta)$ for some positive constants α and β.

8. Product of consecutive integers: rational solutions

Consider again the projective curve $X_a/\mathbb{Q}$ defined by the equation

$$x(x+1)(x+2)(x+3) = ay(y+1)(y+2).$$

and associated with the equation (4.1) discussed in section 4 and section 5. For most values of a, the set $X_a(\mathbb{Q})$ contains at least 16 rational points, including $(a-3, a-3)$, $(a, a+1)$, and their images under the involution $x \mapsto -x-3$. It also always contains the point at infinity $(0:1:0)$. In this section, independent of the rest of this article, we make some remarks on the rational points of $X_a/\mathbb{Q}$.

The curve $X_{-a}/\mathbb{Q}$ is isomorphic to the curve $X_a/\mathbb{Q}$, with an isomorphism given by $y \mapsto -y-2$ and $a \mapsto -a$. When $a = 243/182$, Magma finds 44 rational points[2] in $X_a(\mathbb{Q})$; the point $(-3/2, 1/6)$ is fixed by the involution and so is the point at infinity. When $a = 247/7$, Magma finds 43 rational solutions, three of them integral with $y > x$: $(38, 40)$, $(75, 98)$, and $(492, 1188)$.

Proposition 8.1. *There exist infinitely many values of $a \in \mathbb{Q}$ such that $X_a(\mathbb{Q})$ contains at least 23 points.*

Proof. Given any two rational points in $X_a(\mathbb{Q})$, we can consider the line between these two points. This line will intersect the curve in two additional points, with either both points in $X_a(\mathbb{Q})$, or both points in $X_a(K) \setminus X_a(\mathbb{Q})$ for some quadratic extension $K/\mathbb{Q}$.

Let us consider first the points of intersection of the line $y = a+1$ with the curve X_a. We already know two points on that line, $(a, a+1)$ and $(-a-3, a+1)$. Thus the expression $x(x+1)(x+2)(x+3) - ay(y+1)(y+2)$ factors when $y = a+1$ as

$$(x-a)(x+a+3)(x^2+3x+a^2+3a+2).$$

Using the point $(-1, 0)$, we parameterize the conic $x^2+3x+a^2+3a+2 = 0$ using $a = s(x+1)$ and $x = -\frac{s^2+3s+2}{s^2+1}$. Thus when $a = -\frac{3s^2+s}{s^2+1}$, we find that the point $P = (x(s), y(s) = 1+a)$ is on the curve X_a.

We have expressed below a parametric solution (x, y, a) in terms of t for the points $Q_1 = (x_1, y_1)$ and $Q_2 = (x_2, y_2)$ on the intersection of X_a with the line passing through $(a, a+1)$ and $(a-3, a-3)$ (that is, the line $y = 4x/3 - (a-3)/3$).

$$\begin{aligned} a &= 27t/(1+2t^2) \\ x_1 &= -(2t^2-5t+2)/(1+2t^2) \\ x_2 &= -(4t^2-5t+1)/(1+2t^2) \end{aligned}$$

We claim that it is possible to find infinitely many instances where the points P, Q_1 and Q_2 are rational at the same time. More precisely, we can find infinitely many rational pairs (s, t) such that

$$-\frac{3s^2+s}{s^2+1} = \frac{27t}{(1+2t^2)}.$$

In other words, we are looking at the rationals points on the plane curve $Z/\mathbb{Q}$ defined by the equation $h(s, t) = 0$, where

$$h(s,t) = 6s^2t^2 + 27s^2t + 2st^2 + 3s^2 + s + 27t.$$

[2]It was shown by Kulesz [11] that there exists a parametric family of plane quartic curves with at least 37 rational points. W. Keller and L. Kulesz [10] exhibited a plane quartic curve with at least 16 automorphisms and at least 176 rational points.

This curve has two singular rational points at infinity, $(0:1:0)$ and $(1:0:0)$. Its arithmetic genus is 3, and its geometric genus is 1. Let $E/\mathbb{Q}$ denote the desingularization of the plane projective model $\overline{Z}$ of Z. Then E is an elliptic curve (once a rational point has been fixed as the origin), and is thus isomorphic over $\mathbb{Q}$ to its Jacobian. Rewriting the equation as a quadratic polynomial in s, we can complete the square and get an equation of the form

$$w^2 = 4(t^4 - 162t^3 - 4372t^2 - 486t + 1).$$

Using the function Jacobian in Magma, we find that the Jacobian of the curve given by the above equation has the Weierstrass equation

$$y^2 = x^3 - 4372x^2 + 78728x + 279928.$$

The rank of this curve is 3. The command PointSearch(Z, 10^7) finds 19 points in $Z(\mathbb{Q})$, also indicating that the rank is positive. □

Remark 8.2 In the table below, all curves X_a, with $a = b/c$ and b and c in the range given in the first column, were tested with the Magma command PointSearch. We ran PointSearch(X_a, 10000) for the first three ranges, and PointSearch(X_a, 20000) for the last three ranges. The last column reports on how many such curves produced PointSearch(X_a, h) ≥ 23. The other columns tabulate the frequency of the odd values between 23 and 37. Very few curves with an even number of points were found.

We only found two curves with PointSearch(X_a,h)> 37, the curves with $a = 243/182$ and $a = 247/7$ mentioned at the beginning of this section. Recall that it follows from the Bombieri-Lang conjecture that the number of rational points on any smooth curve $X_a/\mathbb{Q}$ should be bounded by an absolute constant independent of a (see e.g., [9], F.4.3.5).

height	23	25	27	29	31	33	35	37	≥ 23
$0 - 99$	132	76	31	16	5	4	2	4	274
$100 - 199$	147	64	27	10	5	2	1	0	259
$200 - 299$	154	60	14	6	2	2	1	1	246
$300 - 399$	144	45	19	6	3	2	3	0	223
$400 - 499$	153	39	13	6	2	1	0	1	215
$500 - 599$	130	44	13	11	1	0	1	0	202
$600 - 699$	124	47	15	8	2	1	0	1	198

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