Field Theories with "Superconductor" Solutions

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ABSTRACT

The conditions for the existence of non-perturbative type "superconductor" solutions of field theories are examined. A non-covariant canonical transformation method is used to find such solutions for a theory of a fermion interacting with a pseudoscalar boson. A covariant renormalisable method using Feynman integrals is then given. A "superconductor" solution is found whenever in the normal perturbative-type solution the boson mass squared is negative and the coupling constants satisfy certain inequalities. The symmetry properties of such solutions are examined with the aid of a simple model of self-interacting boson fields. The solutions have lower symmetry than the Lagrangian, and contain mass zero bosons.

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Introduction.

This paper reports some work on the possible existence of field theories with solutions analogous to the Bardeen model of a superconductor. This possibility has been discussed by Nambu [1] in a report which presents the general ideas of the theory which will not be repeated here. The present work merely considers models and has no direct physical applications but the nature of these theories seems worthwhile exploring.

The models considered here all have a boson field in them from the beginning. It would be more desirable to construct bosons out of fermions and this type of theory does contain that possibility [1]. The theories of this paper have the dubious advantage of being renormalisable, which at least allows one to find simple conditions in finite terms for the existence of "super-conducting" solutions. It also appears that in fact many features of these solutions can be found in very simple models with only boson fields, in which the analogy to the Bardeen theory has almost disappeared. In all these theories the relation between the boson field and the actual particles is more indirect than in the usual perturbation type solutions of field theory.

Non-covariant theory.

The first model has a single fermion interacting with a single pseudo-scalar boson field with the Lagrangian

$$L = \bar{\psi}\left(i\gamma^{\mu}\frac{\partial\psi}{\partial x^{\mu}} - m\psi\right) + \frac{1}{2}\left(\frac{\partial\phi}{\partial x^{\mu}}\frac{\partial\phi}{\partial x_{\mu}} - \mu_0^2\phi^2\right)$$
$$- g_0\bar{\psi}\gamma^5\psi\phi - \frac{1}{24}\lambda_0\phi^4$$

(The last term is necessary to obtain finite results, as in perturbation theory). The new solutions can be found by a non-covariant calculation which perhaps may show more clearly what is happening than the covariant theory which follows.

Let $\phi(x) = \frac{1}{\sqrt{V}} \sum q_k \, e^{ik.x}$ and let p_k be the conjugate momentum to q_k. Let $a^+_{k\sigma}$, $b^+_{k\sigma}$ be the creation operators for Fermi particles of momentum k spin σ and antiparticles of momentum $-k$, spin $-\sigma$ respectively. Retain only the mode $k=0$ of the boson field in the Hamiltonian. (The significance of this approximation appears below). Then

$$H = H_F + H_B + H_I \tag{1}$$

$$H_F = \sum_i E_i (a^+_i a_i + b^+_i b_i)$$

[k, σ is replaced by a single symbol i]

$$H_B = \tfrac{1}{2}(p_o^2 + \mu_o^2 q_o^2)$$

$$H_I = \frac{g_o}{\sqrt{V}} q_o \sum_i (a^+_i b^+_i + b_i a_i) + \frac{\lambda_o}{24V} q_o^4$$

When H_I is treated as a perturbation, its only finite effects are to alter the boson mass and to scatter fermion pairs of zero total momentum. These effects can be calculated exactly (when $V \to \infty$) by writing $a^+_i b^+_i = c^+_i$ and treating c^+_i as a boson creation operator [2]. The Hamiltonian becomes

$$H' = \sum 2E_i c^+_i c_i + \tfrac{1}{2}(p_o^2 + \mu_o^2 q_o^2) + \frac{g_o}{\sqrt{V}} q_o \sum (c^+_i + c_i)$$

The $\frac{1}{V} q_o^4$ term has no finite effects. Let

$$\frac{c^+_i + c_i}{\sqrt{4E_i}} = q_i \; ; \quad i\sqrt{E}\,(c^+_i - c_i) = p_i$$

then

$$H' = \tfrac{1}{2}(p_o^2 + \mu_o^2 q_o^2) + \sum_i \tfrac{1}{2}(p_i^2 + 4E_i^2 q_i^2) + \frac{g_o}{\sqrt{V}} q_o \sum_i \sqrt{4E_i}\, q_i - \sum_i E_i$$

H' represents a set of coupled oscillators and is easily diagonalised. The frequencies of the normal modes are ω_o, ω_i, given by the roots of the equation

$$\mu_o^2 - \omega^2 = \frac{g_o^2}{V} \sum_i \frac{4E_i}{4E_i^2 - \omega^2}$$

In the limit as $V \to \infty$, this becomes

$$\mu_o^2 - \omega^2 = \frac{2g_o^2}{(2\pi)^3} \int \frac{4E_k}{4E_k^2 - \omega^2}\, d^3k$$

$$= \frac{2g_o^2}{(2\pi)^3} \int d^3k \left\{ \frac{1}{E_k} + \frac{\omega^2}{4E_k^3} + \frac{\omega^4}{4E_k^3(4E_k^2 - \omega^2)} \right\} \qquad (2)$$

The first two terms in the integral diverge. This procedure corresponds exactly to the covariant procedure of calculating the poles of the boson propagator

$$D(k,\omega) = \frac{1}{\omega^2 - k^2 - \mu_o^2 - \Pi(k,\omega)}$$

and including in Π only the lowest polarisation part shown in Fig. 1.

