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On De Giorgi's conjecture in dimensions 4 and 5

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1. Introduction

In this paper, we develop an approach for establishing in some important cases, a conjecture made by De Giorgi more than 20 years ago. The problem originates in the theory of phase transition and is so closely connected to the theory of minimal hypersurfaces that it is sometimes referred to as "the ϵ-version of Bernstein's problem for minimal graphs". The conjecture has been completely settled in dimension 2 by the authors [15] and in dimension 3 in [2], yet the approach in this paper seems to be the first to use, in an essential way, the solution of the Bernstein problem stating that minimal graphs in Euclidean space are necessarily hyperplanes provided the dimension of the ambient space is not greater than 8. We note that the solution of Bernstein's problem was also used in [18] to simplify an argument in [9]. Here is the conjecture as stated by De Giorgi [12].

Conjecture 1.1. *Suppose that u is an entire solution of the equation*

$$\Delta u + u - u^3 = 0, \quad |u| \leq 1, \quad x = (x', x_n) \in \mathbf{R}^n \tag{1.1}$$

satisfying

$$\frac{\partial u}{\partial x_n} > 0, \quad x \in \mathbf{R}^n. \tag{1.2}$$

Then, at least for $n \leq 8$, the level sets of u must be hyperplanes.

The conjecture may be considered together with the following natural, but not always essential condition:

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1. \tag{1.3}$$

The nonlinear term in the equation is a typical example of a two well potential and the PDE describes the shape of a transitional layer from one

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phase to another of a fluid or a mixture. The conjecture essentially states that the basic configuration near the interface should be unique and should depend solely on the distance to that interface.

One could consider the same problem with a more general nonlinearity

$$\Delta u - F'(u) = 0, \quad |u| \leq 1, \quad x \in \mathbf{R}^n \tag{1.4}$$

where $F \in C^2[-1,1]$ is a double well potential, i.e.

$$\begin{cases} F(u) > 0, \quad u \in (-1,1), \quad F(-1) = F(1) = 0 \\ F'(-1) = F'(1) = 0, \quad F''(-1) > 0, \quad F''(1) > 0. \end{cases} \tag{1.5}$$

Most of the discussion in this paper only needs the above conditions on F. However, Theorem 1.2 below requires the following additional symmetry condition:

$$F(-u) = F(u), \quad u \in (-1,1). \tag{1.6}$$

Note that equation (1.4) with $F(u) = \frac{1}{4}(1-u^2)^2$, reduces to (1.1).

Recent developments on the conjecture can be found in [15], [4], [7], [14], [2], [1]. Some earlier works on this subject can be found in [12], [20]–[24].

Modica was first to obtain (partial) results for $n = 2$. A strong form of the De Giorgi Conjecture was proved for $n = 2$ by the authors [15], and later for $n = 3$ by Ambrosio-Cabre [2]. If one replaces (1.2) and (1.3) by the following uniform convergence assumption:

$$u(x', x_n) \to \pm 1 \quad \text{as} \quad x_n \to \pm\infty \quad \text{uniformly in} \quad x' \in \mathbf{R}^n, \tag{1.3$'$}$$

one may then ask whether

$$u(x) = g(x_n + T) \quad \text{for some} \quad T \in \mathbf{R},$$

where g is the solution of the corresponding one-dimensional ODE.

This is referred to as the Gibbons conjecture, which was first established by the authors in [15] for $n = 3$, and later proved for all dimensions in [4], [7] and [14] independently. The ideas used in [15] for the proof of the Gibbons conjecture in dimension 3, were refined and used in two separate directions: First in [4] where a general Liouville theorem for divergence-free, degenerate operators was established and used to show that the De Giorgi conjecture holds in all dimensions, provided all level sets of u are equi-Lipschitzian. They were also used in [2], in combination with a new energy estimate in order to settle the De Giorgi conjecture in dimension 3.

In order to state our main results, we note first that equation (1.4) in any bounded domain Ω is the Euler-Lagrange equation of the functional

$$E_\Omega(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + F(u) \right) dx \tag{1.7}$$

defined on $H^1(\Omega)$. In particular, when Ω is the ball $B_R(0)$ centered at the origin and with radius R, we write $E_R(u) = E_{B_R}(u)$ and we consider the functional

$$\rho(R) = \frac{E_R(u)}{R^{n-1}}, \tag{1.8}$$

which satisfies the following important monotonicity and boundedness properties.

Proposition 1.1. *Assume that F satisfies* (1.5) *and that u is a solution of* (1.4); *then*,

1. (Modica [22]) *The function $\rho(R)$ is an increasing function of R.*
2. (Ambrosio-Cabre [2]) *There is a constant $c > 0$ such that $\rho(R) \le c$ for all $R > 0$.*

If the dimension is less than 8, then the best constant c above can be made explicit. It is proved in [1] (see §2 below) that if u satisfies (1.2)–(1.4), then

$$\lim_{R\to\infty} \rho(R) = \gamma_F \omega_{n-1}, \tag{1.9}$$

where $\gamma_F = \int_{-1}^{1} \sqrt{2F(u)}\, du$ and ω_{n-1} is the volume of the $n-1$ dimensional unit ball.

Here is our main result.

Theorem 1.1. *Assume that F satisfies* (1.5) *and that u is a solution of* (1.2) *and* (1.4) *such that for some $q, c > 0$*:

$$\gamma_F\omega_{n-1} - cR^{-q} \le \rho(R) \le \gamma_F\omega_{n-1} \quad \textit{for } R \textit{ large}. \tag{1.10}$$

If the dimension $n \le q+3$, then $u(x) = g(x \cdot a)$ for some $a \in S^{n-1}$, where g is the solution of the corresponding one-dimensional ODE.

If $n = 3$, this clearly recaptures the result of [2] with $q = 0$ in (1.10). Under the uniform convergence condition (1.3)′, we shall see that (1.10) is satisfied for $q = 2$ and hence will lead to another proof of the Gibbons conjecture up to dimension 5. But our main application is that the De Giorgi conjecture is true in dimensions $n = 4, 5$ provided the solutions are also assumed to satisfy an anti-symmetry condition. This is done by establishing (1.10) with $q = 2$ under such an assumption. More precisely, we have:

Theorem 1.2. *Assume F satisfies* (1.5) *and* (1.6). *Suppose u is a solution to* (1.2)–(1.4) *which –after a proper translation and rotation– satisfies*:

$$u(y,z) = -u(y,-z) \textit{ for } x = (y,z) \in \mathbf{R}^{n-k} \times \mathbf{R}^k, \tag{1.11}$$

where k is an integer with $1 \le k \le n$. If the dimension $n \le 5$, then $u(x) = g(x \cdot a)$ for some $a \in S^{n-1}$.

