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On some identities for balancing and cobalancing numbers

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Abstract

As a consequence of the Binet formula for balancing, cobalancing, square triangular, Lucas-balancing and Lucas-cobalancing numbers, we provide some formulas for these sequences explicitly, which can have certain importance or applications in most recently investigations in this area. Also we give another expression for the general term of each sequence, using the ordinary generating function.

Keywords: Balancing number, cobalancing number, Binet's formula, generating function.

MSC: 11B39, 11B83, 05A15.

1. Introduction

Some sequences of integer numbers have been studied over several years, with emphasis on studies of the well known Fibonacci sequence (and then the Lucas sequence) which is related to the golden ratio and of the Pell sequence which is related to the silver ratio. Behera and Panda [1] introduced the sequence $(B_n)_{n=0}^{\infty}$ of balancing numbers and give some interesting properties of this sequence. According Behera and Panda [1] a positive integer n is a *balancing number* with *balancer* r,

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if it is the solution of the Diophantine equation $1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$.

The sequence $(B_n)_{n=0}^{\infty}$ is defined by the following recurrence relation of second order given by

$$B_{n+1} = 6B_n - B_{n-1},\ n \geq 1, \tag{1.1}$$

with initial terms $B_0 = 0$ and $B_1 = 1$, where B_n denotes the nth balancing number.

On the other hand, following Panda and Ray [13] a positive integer n is a *cobalancing number* with *cobalancer* r, if it is the solution of the Diophantine equation $1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r)$. The sequence $(b_n)_{n=1}^{\infty}$ is defined by the following recurrence relation of second order given by

$$b_{n+1} = 6b_n - b_{n-1} + 2,\ n \geq 2, \tag{1.2}$$

with initial terms $b_1 = 0$ and $b_2 = 2$, where b_n denotes the nth cobalancing number. Panda and Ray [13, Theorem 6.1] proved that every balancer is a cobalancing number and every cobalancer is a balancing number. Many authors have dedicated their research to the study of these sequences and also to the generalisations of the theory of the sequences of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers [2, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. Behera and Panda [1] observed that n is a balancing number if and only if n^2 is a triangular number, that is $8n^2+1$ is a perfect square and the square of a balancing number is a square triangular number, that is, $B_n^2 = ST_n$, where ST_n denotes the nth square triangular number. Also, we know that n is a cobalancing number if and only if $8n^2+8n+1$ is a perfect square. The same way which balancing numbers are related to square triangular numbers, also, the cobalancing numbers are related to pronic triangular numbers (triangular numbers that are expressible as a product of two consecutive natural numbers) [22, 21, 23, 24]. The sequence $(ST_n)_{n=0}^{\infty}$ is defined by the following recurrence relation of second order given by

$$ST_{n+1} = 34ST_n - ST_{n-1} + 2,\ n \geq 1, \tag{1.3}$$

with initial terms $ST_0 = 0$ and $ST_1 = 1$, where ST_n denotes the nth square triangular number. Panda [12] gives us the identity $C_n = \sqrt{8B_n^2+1}$ that involves the nth balancing number and the nth Lucas-balancing number C_n. Also the sequence $(C_n)_{n=0}^{\infty}$ is defined by the following recurrence relation of second order given by

$$C_{n+1} = 6C_n - C_{n-1},\ n \geq 1, \tag{1.4}$$

with initial terms $C_0 = 1$ and $C_1 = 3$.

In addition, Panda and Ray [14] give the identity $c_n = \sqrt{8b_n^2 + 8b_n + 1}$ that involves the nth cobalancing number and the nth Lucas-cobalancing number c_n. The sequence $(c_n)_{n=1}^{\infty}$ is defined by the following recurrence relation of second order given by

$$c_{n+1} = 6c_n - c_{n-1},\ n \geq 2, \tag{1.5}$$

with initial terms $c_1 = 1$ and $c_2 = 7$.