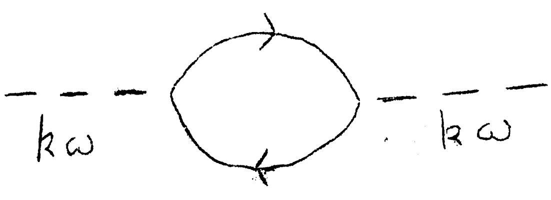

Fig. 1

This comparison shows that the two divergent terms can be absorbed into the renormalised mass and coupling constant. The renormalisation is carried out at the point $k_\mu = 0$ instead of as usual at the one-boson pole of D. Thus if

$$\Pi(0,\omega) = A + B\omega^2 + g_0^2\,\Pi_1(\omega^2) \quad ,$$

then

$$D(0,\omega) = \frac{Z}{\omega^2 - \mu_1^2 - g_1^2\,\Pi_1(\omega^2)}$$

$$Z = \frac{1}{1-B}\ , \quad \frac{\mu_1^2}{Z} = \mu_0^2 - A\ , \quad g_1^2 = \frac{g_0^2}{Z}$$

Equation (2) becomes

$$\mu_1^2 - \omega^2 = \frac{2g_1^2}{(2\pi)^3}\,\omega^4 \int \frac{d^3k}{4E_k^3\,(4E_k^2 - \omega^2)} = F(\omega^2)$$

The isolated root of this equation, ω_0^2, is found as shown in Fig. 2.

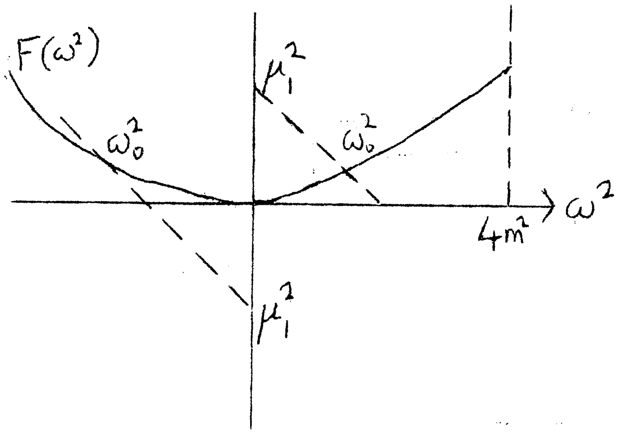

Fig. 2

When $\mu_1^2 > 0$, ω_0^2 is the square of the physical boson mass. When $\mu_1^2 \sim 4m^2$, this root disappears. This is the case when the boson can decay into a fermion pair. There is always another root for ω^2 large and negative. This corresponds to the well-known "ghost" difficulties and will be ignored here. When $\mu_1^2 < 0$ (but not too large), there is a negative root for ω^2. This is usually taken to mean simply that the theory with $\mu_1^2 < 0$ does not exist. Here it is taken to indicate that the approximation used is wrong and that the Hamiltonian (1) must be investigated further.

This is done by a series of canonical transformations based on the idea that in the vacuum state the expectation value of the boson field is not zero. Let

$$q_0 = q_0' + \frac{\Delta}{g_0}\sqrt{V}, \qquad p_0 = p_0'$$

Δ is a parameter to be fixed later. Also make a Bogoliubov transformation on the fermion field

$$a_i = \cos\frac{\theta_i}{2}\,\alpha_i - \sin\frac{\theta_i}{2}\,\beta_i^+$$

$$b_i = \sin\frac{\theta_i}{2}\,\alpha_i^+ + \cos\frac{\theta_i}{2}\,\beta_i$$

$$\tan\theta_i = \frac{\Delta}{E_i}$$

Then

$$H = H_0 + H_1 + H_2 + H_3$$

$$H_0 = -\sum_l \left(\sqrt{E_l^2+\Delta^2} - E_l\right) + \frac{\mu_0^2}{2}\frac{\Delta^2}{g_0^2}V + \frac{\lambda_0}{24}\frac{\Delta^4}{g_0^4}V$$