Remark 1.1. a) It is easy to see that in Theorem 1.2 $a \in \{0\} \times \mathbf{R}^k$ since $u(y,0) = 0$ for $y \in \mathbf{R}^k$. Also note that if $k = 1$, then $u(y,0) = 0$ for $y \in \mathbf{R}^{n-1}$. This case may be regarded as a symmetry result in half-space which was essentially proved in [6] for all dimensions. Our approach is also a bit easier in this case and will be dealt with in Section 6.

b) Note that here we do not assume any growth control on the level sets of the solutions.

c) It is natural to attempt to construct counterexamples with a certain anti-symmetry, similar to those satisfied by Simon's cones that led to the complete solution of the Bernstein problem. Theorem 1.2 implies that such counterexamples do not exist for $n = 4, 5$. However, they may still exist for $n > 8$.

The basic idea behind the proofs in dimension 2 and 3 is the observation that any solution u of (1.4) satisfying an energy estimate of the form

$$\int_{B_R} |\nabla u|^2 \, dx \le cR^2, \tag{1.12}$$

where B_R is the ball of radius $R > 0$, must necessarily have hyperplanes for level sets. Our approach is based on the observation that (1.12) can actually be replaced by

$$\int_{C_{R_k}} |\nabla_{x'} u|^2 \, dx \le c{R_k}^2, \tag{1.13}$$

where C_R are cylinders of the form

$$C_R := \left\{ (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R};\ |x'| \le R,\ |x_n| \le R \right\},$$

R_k is a sequence going to $+\infty$ and $\nabla_{x'}$ is the gradient in the x'-direction.

Here is the strategy: Set

$$h(R) = \frac{1}{R^{n-1}} \int_{C_R} \left(\frac{1}{2} |\nabla u|^2 + F(u) \right) dx. \tag{1.14}$$

We shall see in Section 2 that if u satisfies (1.2)–(1.4) then, after a proper rotation of the coordinates,

$$\lim_{R \to \infty} h(R) = \gamma_F \omega_{n-1}. \tag{1.15}$$

Actually the main axis of the cylinders C_R for which (1.15) holds may not necessarily be the x_n-direction. Even though the x_n-direction is special due to (1.2), the above assumption will not cause a loss of generality in the discussions below. Indeed, if we replace (1.2) by a –probably equivalent– local minimizing condition (see §2 below), then all the main results in this paper would still hold.

Key to our approach is the following result:

THEOREM 1.3. *Suppose u is a solution of* (1.2)–(1.4) *such that for some* $q, c > 0$, *there is a sequence* $R_k \uparrow +\infty$ *so that*:

$$h(R_k) \leq \gamma_F \omega_{n-1} + cR_k^{-q} \quad \text{for all } k. \tag{1.16}$$

If the dimension $n \leq q+3$, *then* $u(x) = g(x_n + T)$ *for some constant* T.

We shall first establish Theorem 1.3 in Section 3. We then show in Section 4 how it implies Theorem 1.1. In Section 5, we show how the latter implies Theorem 1.2. Finally, in Section 6, we give a simpler proof of Theorem 1.2, in the case where the anti-symmetry condition reduces the conjecture to a half-space setting, i.e., in $\mathbf{R}_+^{n-1}$. We also point out some cases where our results can be generalized.

Finally, we believe that the approach is quite promising and has the potential to lead to a resolution of the conjecture in all dimensions below 8, or at least to a complete solution in dimensions 4 and 5. The latter would depend on the improvement of our estimates below or –more specifically– on a positive solution of a conjecture that we formulate in Section 5.

2. De Giorgi's conjecture and Bernstein's problem for minimal graphs

In this section, we introduce notation while collecting all needed known facts, especially those connecting De Giorgi's conjecture with the Bernstein problem for minimal graphs. Unless specifically stated otherwise, we shall assume throughout that the nonlinear term F satisfies (1.5).

PROPOSITION 2.1. *When* $n = 1$, *problem* (1.3)–(1.4) *has a unique solution up to translation, denoted* $g(t)$, *which satisfies*: $g'(t) > 0$ *and* $g(t) = -g(-t)$ *for all* $t \in \mathbf{R}$. *Moreover*,

$$0 < 1 - g(t) < ce^{-\mu t}, \quad t \geq 0 \tag{2.1}$$

for some constant $c, \mu > 0$.

The De Giorgi conjecture may therefore be stated as claiming that any solution u for (1.2)–(1.4) can be written as $u(x) = g(x \cdot a)$ for some $a \in S^{n-1}$.

PROPOSITION 2.2 (Modica [20]). *Suppose u is a solution of* (1.4); *then*

$$|\nabla u(x)|^2 \leq 2F(u(x), \quad \forall x \in \mathbf{R}^n. \tag{2.2}$$

It is also known (see [23] and [1]) that solutions of (1.4) and (1.2) are local minimizers of the functional E in the following sense.

Proposition 2.3. *For any solution u of* (1.2)–(1.4) *and any bounded smooth domain* $\Omega \subset \mathbf{R}^n$,

$$E_\Omega(u) = \min\left\{E_\Omega(v);\ v = u \ \text{ on } \partial\Omega, \quad |v| \le 1, \quad v \in C^1(\bar{\Omega})\right\}. \tag{2.3}$$

This easily yields the estimate $E_R(u) \le cR^{n-1}$ mentioned in Proposition 1.1 above.

Actually, in all the results stated below, one can replace condition (1.2) by the possibly weaker condition that u is a local minimizer, i.e., that (2.3) holds for all bounded smooth domains. However, there are reasons to believe that conditions (1.2) and (2.3) are actually equivalent and we propose the following:

Conjecture 2.1. *Assume that u is a local minimizer of E, i.e., that* (2.3) *holds for all bounded smooth domains Ω. Then after appropriate rotation of the coordinates,* (1.2) *holds.*

Indeed, it is observed in [1] and [10] that Conjecture 2.1 holds for $n = 2$ and 3 since arguments similar to those in the proof of De Giorgi's conjecture in these dimensions apply under condition (2.3) and lead to the one-dimensional symmetry of the solution and therefore to the monotonicity property (1.2).

We note that Sternberg also raised a similar question for minimizers in bounded convex domains with mean 0.

Modica also studied the De Giorgi conjecture by using the Γ-convergence approach. Namely, for any $\varepsilon > 0$, one considers the following scaling of u. For a fixed $K > 0$, set

$$u_\varepsilon(x) = u\left(\frac{x}{\varepsilon}\right), \quad x \in B_K$$

and its energy on B_K,

$$E^\varepsilon(u_\varepsilon) = \int_{B_K} \left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}F(u_\varepsilon)\right) dx. \tag{2.4}$$

Since for any $K > 0$, we have

$$E^\varepsilon(u_\varepsilon, B_K) = \varepsilon^{n-1}E^1(u, B_{\frac{K}{\varepsilon}}) \le cK^{n-1},$$

there are a subsequence (u_{ε_k}) and a set D with a locally finite perimeter in $\mathbf{R}^n$, such that:

- $u_{\varepsilon_k} \to \chi_D - \chi_D^c$ in L^1_{loc} and
- $\lim_k D^{\varepsilon_k}(u_{\varepsilon_k}, A) = \gamma_F P(D, A)$ for any open bounded subset A in $\mathbf{R}^n$.