The Binet formula is well known for several sequences of integer numbers. The general Binet formula for a mth order linear recurrence was deduced in 1985 by Levesque in [6]. Sometimes this formula is used in the proof of basic properties of integer sequences. In the case of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing sequences, their Binet formulas are respectively,

$$B_n = \frac{r_1^n - r_2^n}{r_1 - r_2}, \tag{1.6}$$

$$C_n = \frac{r_1^n + r_2^n}{2}, \tag{1.7}$$

$$b_n = \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2}, \tag{1.8}$$

$$c_n = \frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2}, \tag{1.9}$$

and using the relation between balancing numbers, square triangular numbers and (1.6) we obtain

$$ST_n = \frac{r_1^{2n} + r_2^{2n}}{32} - \frac{1}{16}, \tag{1.10}$$

where $r_1 = {\alpha_1}^2 = 3+2\sqrt{2}$ and $r_2 = {\alpha_2}^2 = 3-2\sqrt{2}$ are the roots of the characteristic equation $x^2 = 6x - 1$, associated with the recurrence relations of the sequences and α_1 and α_2 are the roots of the characteristic equation, $x^2 = 2x + 1$, associated with the Pell sequence [3, 4, 14].

There is a large number of sequences indexed in *The Online Encyclopedia of Integer Sequences*, being in this case

$$\begin{aligned}
\{(B_n)_{n=0}^{\infty}\} &= \{0, 1, 6, 35, 204, 1189, 6930, \ldots\} : A001109\\
\{(b_n)_{n=1}^{\infty}\} &= \{0, 2, 14, 84, 492, 2870, 16730, \ldots\} : A053141\\
\{(ST_n)_{n=0}^{\infty}\} &= \{0, 1, 36, 1225, 41616, 1413721, \ldots\} : A001110\\
\{(C_n)_{n=0}^{\infty}\} &= \{1, 3, 17, 99, 577, 3363, 19601, \ldots\} : A001541\\
\{(c_n)_{n=1}^{\infty}\} &= \{1, 7, 41, 239, 1393, 8119, 47321, \ldots\} : A002315.
\end{aligned}$$

Many interesting properties and important identities about these sequences are available in the literature. Interested readers can follow [2, 5, 8, 10, 12, 16, 18], among many others scientific papers. The purpose of this paper is to provide some formulas of the sequences stated above to help possible applications. In the next two sections we obtain new identities and properties for these sequences, such as the famous Catalan, Cassini and d'Ocagne identities and the sums formulae for each one. The last section is devoted to explicitly give the ordinary generating function of these sequences, as well as another expression for the general term of them.

2. Balancing, Lucas-balancing and square triangular numbers: some identities

According with recurrence relations (1.1), (1.3) and (1.4) and using the well known results involving recursive sequences, consider the respective characteristic equation and note that $r_1 r_2 = 1,\ r_1 - r_2 = 4\sqrt{2},\ r_1 + r_2 = 6$.

As a consequence of the Binet formulas (1.6), (1.7) and (1.10) we get for these sequences the following interesting identities. The first one and its proof can be found in Panda [12].

Proposition 2.1 (Catalan's identities). *For the sequences* $(B_n)_{n=0}^{\infty}$, $(C_n)_{n=0}^{\infty}$ *and* $(ST_n)_{n=0}^{\infty}$ *if* $n \geq r$ *we have*

$$B_{n-r}B_{n+r} - B_n^2 = -B_r^2, \tag{2.1}$$

$$C_{n-r}C_{n+r} - C_n^2 = C_r^2 - 1 \tag{2.2}$$

and

$$ST_{n-r}ST_{n+r} - ST_n^2 = ST_r^2 - 2ST_nST_r, \tag{2.3}$$

respectively.

Proof. For the second identity, using the Binet formula (1.7)

$$\begin{aligned} C_{n-r}C_{n+r} - C_n^2 &= \left(\frac{r_1^{n-r} + r_2^{n-r}}{2}\right)\left(\frac{r_1^{n+r} + r_2^{n+r}}{2}\right) - \left(\frac{r_1^n + r_2^n}{2}\right)^2 \\ &= \frac{(r_1 r_2)^{n-r}\left(r_2^{2r} + r_1^{2r} - 2r_1^r r_2^r\right)}{2^2} \\ &= \frac{(r_1^r + r_2^r)^2 - 4r_1^r r_2^r}{2^2} \\ &= C_r^2 - 1 \end{aligned}$$

and then the result follows. To obtain the last equality we use the definition of square triangular number, that is, $B_n^2 = ST_n$ and (2.1)

$$\begin{aligned} ST_{n-r}ST_{n+r} - ST_n^2 &= B_{n-r}^2 B_{n+r}^2 - B_n^4 \\ &= (B_n^2 - B_r^2)^2 - B_n^4 \\ &= B_r^4 - 2B_n^2 B_r^2 \\ &= ST_r^2 - 2ST_nST_r. \end{aligned}$$ □