Here $\gamma_F = \int_{-1}^1 \sqrt{2F(t)}\, dt$ and the perimeter functional (of D in A) is defined as

$$P(D, A) := \sup\left\{\int_D \operatorname{div} g\, dx;\ g \in C_0^1(A, \mathbf{R}^n),\ |g| \le 1\right\}.$$

Moreover, the set D is a local minimizer of the perimeter, i.e., for each $K > 0$.

$$P(D, B_K) = \min\{P(F, B_K); D\Delta F \subset B_K\}. \tag{2.5}$$

The results on minimal sets ([13], [19]) yield that ∂D is a hyperplane, provided the dimension $n \leq 8$. In other words, the subsequence u_{ε_k} converges in $L^1(B_K)$ to $\chi_D - \chi_{B_K \setminus D}$ and

$$D \cap B_K = B_K^+ = \{x \cdot \mathbf{a} > 0;\, x \in B_K\} \text{ for some } \mathbf{a} \in S^{n-1}. \tag{2.6}$$

See also [23] and [1] for more details.

By combining the monotonicity formula and the Γ-convergence result as well as the minimality property of u, one then obtains that for $n \leq 8$:

$$D_R(u) \leq \gamma_F w_{n-1} R^{n-1} \quad \text{for all R.} \tag{2.7}$$

Finally, we restate the uniform convergence result of Caffarelli and Cordoba [8] on the level sets of u_ε.

Proposition 2.4. *Choose the subsequence ε_k along which the above Γ-convergence holds and let $\mathbf{a}$ be the normal direction to the associated limiting hyperplane. Let*

$$d_{\varepsilon_k}(\delta) = \sup \left\{ |x \cdot \mathbf{a}|;\, |u_{\varepsilon_k}(x)| < \delta,\, x \in B_{K/2} \right\}.$$

Then, for any $\delta \in (0, 1)$,

$$\lim_{\varepsilon_k \to 0} d_{\varepsilon_k}(\delta) = 0.$$

An easy consequence of Proposition 2.4 and the maximum principle is the following:

Proposition 2.5. *Let $d > 0$, ε_k and $\mathbf{a}$ as above. Then*

$$1 - |u_{\varepsilon_k}(x)|^2 < ce^{-\mu/\varepsilon_k} \quad \textit{for } |x \cdot \mathbf{a}| > d \textit{ and } x \in B_{K/2}, \tag{2.8}$$

where c, μ are independent of ε_k.

See e.g. [15] for a proof of a similar estimate.

3. Energy estimates on cylinders

In this section, we prove Theorem 1.3 and some of its direct applications. Again, we consider cylinders of the form:

$$C_R := \left\{ (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R};\, |x'| \leq R,\, |x_n| \leq R \right\}.$$

We are assuming here, for simplicity, that the main axis $\mathbf{a}$ that is normal to the "limiting" hyperplane described in Section 2 is the x_n-direction. Even though the x_n-direction is special due to (1.2), we do not use (1.2) for this special direction and therefore the above assumption will not lose the generality in the discussions below. Indeed, we can replace (1.2) by the local minimizing condition (2.3). See Remark 3.1 below.

LEMMA 3.1. *Let u be a solution of* (1.2)–(1.4), *and consider the subsequence ϵ_k along which the above Γ-convergence holds as in* (2.8). *Then*:

$$\int_{C_{R_k}} \left(\frac{1}{2}|u_{x_n}|^2 + F(u)\right) dx \geq \gamma_F \omega_{n-1} R_k^{n-1} - ce^{-\mu R_k} \tag{3.1}$$

for some $c, \mu > 0$, where $R_k = \frac{1}{\varepsilon_k} \to +\infty$ as $k \to \infty$.

Proof. Use Proposition 2.5, with $K = 2R$, $d = \frac{1}{4}$ and note that $C_{R_k} \subset B_{2R_k}$. Then

$$\begin{aligned} \int_{C_{R_k}} \left(\frac{1}{2}|u_{x_n}|^2 + F(u(x))\right) dx_n &\geq \int_{B_{R_k^{n-1}}} \int_{-R_k}^{R_k} |u_{x_n}| \cdot \sqrt{2F(u(x))}\, dx_n\, dx' \\ &\geq \int_{B_{R_k^{n-1}}} \int_{1+ce^{-\mu R_k}}^{1-ce^{-\mu R_k}} \sqrt{2F(u)}\, du\, dx' \\ &\geq \omega_{n-1} R_k^{n-1} \left(\gamma_F - ce^{-\mu R_k}\right), \end{aligned}$$

where c, μ may have changed from line to line. We note that here we have only used the fact that $\sqrt{2F(u)} = O(1-u^2)$ as $u^2 \to 1$.

Proof of Theorem 1.3. Consider

$$h(R) = \frac{1}{R^{n-1}} \int_{C_R} \left(\frac{1}{2}|\nabla u|^2 + F(u)\right) dx.$$

The discussion in Section 2 yields that

$$\lim_{R_k \to \infty} h(R_k) = \gamma_F \omega_{n-1}. \tag{3.2}$$

Assume now that for some $q, c > 0$,

$$h(R_k) \leq \gamma_F \omega_{n-1} + cR_k^{-q} \quad \text{for all } k. \tag{3.3}$$

We need to prove that for $n \leq \min\{q+3, 8\}$, the solution u depends only on one variable.

Estimates (3.1) and (3.3) lead to

$$\int_{C_{R_k}} |\nabla_{x'} u|^2\, dx \leq c{R_k}^{-q+n-1}. \tag{3.4}$$

Now we follow an idea already used in [6], [15] and later in [2]. Let $\sigma = \frac{\partial u}{\partial x_n} > 0$, $\varphi = \nabla u \cdot \nu$ for any fixed $\nu = (\nu', 0) \in \mathbf{R}^{n-1} \times \{0\}$. Then $\psi = \frac{\varphi}{\sigma}$ satisfies

$$\operatorname{div}(\sigma^2 \nabla \psi) = 0, \quad x \in \mathbf{R}^n. \tag{3.5}$$

Choose a proper cut-off function $\chi(x)$ such that

$$\chi(x) = \begin{cases} 1 & x \in C_{1/2} \\ 0 & x \in \mathbf{R}^n \setminus C_1 \end{cases}$$

and $\chi_R(x) = \chi(x/R)$. Then

$$\int_{C_R} \chi_R^2 \sigma^2 |\nabla\psi|^2 \, dx \le b \left(\int\limits_{C_R \setminus C_{R/2}} \chi_R^2 \sigma^2 |\nabla\psi|^2 \, dx \right)^{1/2} \cdot \left(\frac{1}{R^2} \int_{C_R} \varphi^2 \, dx \right)^{1/2} \tag{3.6}$$

for some $b > 0$. Since

$$\int_{C_{R_k}} \varphi^2 \, dx \le c \int_{C_{R_k}} |\nabla_{x'} \cdot u|^2 \, dx \le c R_k^{-q+n-1},$$

then we have by (3.6) that:

$$\int_{C_{R_k}} \chi_{R_k}^2 \sigma^2 |\nabla\psi|^2 \le c R_k^{-q+n-3} < \alpha < \infty \tag{3.7}$$

as long as $n \le q + 3$.

By letting $R_k \to \infty$, (3.6) and (3.7) lead to

$$\int_{\mathbf{R}^n} \sigma^2 |\nabla\psi|^2 \, dx \le 0.$$

Therefore $\psi \equiv c$ and $\varphi \equiv c\sigma(x)$ for $x \in \mathbf{R}^n$. Since $\nu = (\nu', 0)$ is arbitrary in $\nu' \in \mathbf{R}^{n-1}$, the solution $u(x)$ is independent of at least $n-2$ dimensions and therefore can be regarded as a function in $\mathbf{R}^2$. If the direction $\mathbf{a}$ happens to be the same as the x_n-direction, we will then have u independent of $n-1$ dimensions. In any case, the validity of De Giorgi's conjecture in two dimensions completes the proof of Theorem 1.3.

Remark 3.1. If we replace (1.2) by the local minimizing condition (2.3), we have to replace σ in the above argument by the "first eigenfunction" of the linearized equation of (1.4) (see [15] for the existence of such an eigenfunction in general). Note that the minimizing condition implies that the "first eigenvalue" λ_1 is 0.

COROLLARY 3.1. *Assume the uniform convergence condition* (1.3)$'$. *Then* (1.16) *holds for* $q = 2$; *that is*:

$$h(R) \le \gamma_F \omega_{n-1} + cR^{-2} \quad \textit{for all } R > 0. \tag{3.8}$$

In other words, the above approach yields another proof of the Gibbons conjecture up to dimension 5.

Proof. Following [22], we can derive the following formula for $h(r)$:

$$\begin{aligned} h'(r) &= \frac{1}{2}r^{-n}\int_{C_r}\left(2F(u)-|\nabla u|^2\right)dx \\ &\quad + r^{-(n+1)}\int_{\partial C_r}(\nabla u\cdot\nu)(\nabla u\cdot x)\,dS_x \\ &\geq r^{-(n+1)}\int_{\partial C_r\cap\{|x_n|<r\}}\langle\nabla u, x'\rangle^2+\langle\nabla u, x'\rangle\cdot\left(\frac{\partial u}{\partial x_n}x_n\right)dS_x \\ &\quad + r^{-(n+1)}\int_{\partial C_r\cap\{|x_n|=r\}}\langle\nabla u, x\rangle\cdot\left(\frac{\partial u}{\partial x_n}x_n\right)dS_x \\ &\quad -\frac{1}{4}r^{-(n+1)}\int_{\partial C_r\cap\{|x_n|<r\}}\left(\frac{\partial u}{\partial x_n}x_n\right)^2 dS_x \\ &\quad + r^{-(n+1)}\int_{\partial C_r\cap\{|x_n|=r\}}\langle\nabla u, x\rangle\cdot\left(\frac{\partial u}{\partial x_n}x_n\right)dS_x. \end{aligned}$$

According to [15], the uniform convergence condition $(1.3)'$ implies:

$$|\nabla u|\leq ce^{-\mu|x_n|},\quad x\in\mathbf{R}^n, \tag{3.9}$$

for some constant $c,\mu>0$. It follows that

$$\begin{aligned} \int_{\partial C_r\cap\{|x_n|<r\}}\left(\frac{\partial u}{\partial x_n}x_n\right)^2 dS_x &\leq \int_{\partial B_r^{n-1}}\int_{-r}^{r}\left(\frac{\partial u}{\partial x_n}\cdot x_n\right)^2 dx_n\,dS_{x'} \\ &\leq \int_{\partial B_r^{n-1}}\int_{-r}^{r}\left(ce^{-\mu|x_n|}x_n\right)^2 dx_n\,dS_{x'}\leq cr^{n-2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \gamma_F\omega_{n-1}-h(R) &= \int_R^\infty h'(r)\,dr \\ &\geq -c\int_R^\infty r^{-3}+e^{-2\mu r}\,dr\geq -cR^{-2}, \end{aligned}$$

which establishes (3.8).

4. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. The idea here is to use the lower estimate on balls to get an upper estimate on cylinders.

PROPOSITION 4.1. *Assume a solution u to* (1.2)–(1.4) *satisfies*

$$\gamma_F\omega_{n-1}-c_1R^{-q}\leq\rho(R)\leq\gamma_F\omega_{n-1} \tag{4.1}$$

for some $q>0$ and $c_1>0$. Let $R_k=\frac{1}{\epsilon_k}$ be a sequence such that the Γ-convergence holds toward a hyperplane with normal $\mathbf{a}$ as in (2.8). *Let $h(R)$ be the normalized energy associated to the cylinder C_R in the $\mathbf{a}$-direction. Then*

$$\gamma_F\omega_{n-1}-c_2e^{-\mu R}\leq h(R)\leq\gamma_F\omega_{n-1}+c_2R^{-q},\qquad R\geq 1 \tag{4.2}$$

for some $c_2 > 0$ and $\mu > 0$ independent of R.

Consequently, the asymptotic direction $\mathbf{a}$ *is unique and does not depend on the choice of the subsequence.*

Proof. Note first that by Lemma 3.1,

$$h(R_k) \geq \gamma_F \omega_{n-1} - ce^{-\mu R_k}. \tag{4.3}$$

Following [22], we have the monotonicity formula

$$\rho'(r) = \frac{1}{2} r^{-n} \int_{B_r} \left(2F(u) - |\nabla u|^2\right) dx + r^{-(n+1)} \int_{\partial B_r} (\nabla u \cdot x)^2 \, dS_x. \tag{4.4}$$

Integrating the above equality from R to ∞, we obtain

$$\gamma_F \omega_{n-1} - \rho(R) \geq \int_R^\infty r^{-(n+1)} \int_{\partial B_r} (\nabla u \cdot x)^2 \, dS_x dr \quad \geq \int_{B_R^c} \frac{(\nabla u \cdot x)^2}{|x|^{n+1}} dx. \tag{4.5}$$

Then, by (4.1),

$$\int_{B_R^c} \frac{(\nabla u \cdot x)^2}{|x|^{n+1}} \leq c_1 R^{-q}. \tag{4.6}$$

On the other hand,

(4.7)
$$\begin{aligned}
h'(r) \quad &= \quad \frac{1}{2} r^{-n} \int_{C_r} \left(2F(u) - |\nabla u|^2\right) dx \\
&\quad + r^{-n} \int_{\partial C_r} (\nabla u \cdot \nu)(\nabla u \cdot x) \, dS_x \\
&\geq \quad r^{-(n+1)} \int_{\partial C_r \cap \{|x_n| < r\}} (\nabla u \cdot x')(\nabla u \cdot x) \, dS_x \\
&\qquad + r^{-(n+1)} \int_{\partial C_r \cap \{|x_n| = r\}} (\nabla u \cdot x) \cdot \left(\frac{\partial u}{\partial x_n} x_n\right) dS_x \\
&\geq \quad -r^{-(n+1)} \int_{\partial C_r} |(\nabla u \cdot x')(\nabla u \cdot x)| \, dS_x \\
&\geq \quad -\frac{1}{2} r^{-(n+1)} \int_{\partial C_r} \left(r^{-\alpha}(\nabla u \cdot x')^2 + r^{\alpha}(\nabla u \cdot x)^2\right) dS_x \\
&\geq \quad -\frac{1}{2} r^{-(n-1+\alpha)} \int_{\partial C_r} |\nabla_{x'} u|^2 \, dS_x - \frac{1}{2} r^{-(n+1-\alpha)} \int_{\partial C_r} (\nabla u \cdot x)^2 \, dS_x.
\end{aligned}$$