Note that for $r = 1$ in Catalan's identities obtained, we get the Cassini identities for these sequences. In fact, the equations (2.1), (2.2) and (2.3), for $r = 1$, yields, respectively

$$B_{n-1}B_{n+1} - B_n^2 = -B_1^2,$$

$$C_{n-1}C_{n+1} - C_n^2 = C_1^2 - 1$$

and

$$ST_{n-1}ST_{n+1} - ST_n^2 = ST_1^2 - 2ST_nST_1.$$

Now, using one of the initial terms of these sequences, we obtain

Proposition 2.2 (Cassini's identities). *For the sequences* $(B_n)_{n=0}^{\infty}$, $(C_n)_{n=0}^{\infty}$ *and* $(ST_n)_{n=0}^{\infty}$ *we have*

$$B_{n-1}B_{n+1} - B_n^2 = -1, \tag{2.4}$$

$$C_{n-1}C_{n+1} - C_n^2 = 8 \tag{2.5}$$

and

$$ST_{n-1}ST_{n+1} - ST_n^2 = 1 - 2ST_n,$$

respectively.

Note that an equivalent identity of (2.4) can be found in Behera and Panda [1] where its proof was done using induction on n.

The d'Ocagne identity for each of these sequences can also be obtained using the Binet formula for these type of sequences. We get

Proposition 2.3 (d'Ocagne's identities). *For the sequences* $(B_n)_{n=0}^{\infty}$, $(C_n)_{n=0}^{\infty}$ *and* $(ST_n)_{n=0}^{\infty}$ *if* $m > n$ *we have*

$$B_mB_{n+1} - B_{m+1}B_n = B_{m-n}, \tag{2.6}$$

$$C_mC_{n+1} - C_{m+1}C_n = -8B_{m-n}$$

and

$$ST_mST_{n+1} - ST_{m+1}ST_n = \frac{1}{8}B_{m-n}\left(C_{m+n+1} - 3C_{m-n}\right),$$

respectively.

Proof. Once more, using the Binet formula (1.6), the fact that $r_1r_2 = 1$ and $m > n$, we get that $B_mB_{n+1} - B_{m+1}B_n$ is

$$\begin{aligned}
&\left(\frac{r_1^m - r_2^m}{r_1 - r_2}\right)\left(\frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2}\right) - \left(\frac{r_1^{m+1} - r_2^{m+1}}{r_1 - r_2}\right)\left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right) = \\
&= (r_1r_2)^n \frac{(r_1 - r_2)(r_1^{m-n} - r_2^{m-n})}{(r_1 - r_2)^2} \\
&= \frac{r_1^{m-n} - r_2^{m-n}}{r_1 - r_2} \\
&= B_{m-n}.
\end{aligned}$$

For the Lucas-balancing numbers, the proof of the statement is similar to the previous one.

For the equality involving the square triangular numbers, first we apply the fact that $B_n^2 = ST_n$ and then we obtain

$$ST_mST_{n+1} - ST_{m+1}ST_n = B_m^2B_{n+1}^2 - B_{m+1}^2B_n^2$$

$$= (B_m B_{n+1} - B_{m+1} B_n)(B_m B_{n+1} + B_{m+1} B_n).$$

Now, we write $B_m B_{n+1} + B_{m+1} B_n$ as $\frac{1}{8}(C_{m+n+1} - 3C_{m-n})$ using the Binet formulas (1.6) and (1.7), $r_1 r_2 = 1$, $r_1 + r_2 = 6$ and doing some calculations. Finally using (2.6) the result follows. □

Once more, using the Binet formulas (1.6), (1.7) and (1.10) we obtain another property of the balancing, Lucas-balancing and square triangular sequences which is stated in the following proposition.

Proposition 2.4. *If B_n and C_n are the nth terms of the balancing sequence and Lucas-balancing sequence, respectively, then*

$$\lim_{n\to\infty} \frac{B_n}{B_{n-1}} = r_1 \tag{2.7}$$

and

$$\lim_{n\to\infty} \frac{C_n}{C_{n-1}} = r_1. \tag{2.8}$$

Consequently, if ST_n is the nth term of the square triangular sequence then

$$\lim_{n\to\infty} \frac{ST_n}{ST_{n-1}} = r_1^2. \tag{2.9}$$

Proof. We have that

$$\lim_{n\to\infty} \frac{B_n}{B_{n-1}} = \lim_{n\to\infty} \left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right)\left(\frac{r_1 - r_2}{r_1^{n-1} - r_2^{n-1}}\right) = \lim_{n\to\infty} \left(\frac{r_1^n - r_2^n}{r_1^{n-1} - r_2^{n-1}}\right). \tag{2.10}$$

Since $\left|\frac{r_2}{r_1}\right| < 1$, $\lim_{n\to\infty} \left(\frac{r_2}{r_1}\right)^n = 0$. Next we use this fact writing (2.10) in an equivalent form, obtaining

$$\lim_{n\to\infty} \frac{B_n}{B_{n-1}} = \lim_{n\to\infty} \frac{1 - \left(\frac{r_2}{r_1}\right)^n}{\frac{1}{r_1} - \left(\frac{r_2}{r_1}\right)^n \frac{1}{r_2}} = \frac{1}{\frac{1}{r_1}} = r_1.$$

Proceeding in a similar way with C_n we get the analogous result for the Lucas-balancing sequence. For the square triangular sequence, taking into account that $ST_n = B_n^2$ and using (2.7) the results follows. □

Note that (2.7) and (2.8) are presented in Behera and Panda [1], but the authors used different methods in their proofs.

In what follows, we can easily show the next result using basic tools of calculus of limits, (2.7), (2.8) and (2.9).

Corollary 2.5. *If B_n, C_n and ST_n are the nth terms of the balancing sequence, Lucas-balancing sequence and square triangular sequence, respectively, then*

$$\lim_{n\to\infty} \frac{B_{n-1}}{B_n} = \frac{1}{r_1} = r_2,$$

$$\lim_{n\to\infty} \frac{C_{n-1}}{C_n} = \frac{1}{r_1} = r_2$$

and

$$\lim_{n\to\infty} \frac{ST_{n-1}}{ST_n} = \frac{1}{r_1^2} = r_2^2.$$

Ray [17] establishes some new identities for the common factors of both balancing and Lucas-balancing numbers. Also we can establish more identities listed in the following proposition. Some of these identities involve both type of numbers, sums of terms, products of terms, among other relations between terms of these sequences.

Proposition 2.6. *If B_j, C_j and ST_j are the jth terms of the balancing sequence, Lucas-balancing sequence and square triangular sequence, respectively, then*

1. $B_{2n} = 2C_n B_n$;
2. $ST_n^2 = B_n^4$;
3. $C_n^2 = 8B_n^2 + 1 = 8ST_n + 1$;
4. $C_{2n} = 16B_n^2 + 1$;
5. $B_{n+2} - B_{n-2} = 12C_n$;
6. $\sum_{j=0}^{n} B_j = \frac{-1-B_n+B_{n+1}}{4}$;
7. $\sum_{j=0}^{n} C_j = \frac{2-C_n+C_{n+1}}{4}$;
8. $\sum_{j=0}^{n} ST_j = \frac{-1-ST_n+ST_{n+1}-2n}{32}$.