Now let

$$l(r) := \int_{C_r} |\nabla_{x'} u|^2 dx \tag{4.8}$$

and

$$k(r) := \int_{\mathbf{R}^n \setminus C_r} \frac{(\nabla u \cdot x)^2}{|x|^{n+1}} dx. \tag{4.9}$$

We know by (1.9) that

$$l(r) \leq c r^{n-1} \tag{4.10}$$

and by (4.6) that

$$k(r) \leq c_1 r^{-q}. \tag{4.11}$$

Therefore, when $\alpha > 0$ we have

$$\begin{aligned} \int_R^\infty r^{-(n-1+\alpha)} \int_{\partial C_r} |\nabla_{x'} u|^2 \, dS_x \, dr \ &\leq \ \int_R^\infty r^{-(n-1+\alpha)} l'(r) dr \\ &\leq \ (n-1+\alpha) \int_R^\infty r^{-(n+\alpha)} l(r) dr \leq c R^{-\alpha} \end{aligned}$$

for some positive constant c.

We also have for $0 < \alpha < q$,

$$\begin{aligned} \int_R^\infty r^{-(n+1-\alpha)} \int_{\partial C_r} (\nabla u \cdot x)^2 \, dS_x dr \quad &\leq \quad (\sqrt{2})^{n+1} \int_R^\infty r^\alpha (-k'(r)) dr \\ &\leq \quad c R^{\alpha - q}. \end{aligned} \tag{4.12}$$

for some constant $c > 0$.

Integrating from R to R_k and letting $k \to \infty$, we conclude from (4.7), (4.12) and (4.13) that

$$\gamma_F \omega_{n-1} - h(R) = \int_R^\infty h'(r) dr \geq -c(R^{-\alpha} + R^{\alpha - q}). \tag{4.13}$$

Choose $\alpha = q/2$ to obtain

$$h(R) \leq \gamma_F \omega_{n-1} + c R^{-q/2} \tag{4.14}$$

for some $\mu, c > 0$ independent of $R \geq 1$.

The inequality (4.15) implies that for any sequence $(R_m = \frac{1}{\varepsilon_m})_m$ tending to infinity, the Γ-limit of u_{ε_m} defined in (2.6) will always be the same. In other words, the direction $\mathbf{a}$ defined in (2.6) does not depend on the choice of the sequence $(R_m)_m$. Otherwise the limit hyperplane would intersect the limit cylinder at an angle other than $\pi/2$, which would lead to $\lim_{R_m \to \infty} h(R_m) > \gamma_F \omega_{n-1}$, therefore contradicting (4.15). This means that estimate (4.15) is actually a rigidity result, since it allows only one asymptotic orientation for the level set at infinity.

From this, we conclude that (3.1) holds for all $r > 0$; that is,

$$h(r) \geq \gamma_F \omega_{n-1} - c_1 e^{-\mu r} + \frac{1}{2} r^{-(n-1)} l(r), \qquad r \geq 1, \tag{4.15}$$

for some c_1, μ independent of r.

Combine now (4.15) and (4.16) to obtain

$$l(r) \leq c r^{n-1-q/2}. \tag{4.16}$$

We also obtain from (4.7) that

$$h'(r) \geq -\delta r^{-(n-1)} \int_{\partial C_r} |\nabla_{x'} u|^2 \, dS_x - \frac{1}{4\delta} r^{-(n+1)} \int_{\partial C_r} (\nabla u \cdot x)^2 \, dS_x \tag{4.17}$$

where $\delta > 0$ is chosen so that $\delta < q/4(n-1)$.

Repeating estimates (4.12) and (4.13) with $\alpha = 0$, we get

$$\gamma_F \omega_{n-1} - h(R) \geq -(n-1)\delta \int_R^\infty r^{-n} l(r) dr - cR^{-q}, \qquad R \geq 1. \tag{4.18}$$

Let now

$$L(R) := \int_R^\infty r^{-n} l(r) dr, \qquad R \geq 1.$$

Then (4.16) and (4.19) yield the differential inequality

$$-rL'(r) \leq 2(n-1)\delta L(r) + cr^{-q}, \qquad r \geq 1. \tag{4.19}$$

Solving the above inequality leads to

$$L(r) \leq Cr^{-q}, \qquad r \geq 1.$$

Therefore we obtain (4.2) as well as

$$l(r) \leq cr^{n-1-q}, \qquad r \geq 1. \tag{4.20}$$

This proves Proposition 4.1.

Theorem 1.1 now follows immediately from Theorem 1.3 and Proposition 4.1.

Remark 4.1. From the proof of Proposition 4.1, it is clear that Theorem 1.1 holds if the condition (1.10) is replaced by

$$\int_{\mathbf{R}^n \setminus B_R} \frac{(\nabla u \cdot x)^2}{|x|^{n+1}} dx \leq cR^{-q} \tag{4.21}$$

for some positive constant c. This quantity might be estimated directly. Again, the best possible q in the estimate is 2.

5. Lower estimates on balls for the anti-symmetric case

Estimate (1.9) gives a good upper bound for the energy $E_R(u)$ on balls, which was sufficient to prove De Giorgi's conjecture in dimension 3 ([2]). However, in order to deal with higher dimensions via the approach outlined above, we need, in view of Theorem 1.1, to establish good lower estimates on $E_R(u)$. We shall do so in this section, under the assumption that F satisfies (1.5) and (1.6).

For this purpose, we consider the following minimizing problem in a given ball B_R:

$$e_R := E_R(v_R) = \min\left\{E_R(v);\, v \in H^1(B_R),\, |v| \le 1,\, \int_{B_R} v = 0\right\}. \tag{5.1}$$

It is easy to see that v_R exists and satisfies for some constant a_R

$$\begin{cases} \Delta v_R - F'(v_R) & = a_R, \quad x \in B_R \\ |v_R| & < 1, \quad\;\; x \in B_R \\ \frac{\partial v_R}{\partial n} & = 0 \quad\;\;\; \text{on } \partial B_R. \end{cases} \tag{5.2}$$

Now we formulate the following:

Conjecture 5.1. *At least for R large enough, $a_R = 0$ and, after proper rotations, $v_R(x', x_n) = v_R(|x'|, x_n) = -v_R(|x'|, -x_n)$.*

If we write x in its spherical coordinates $x = (r, \theta, \varphi)$, with $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\varphi \in (0, \pi)^{n-2}$, then the Steiner symmetrization argument in the spherical coordinates yields the following partial answer. (See [17, Th. 2.31 on p. 83 under condition (A 2.7f) on p. 82]).