Proof. The first five identities are easily proved using the Binet formulas for B_n, C_n and ST_n, respectively. For the first identity, we easily have that

$$2C_n B_n = 2\left(\frac{r_1^n + r_2^n}{2}\right)\left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right) = B_{2n}$$

after doing some calculations. However, we can find in Panda [11] a different proof of this identity. The second identity is easily obtained using the well know fact that $B_n^2 = ST_n$. The third equality is a well known relation between the balancing and Lucas-balancing numbers [12]. We add one more identity using the fact that $C_n = \sqrt{8ST_n + 1}$. In order to obtain the fourth identity we only need adding and subtracting appropriate terms in order to get B_n^2. About the fifth identity we use the Binet formula for the sequences involved and we immediately get the result. Next we obtain the sum of the terms of the sequences, starting with the balancing sequence. Since $B_{n+1} = 6B_n - B_{n-1}$ for every $n \geq 1$, we have

$$6B_1 - B_0 = B_2$$

$$6B_2 - B_1 = B_3$$

$$6B_3 - B_2 = B_4$$

$$\cdots$$

$$6B_n - B_{n-1} = B_{n+1}.$$

Consequently,

$$6(B_1+B_2+B_3+\cdots+B_n)-B_0-B_1-B_2-\cdots-B_{n-1} = B_2+B_3+B_4+\cdots+B_{n+1}$$

which is equivalent to

$$6\sum_{j=1}^{n} B_j - 2\sum_{j=2}^{n-1} B_j = B_0 + B_1 + B_n + B_{n+1}.$$

Therefore

$$4\sum_{j=0}^{n} B_j = 5B_0 - B_1 - B_n + B_{n+1}.$$

Now if we consider the initial terms the result follows. For the Lucas-balancing and square triangular sequences proceeding in a similar way we obtain the required results. □

Remark 2.7. One of the most usual methods for the study of the recurrence sequences is to define the so-called generating matrix. Ray [15] introduces the matrix Q_B called $Q-$matrix and defined by $Q_B = \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}$. This matrix is a generating matrix for balancing sequence. Ray [15, Theorem 1] gives the result $Q_B^n = \begin{bmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{bmatrix}$. Also, Ray [15] defines the $R-$matrix as $R_B = \begin{bmatrix} 3 & -1 \\ 1 & -3 \end{bmatrix}$ and the author shows that $R_B Q_B^n = \begin{bmatrix} C_{n+1} & -C_n \\ C_n & -C_{n-1} \end{bmatrix}$, where C_n denotes the nth Lucas-balancing number. Note that $|Q_B| = 1$ and so $|Q_B^n| = |Q_B|^n = 1$. On the other hand $|Q_B^n| = -B_{n-1}B_{n+1} + B_n^2$ and so we obtain, for balancing sequence, the respective Cassini identity given in the equation (2.4). For the Lucas-balancing sequence, using a similar argument as we did for balancing sequence, we can obtain the respective Cassini identity given in the equation (2.5). In fact, we know that $|R_B| = -8$ and $|R_B Q_B^n| = |R_B||Q_B|^n = -8$. On the other hand, $|R_B Q_B^n| = -C_{n+1}C_{n-1} + C_n^2$ and then the Cassini identity for Lucas-balancing sequence follows.

3. Cobalancing, Lucas-cobalancing: some identities

In this section we present new identities and properties for cobalancing and Lucas-cobalancing sequences previously given by the recurrence relations (1.2) and (1.5). These identities are easily proved using the Binet formulas, (1.6), (1.8) and (1.9) for the involved sequences.

Proposition 3.1 (Catalan's identities). *For the sequences* $(b_n)_{n=1}^{\infty}$ *and* $(c_n)_{n=1}^{\infty}$ *if* $n > r$ *we have*

$$b_{n-r}b_{n+r} - b_n^2 = B_r^2 - \frac{1}{2}\left(b_{n-r} + b_{n+r} - 2b_n\right) \tag{3.1}$$

and

$$c_{n-r}c_{n+r} - c_n^2 = -8B_r^2. \tag{3.2}$$

Note that for $r = 1$ in Catalan's identities obtained, we have the Cassini identities for these sequences. In fact, using the equations (3.1) and (3.2), for $r = 1$, and the initial terms of these sequences, we proved the following result:

Proposition 3.2 (Cassini's identities). *For the sequences* $(b_n)_{n=1}^{\infty}$ *and* $(c_n)_{n=1}^{\infty}$ *if* $n \geq 2$ *we have*

$$b_{n-1}b_{n+1} - b_n^2 = -2b_n$$

and

$$c_{n-1}c_{n+1} - c_n^2 = -8.$$

Panda and Ray [13] obtained an equivalent identity using different arguments in their proof.