Lemma 5.1. *After proper rotations, $v_R(x) = v_R(r, \theta)$ and $v_R(r, \theta)$ is increasing in θ. In particular, $v_R(x', x_n) = v_R(|x'|, x_n)$ in the cartesian coordinates.*

Remark 5.1. If Conjecture 5.1 is true, one can then proceed as below to obtain the following estimates for e_R

$$\gamma_F \omega_{n-1} R^{n-1} - c_1 R^{n-3} \le e_R \le \gamma_F \omega_{n-1} R^{n-1} - c_2 R^{n-3} \tag{5.3}$$

for some $c_1, c_2 > 0$. These would be useful to resolve the De Giorgi conjecture in dimensions 4 and 5. We shall do so below under additional anti-symmetry conditions. In this case, we minimize E_R under extra constraints, such as anti-symmetry. Write $x = (y, z) \in \mathbf{R}^{n-k} \times \mathbf{R}^k$, $1 \le k \le n$ and consider the following minimization problem:

$$e_R^k := E_R(v_R^k) = \min\left\{E_R(v); v \in H^1(B_R), \quad |v| \le 1, \quad v(y,z) = -v(y,-z)\right\}. \tag{5.4}$$

Again, by the Steiner symmetrization argument (same reference as above), we have:

Lemma 5.2. *For any $1 \le k \le n$, there exists a minimizer v_R^k of (5.4) which satisfies $v_R^k \equiv v_R^1$. Moreover, in spherical coordinates, $v_R^1(r, \theta, \phi) = v_R^1(r, \theta, 0) = v_R^1(r, -\theta, 0)$ is decreasing in $\theta \in (0, \pi)$ and is independent of ϕ. Furthermore, $v_R^1(x', x_n) = v_R^1(|x'|, x_n) = -v_R^1(|x'|, -x_n)$ in cartesian coordinates and in particular,*

$$e_R^k = e_R^1 \quad for \quad 1 < k \le n.$$

Note that the anti-symmetry in x_n of the minimizer v_R^k follows automatically from the anti-symmetry in $z \in \mathbf{R}^k$ in the Steiner symmetric rearrangment in spherical coordinates.

It is also obvious that v_R^1 satisfies

$$\left\{ \begin{array}{rll} \Delta v_R^1 - F'(v_R^1) & = 0, & x \in B_R \\ |v_R^1| & < 1, & x \in B_R \\ \frac{\partial v_R^1}{\partial n} & = 0 & \text{on} \quad \partial B_R \end{array} \right. \tag{5.5}$$

and

$$v_R^1(x', x_n) > 0, \quad \text{for } x_n > 0 \quad \text{and } v_R^1(x', 0) = 0. \tag{5.6}$$

We also consider the following minimizing problem with vanishing Dirichlet boundary condition on balls

$$e_R^D := E_R(u_R) = \min \left\{ E_R(v);\, v \in H_0^1(B_R),\, |v| \le 1 \right\}. \tag{5.7}$$

Here are some basic facts about this minimizing problem.

Lemma 5.3. a) *There exists a minimizer* u_R *to* (5.7) *for* $R > 0$, *which does not change sign and therefore can be chosen as nonnegative. Naturally,* u_R *also satisfies*

$$\left\{ \begin{array}{rll} \Delta u_R - F'(u_R) & = 0, & x \in B_R \\ 0 \le u_R & < 1, & x \in B_R \\ u_R & = 0 & \text{on} \quad \partial B_R. \end{array} \right. \tag{5.8}$$

b) *There is a positive constant* $c > 0$ *such that*

$$e_R^D \le cR^{n-1}, \quad R > 0.$$

c) *There is a constant* $R_0 > 0$ *such that* $u_R(x) > 0$, *for all* $x \in B_R$ *when* $R > R_0$, *and* $u_R \equiv 0$ *when* $0 < R \le R_0$.

d) *Furthermore,* $u_R(x) = u_R(|x|)$ *and* $u_R(r)$ *is strictly decreasing in* $r > 0$ *and increasing in* R *when* $R > R_0$.

Proof. Part a) follows from standard variational arguments, while part b) only requires choice of a proper test function that vanishes on the ball B_{R-1}.

To prove part c), one first notes that the trivial solution $u \equiv 0$ has energy $E_R(0) = \omega_n F(0) R^n$. Therefore, because of b), it could not be the minimizer for $R > R_1$ when R_1 is sufficiently large. The strong maximum principle then implies that $u_R(x) > 0$, for all $x \in B_R$. It now suffices to choose R_0 as the smallest radius such that u_R is nontrivial.

Finally, the radial symmetry and monotonicity in r of u_R claimed in part d) is nothing but the classical result of Gidas-Ni-Nirenberg. Indeed, the monotonicity in R may be shown as follows. Suppose that for some $R_2 > R_1 > R_0$, there is $r_0 \in (0, R_1)$ such that $u_{R_2}(r_0) = u_{R_1}(r_0)$. Define a function

$$\bar{u}(r) = \begin{cases} u_{R_2}(r), & r \le r_0, \\ u_{R_1}(r), & r_0 < r < R_1. \end{cases}$$

Since u_{R_1} is the minimizer of (5.7) in B_{R_1}, we get that $E_{r_0}(u_{R_1}) < E_{r_0}(u_{R_2})$ by comparing $E_{R_1}(u_{R_1})$ with $E_{R_1}(\bar{u})$. Note that the strict inequality follows from the regularity of minimizers and uniqueness of initial value problems for ODEs. This is similar when we define another function

$$\underline{u}(r) = \begin{cases} u_{R_1}(r), & r \le r_0, \\ u_{R_2}(r), & r_0 < r < R_2. \end{cases}$$

Since u_{R_2} is the minimizer of (5.7) in B_{R_2}, we have that $E_{r_0}(u_{R_2}) < E_{r_0}(u_{R_1})$ by comparing $E_{R_2}(u_{R_2})$ with $E_{R_1}(\underline{u})$. This contradiction implies $u_{R_1}(r) < u_{R_2}(r)$ for $r < R_1$ and therefore the strict monotonicity of $u_R(r)$ in R.

Remark 5.2. For some nonlinearities F which include the original $F(u) = (1 - |u|^2)^2/4$ in the De Giorgi conjecture, it can be proved that the minimizer u_R is actually unique. This will be discussed more generally in a forthcoming paper [16]. We also note that for general positive radial solutions of (5.8) other than minimizers, the monotonicty in R may not hold.

We also have the following estimate for $u_R(0)$.

LEMMA 5.4. *There exists a positive constant c and μ such that*

$$u_R(0) \ge 1 - ce^{-\mu R}. \tag{5.9}$$

Proof. By the monotonicity of $u_R(r)$ in r and the energy estimate (5.9),

$$\min\{u_R(r), r < R/2\} \to 1 \quad \text{as } R \to \infty.$$

Therefore, there exist $R_1 > 0, \mu > 0$ such that $F''(u_R(r)) > 150\mu^2$ for all $r < R/2$ when $R > R_1$. Then $w(r) := 1 - u_R(r)$ satisfies $w(r) < 1$ and

$$-\Delta w(|x|) + 150\mu^2 w(|x|) \le 0, \quad \text{for all } |x| < R/2.$$

Let $\eta(x) := e^{12\mu(|x| - R/2)} + e^{12\mu(R/4 - |x|)}$. By a direct computation, we see that

$$-\Delta \eta(x) + 150\mu^2 \eta(x) \le 0, \quad \text{for all } |x| \in [R/4, R/2],$$

when $R > R_2$ is sufficiently large.

Since $\eta(x) > 1$ when $|x| = R/4$ and $|x| = R/2$, the maximum principle then implies that $w(|x|) \le \eta(x)$, for all $|x| \in [R/4, R/2]$. In particular, $w(R/3) < ce^{-\mu R}$. Therefore $u_R(0) > u_R(R/3) > 1 - ce^{-\mu R}$ and the lemma is proved.

It is clear that the solution $v_R^1(x)$ also minimizes the functional

$$E_R^+(u) = \int_{B_R^+} \left(\frac{1}{2}|\nabla u|^2 + F(u)\right) dx \tag{5.10}$$

on the set $H_R^+ := \{u \in H^1(B_R^+) : u(x',0) = 0, \quad x \in B_R^+\}$, where B_R^+ is the upper half ball with radius R.

Setting $u_{R,y}(x) = u_R(x-y)$, for all $x \in B_R(y)$, we have the following lemma.

LEMMA 5.5. *If $y \in B_R^+$, and $r \leq \min\{R/2, y_n\}$, then $u_{r,y}(x) < v_R^1(x)$ for all x in $B_r(y) \cap B_R^+$.*

Proof. First consider $y^0 = (0', R/2) \in B_R^+$. Since $B_r(y^0) \subset B_R^+$ for $r \leq R/2$, one uses the same argument as in the proof of the monotonicity of u_R in R, to conclude that $u_{r,y^0}(x) < v_R^1(x)$ for $x \in B_r(y^0)$. Indeed if not, then there exists a nonempty domain $\Omega \subset B_r(y^0)$ such that $u_{r,y^0}(x) \geq v_R^1(x)$ for $x \in \Omega$. Now define a new function

$$\tilde{u}(x) = \begin{cases} u_{r,y^0}(x), & x \in \Omega, \\ v_R^1(x), & x \in B_R^+ \setminus \Omega. \end{cases}$$

Since v_R^1 is a minimizer of (5.10) in H_R^+, we know that $E_\Omega(u_{r,y^0}) > E_\Omega(v_R^1)$ by comparing $E_R^+(\tilde{u})$ with $E_R^+(v_R^1)$. The strict inequality follows from the Hopf lemma or the strong maximum principle. Similarly, we can conclude that $E_\Omega(u_{r,y^0}) < E_\Omega(v_R^1)$ since u_{r,y^0} is a minimizer of (5.7). This contradiction proves the lemma for $y = y^0$.

To finish the proof of the lemma, one can use ideas similar to those in the moving plane method: Move y from y^0 continuously in B_R^+ while keeping the inequality $u_{r,y}(x) < v_R^1(x)$, for all $x \in B_r(y) \cap B_R^+$ in the process. Now, as long as $r \leq y_n$, the process can only stop at the following two possibilities: Either y reaches the boundary of B_R^+ or $u_{r,y}(x^0) = v_R^1(x^0)$ for some $x^0 \in B_r(y) \cap \overline{B_R^+}$. We note that $u_{r,y}(x) \leq v_R^1(x)$ for $x \in B_r(y) \cap \overline{B_R^+}$ when the process stops at y. We now claim that only the first case can happen. Indeed, if the second case occurs with y in the interior of B_R^+, the strong maximum principle applied to $w(x) = v_R^1(x) - u_{r,y}(x)$, which satisfies a nice linear elliptic equation, would rule out the possibility of x^0 being in the interior of B_R^+. So x_0 must be on the boundary of B_R^+. But then, if ν is the outer normal of ∂B_R^1 at x^0, then the Hopf lemma implies that

$$\frac{\partial w}{\partial \nu}(x_0) = \frac{\partial v_R^+}{\partial \nu}(x_0) - \frac{\partial u_{r,y}}{\partial \nu}(x_0) < 0.$$

However, $\frac{\partial v_R^+}{\partial \nu}(x_0) = 0$ by (5.5) and $\frac{\partial u_{r,y}}{\partial \nu}(x_0) < 0$ by the strict monotonicity of $u_R(s)$ in s. Note also that the vector $x^0 - y$ forms an acute angle with ν when $y \in B_R^+$. This contradiction proves the lemma.

In view of Lemma 5.4, we can now state the following:

Corollary 5.1. *There exist constants $c, \mu > 0$ such that*

$$v_R^1(x) \geq u_{x_n/2}(0) \geq 1 - ce^{-\mu x_n/2}, \quad x \in B_R^+. \tag{5.11}$$

Now we can establish the following estimate on e_R^1.

Lemma 5.6. *Assume that F satisfies* (1.5) *and* (1.6). *Then, there exist constants $c_1, c_2 > 0$ such that for all $R > 0$,*

$$\gamma_F \omega_{n-1} R^{n-1} - c_1 R^{n-3} \leq e_R^1 = E_R(v_R^1) \leq \gamma_F \omega_{n-1} R^{n-1} - c_2 R^{n-3}. \tag{5.12}$$

Proof. We estimate e_R^1 directly as follows.

$$\begin{aligned} e_R^1 = E_R(v_R^1) &= \int_{B_R^+} |\nabla v_R^1|^2 + 2F(v_R^1)\, dx \\ &\geq \int_{B_R^+} 2|\nabla v_R^1|\sqrt{2F(v_R^1)}\, dx \\ &\geq 2\int_{B_R'} \int_0^{\sqrt{R^2-|x'|^2}} \frac{\partial v_R^1}{\partial x_n}\sqrt{2F(v_R^1)}\, dx_n dx' \\ &\geq 2\int_{B_R'} \int_0^{v_R^1(x',\sqrt{R^2-|x'|^2})} \sqrt{2F(s)}\, ds dx' \\ &\geq \omega_{n-1}\gamma_F R^{n-1} - 2\int_{B_R'} \int_{v_R^1(x',\sqrt{R^2-|x'|^2})}^1 \sqrt{2F(s)}\, ds dx' \\ &\geq \omega_{n-1}\gamma_F R^{n-1} - c\int_{B_R'} (1 - v_R^1(x', \sqrt{R^2-|x'|^2}))dx' \\ &\geq \omega_{n-1}\gamma_F R^{n-1} - c\int_0^R r^{n-2} e^{-\mu\sqrt{R^2-r^2}/2} dr \\ &\geq \omega_{n-1}\gamma_F R^{n-1} - c\int_0^R (R^2-z^2)^{\frac{n-3}{2}} e^{-\mu z/2} z dz \\ &\geq \omega_{n-1}\gamma_F R^{n-1} - c_1 R^{n-3}. \end{aligned} \tag{5.13}$$

It is also easy to establish the upper bound,

$$e_R^1 \leq \gamma_F \omega_{n-1} R^{n-1} - c_2 R^{n-3} \quad \text{for some} \quad c_2 > 0. \tag{5.14}$$