Proposition 3.3 (d'Ocagne's identities). *If* $m > n$ *and for the sequences* $(b_n)_{n=1}^{\infty}$ *and* $(c_n)_{n=1}^{\infty}$*, we have*

$$b_m b_{n+1} - b_{m+1}b_n = -B_{m-n} + B_m - B_n$$

and

$$c_m c_{n+1} - c_{m+1}c_n = 16B_{m-n}.$$

As we proceeded in the previous cases for balancing, Lucas-balancing and square triangular sequences we obtain analogous results for cobalancing and Lucas-cobalancing, as a consequence of the Binet formulas (1.8) and (1.9).

Proposition 3.4. *If* b_n *and* c_n *are the nth terms of the cobalancing sequence and Lucas-cobalancing sequence, respectively, then*

$$\lim_{n\to\infty} \frac{b_n}{b_{n-1}} = r_1 \tag{3.3}$$

and

$$\lim_{n\to\infty} \frac{c_n}{c_{n-1}} = r_1. \tag{3.4}$$

Corollary 3.5. *If b_n and c_n are the nth terms of the cobalancing sequence and Lucas-cobalancing sequence, respectively, then*

$$\lim_{n\to\infty} \frac{b_{n-1}}{b_n} = \frac{1}{r_1} = r_2$$

and

$$\lim_{n\to\infty} \frac{c_{n-1}}{c_n} = \frac{1}{r_1} = r_2.$$

As we did for balancing, Lucas-balancing and square triangular numbers we present in the next result new identities, where some of them involve these type of numbers, sums of terms, products of terms, among others.

Proposition 3.6. *If b_j and c_j are the jth terms of the cobalancing sequence and Lucas-cobalancing sequence, respectively, then*

1. $c_n^2 = \frac{1}{2}(C_{2n-1}-1) = 8b_n^2+8b_n+1$, *where C_j is the jth term of Lucas-balancing sequence*;
2. $b_n^2 = \frac{1}{16}C_{2n-1}-b_n-\frac{3}{16}$, *where C_j is the jth term of Lucas-balancing sequence*;
3. $b_n c_n = \frac{1}{2}(B_{2n-1} - c_n)$, *where B_j is the jth term of balancing sequence*;
4. $\sum_{j=1}^{n} b_j = \frac{b_{n+1}-b_n-2n}{4}$;
5. $\sum_{j=1}^{n} c_j = \frac{c_{n+1}-c_n-2}{4}$.

Proof. The first three identities are easily proved using the Binet formulas for the sequences involved. For the first identity, one of the equalities come from a well known relation between the cobalancing and Lucas-cobalancing sequences and the other equality is easily obtained using the Binet formula for $(c_n)_{n=1}^{\infty}$ and doing some calculations. Again using the Binet formula and doing some calculations we get the second and third identities as we refered before. For the last two identities a similar process that we applied for the sum of the first n terms of balancing sequence, can be used and the result follows. □

4. Generating functions

Next we shall give the generating functions for balancing, cobalancing, square triangular numbers, Lucas-balancing and Lucas-cobalancing sequences. These sequences can be considered as the coefficients of the power series expansion of the corresponding generating function. Recall that a sequence $(x_n)_{n=1}^{\infty}$ has a generation function given by $f(x) = \sum_{n=1}^{\infty} a_n x^n$. Behera and Panda [1] obtained

Proposition 4.1. *The ordinary generating function of the balancing sequence can be written as*

$$G(B_n; x) = \frac{x}{1-6x+x^2}. \tag{4.1}$$

Also Panda and Ray [13] established

Proposition 4.2. *The ordinary generating function of the cobalancing sequence can be written as*

$$G(b_n;x) = \frac{2x^2}{(1-x)(1-6x+x^2)}. \tag{4.2}$$

Sloane and Plouffe [20] have already obtained a generating function to the square triangular numbers sequence