This can be proved by calculating directly

$$
\begin{aligned}
e_R^1 &\le E_R\Big(g(x_n)\Big) \\
&= 2\int_0^R \omega_{n-1}(R^2-x_n^2)^{\frac{n-1}{2}}\left[\frac{1}{2}|g'(x_n)|^2+F\Big(g(x_n)\Big)\right]\,dx_n \\
&= 2\omega_{n-1}\cdot R^{n-1}\int_0^R \frac{1}{2}|g'(x_n)|^2+F\Big(g(x_n)\Big)\,dx_n \\
&\quad -2\omega_{n-1}\cdot R^{n-1}\int_0^R\left[1-\left(1-\frac{x_n^2}{R^2}\right)^{\frac{n-1}{2}}\right]\cdot\left[\frac{1}{2}|g'(x_n)|^2+F\Big(g(x_n)\Big)\right]\,dx_n \\
&= \gamma_F\omega_{n-1}R^{n-1}-c_1R^{n-3}.
\end{aligned}
$$

We now have the following lower estimate for the energy of u on balls which, in view of Theorem 1.1, immediately yields Theorem 1.2.

PROPOSITION 5.1. *Assume u is a solution to* (1.2)–(1.4). *In addition, assume that, after a proper translation, u satisfies*:

(5.15) $$u(y,z)=-u(y,-z)\text{ for }x=(y,z)\in\mathbf{R}^{n-k}\times\mathbf{R}^k,$$

where k is an integer with $1\le k\le n$. Then

(5.16) $$\gamma_F\omega_{n-1}-c_1R^{-2}\le\rho(R)\le\gamma_F\omega_{n-1}\quad\textit{for some}\quad c_1>0.$$

6. Comments and remarks

We start by using a slightly simpler energy method to tackle a particular but important case of anti-symmetry where it is assumed that $u(x,x_n)=-u(x,-x_n)$ for $x=(x',x_n)\in\mathbf{R}^n$. This may be regarded as a result for half-space problems already studied in [5]. Our method here for dimensions $n=4,5$ gives a completely different approach from those in [5]. We hope to use this special case to illustrate the strength as well as the limitation of this approach. As we shall see below, the passage from lower estimates on balls to upper estimates on cylinders is simpler in this case.

PROPOSITION 6.1. *Assume that F satisfies* (1.5) *and* (1.6) *and that a solution u to* (1.2)–(1.4) *satisfies $u(x',x_n)=-u(x',-x_n)$ for all $x=(x',x_n)\in\mathbf{R}^n$. If $n\le 5$, then $u(x',x_n)=g(x_n)$ for all $x=(x',x_n)\in\mathbf{R}^n$.*

We start with the following:

LEMMA 6.1. *Let $D_R=B_R\backslash C_{R/2}$ and consider the minimizing problem*

(6.1)
$$\bar e_R:=E_{D_R}(\bar v_R)=\min\Big\{E_{D_R}(v);v\in H^1(D_R),v(x',x_n)=-v(x',-x_n)\Big\}.$$

Then

$$\bar{e}_R \geq \gamma_F \omega_{n-1} R^{n-1} - \gamma_F \omega_{n-1} \left(\frac{R}{2}\right)^{n-1} - cR^{n-3} \tag{6.2}$$

for some $c > 0$.

Proof. We first note that $\bar{v}_R(x) > 0$ for $x \in D_R^+ = \{x;\, x_n > 0, x \in D_R\}$ after possible reflection, since we may replace $\bar{v}_R$ by $|\bar{v}_R|$ in D_R^+ and $|\bar{v}_R|$ is still a minimizer in D_R^+. This implies $\bar{v}_R = |\bar{v}_R|$ in D_R^+, hence the positivity of $\bar{v}_R$ in D_R^+. Similarly to Lemma 5.5 and Corollary 5.1, we have the following.

LEMMA 6.2. *If* $y \in D_R^+$, *and* $r \leq \min\{R/8, y_n\}$, *then* $u_{r,y}(x) < \bar{v}_R(x)$ *for all* x *in* $B_r(y) \cap D_R^+$.

COROLLARY 6.1. *There exist constants* $c, \mu > 0$ *such that*

$$\bar{v}_R(x) \geq u_{x_n/8}(0) \geq 1 - ce^{-\mu x_n/8}, \quad x \in D_R^+. \tag{6.3}$$

A similar calculation to the proof of Lemma 5.6 leads to (6.2).

LEMMA 6.3. *Under the assumption* $u(x', x_n) = -u(x', -x_n)$ *for all* $x = (x', x_n) \in \mathbf{R}^n$, *estimate* (1.10) *holds with* $q = 2$.

Proof. By Lemma 6.1,

$$\begin{aligned} 2^{n-1}\rho(R) - h(R/2) &= \tfrac{2^{n-1}}{R^{n-1}} \textstyle\int_{D_R} (\frac{1}{2}|\nabla u|^2 + F(u))\, dx \\ &\geq 2^{n-1}\gamma_F \omega_{n-1} - \gamma_F \omega_{n-1} - CR^{-2}. \end{aligned}$$

Therefore

$$\begin{aligned} h(R/2) &\leq 2^{n-1}\rho(R) - 2^{n-1}\gamma_F\omega_{n-1} + \gamma_F\omega_{n-1} + CR^{-2} \\ &\leq \gamma_F\omega_{n-1} + CR^{-2}, \end{aligned}$$

where we have used the inequality $\rho(R) \leq \gamma_F \omega_{n-1}$ in (1.9).

Proposition 6.1 now follows from Theorem 1.3.

Remark 6.1. The approach in this paper, allows us to substantially weaken the required hypothesis on the nonlinearity F. For example, the condition $F''(-1) = F''(1) > 0$ may be replaced by

$$F'(u) \sim -\lambda |u^2 - 1|^k, \quad \text{near} \quad u = \pm 1 \tag{6.4}$$

where $1 \leq k < 5$, $\lambda > 0$.

Indeed, by proceeding formally, one has the following asymptotics if $k > 1$,

$$1 - g^2(t) \quad \sim \quad |t|^{-\frac{2}{k-1}} \quad \text{as} \quad |t| \to \infty, \tag{6.5}$$

$$g'(t) = \sqrt{2F\Big(g(t)\Big)} \quad \sim \quad |t|^{-\frac{k+1}{k-1}} \quad \text{as} \quad |t| \to \infty, \tag{6.6}$$

$$F'\Big(g(t)\Big) \quad \sim \quad |t|^{-\frac{2k}{k-1}}, \qquad |t| \to \infty. \tag{6.7}$$

In the calculation in Lemma 5.6 which holds the key estimate, we only need that

$$\int_{-\infty}^{+\infty} |t|\Big(1 - g^2(t)\Big)\sqrt{\Big(2F(g(t)\Big)}\, dt < \infty.$$

This requires $\frac{4}{k-1} > 1$ and hence $k < 5$. We omit the details.

Remark 6.2. One may of course replace the Laplacian in (1.4) by a more general quasilinear operator with variational structure and obtain similar results.

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