Proposition 4.3. *The ordinary generating function of the square triangular numbers sequence can be written as*

$$G(ST_n;x) = \frac{x(x+1)}{(1-x)(1-34x+x^2)}. \tag{4.3}$$

Now, consider the ordinary generating function $G(C_n;x)$ associated with the sequence $(C_n)_{n=0}^{\infty}$ and defined by

$$G(C_n;x) = \sum_{n=1}^{\infty} C_n x^n$$

Using the initial terms, we get

$$\begin{aligned} G(C_n;x) &= 3x + 17x^2 + \sum_{n=3}^{\infty} C_n x^n \\ &= 3x + 17x^2 + \sum_{n=3}^{\infty} (6C_{n-1} - C_{n-2})x^n \\ &= 3x + 17x^2 + \sum_{n=3}^{\infty} 6C_{n-1}x^n - \sum_{n=3}^{\infty} C_{n-2}x^n \\ &= 3x + 17x^2 + 6x\sum_{n=3}^{\infty} C_{n-1}x^{n-1} - x^2\sum_{n=3}^{\infty} C_{n-2}x^{n-2} \end{aligned}$$

and considering $k = n-1$ and $j = n-2$ then the last identity can be written as

$$\begin{aligned} G(C_n;x) &= 3x + 17x^2 + 6x\left(\sum_{k=1}^{\infty} C_k x^k - 3x\right) - x^2\sum_{j=1}^{\infty} C_j x^j \\ &= 3x - x^2 + 6x\sum_{k=1}^{\infty} C_k x^k - x^2\sum_{j=1}^{\infty} C_j x^j. \end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} C_n x^n (1 - 6x + x^2) = 3x - x^2$$

and so we have the following result with respect to the Lucas-balancing sequence.

Proposition 4.4. *The ordinary generating function of the Lucas-balancing sequence can be written as*

$$G(C_n;x)=\frac{3x-x^2}{1-6x+x^2}. \tag{4.4}$$

With a slight modification of the previous proposition we obtain a generating function for Lucas-cobalancing sequence $(c_n)_{n=1}^{\infty}$ as

Proposition 4.5. *The ordinary generating function of the Lucas-cobalancing sequence can be written as*

$$G(c_n;x)=\frac{x+x^2}{1-6x+x^2}. \tag{4.5}$$

Now recall that for a sequence $(a_n)_{n=0}^{\infty}$, if $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=L$, where L is a positive real number, considering the power series $\sum_{n=1}^{\infty}a_nx^k$ its radius of convergence R is equal to $\frac{1}{L}$. Hence, for the balancing, the Lucas-balancing, the cobalancing and the Lucas-cobalancing sequences, using the results (2.7), (2.8), (3.3) and (3.4), respectively, we know that the sequences can be written as a power series with radius of convergence equal to $\frac{1}{r_1}=r_2$. In the case of the square triangular numbers sequence, according to (2.9), the radius of convergence is $\frac{1}{r_1^2}=r_2^2$. For a sequence $(a_n)_{n=0}^{\infty}$ we have $a_n=\frac{h^{(n)}(0)}{n!}$, where the derivation is meant in the convergence domain, and $h(x)$ is the corresponding generating function. Next we give another expression for the general term of all sequences using the ordinary generating function (4.1), (4.2), (4.3), (4.4) and (4.5).

Remark 4.6. Let us consider $F(x)=\sum_{n=1}^{\infty}B_nx^n$, $f(x)=\sum_{n=1}^{\infty}b_nx^n$, $G(x)=\sum_{n=1}^{\infty}C_nx^n$, $g(x)=\sum_{n=1}^{\infty}c_nx^n$ for $x\in]-r_2,r_2[$ and $t(x)=\sum_{n=1}^{\infty}ST_nx^n$ for $x\in]-r_2^2,r_2^2[$. Then we have that

$$B_n=\frac{F^{(n)}(0)}{n!},$$

$$b_n=\frac{f^{(n)}(0)}{n!},$$

$$C_n=\frac{G^{(n)}(0)}{n!},$$

$$c_n=\frac{g^{(n)}(0)}{n!}$$

and

$$ST_n=\frac{t^{(n)}(0)}{n!},$$

where $F^{(n)}(x)$, $f^{(n)}(x)$, $G^{(n)}(x)$, $g^{(n)}(x)$ and $t^{(n)}(x)$ denote the nth order derivative of the functions F, f, G, g and t, respectively.

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