$$H_1 = q_0'\sqrt{V}\left\{\mu_0^2\frac{\Delta}{g_0} + \frac{\lambda_0}{6}\frac{\Delta^3}{g_0^3} - \frac{g_0}{V}\sum_l \frac{\Delta}{\sqrt{E_l^2+\Delta^2}}\right.$$

$$H_2 = \frac{1}{2}p_0'^2 + \frac{1}{2}q_0'^2\left(\mu_0^2 + \frac{\lambda_0}{2}\frac{\Delta^2}{g_0^2}\right) + \sum_l \sqrt{E_l^2+\Delta^2}\left(\alpha_l^\dagger\alpha_l + \beta_l^\dagger\beta_l\right)$$
$$+ \frac{g_0}{\sqrt{V}}q_0'\sum_l \frac{E_l}{\sqrt{E_l^2+\Delta^2}}\left(\alpha_l^\dagger\beta_l^\dagger + \beta_l\alpha_l\right)$$

$$H_3 = \frac{\lambda_0}{24V}q_0'^4 + \frac{\lambda_0}{6}\frac{\Delta}{g_0}\frac{1}{\sqrt{V}}q_0'^3 + \frac{g_0}{\sqrt{V}}q_0'\sum_l \frac{\Delta}{\sqrt{E_l^2+\Delta^2}}\left(\alpha_l^\dagger\alpha_l + \beta_l^\dagger\beta_l\right)$$

Now choose Δ so that $H_1=0$. H_0 is just a constant proportional to V and H_3 has no finite effects. This leaves H_2, which can be diagonalised by the method used above for the original Hamiltonian (1).

One solution of $H_1=0$ is $\Delta=0$. This gives the original approximation, which is no use when $\mu_1^2<0$. Other solutions are given by

$$\mu_0^2+\frac{1}{6}\lambda_0\frac{\Delta^2}{g_0^2} = \frac{g_0^2}{V}\sum_l \frac{1}{\sqrt{E_l^2+\Delta^2}}$$
$$= \frac{2g_0^2}{(2\pi)^3}\int d^3k\left\{\frac{1}{E_k}-\frac{\Delta^2}{2E_k^3}\right\}+g_0^2\,G(\Delta^2) \tag{3}$$

where G is a finite function. Let

$$\mu_0^2-\frac{2g_0^2}{(2\pi)^3}\int\frac{d^3k}{E_k} = \frac{\mu_1^2}{Z}$$
$$\lambda_0+\frac{6g_0^4}{(2\pi)^3}\int\frac{d^3k}{E_k^3} = \frac{\lambda_1}{Z^2}\;, \qquad g_0^2=\frac{g_1^2}{Z}$$

μ_1^2, λ_1, Z can be identified as the lowest order perturbation theoretic values of the renormalised boson mass, four-boson interaction constant and wave function renormalisation, arising from the graphs in Fig. 1 and 3

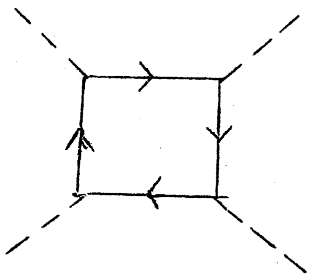

Fig. 3

As before, the renormalisations are carried out at $k_\mu=0$. Equation (3) becomes

$$\mu_1^2+\frac{1}{6}\lambda_1\frac{\Delta^2}{g_1^2} = g_1^2\,G(\Delta^2) \tag{4}$$

Thus an equation for Δ^2 is obtained which is finite in terms of constants which would be the renormalised parameters of the theory in the ordinary solution. It will be shown below that when $\mu_1^2 < 0$, equation (4) has solutions for a certain range of λ_1, and that then there does exist a real boson.

Covariant theory.

A first approach to a covariant theory can be made by calculating the fermion Green's function in a self-consistent field approximation. In perturbation theory the term represented by Fig. 4 vanishes by reflection invariance.

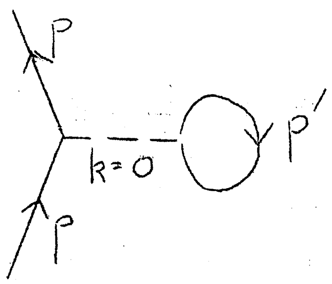

Fig. 4

However, suppose it gives a contribution $\gamma_5 \Delta$ to the fermion selfenergy. Then

$$S(p) = \frac{1}{\not p - m - \gamma_5 \Delta}$$

and evaluating Fig. 4 with this value of S gives

$$\Delta = -\frac{1}{\mu_0^2} \frac{g_0^2}{(2\pi)^4 i} \int d^4p \, \mathrm{tr}\, \gamma_5 \frac{1}{\not p - m - \gamma_5 \Delta} = -\frac{4g_0^2}{\mu_0^2} \frac{1}{(2\pi)^4 i} \int d^4p \frac{\Delta}{p^2 - m^2 - \Delta^2}$$

which is the same as equation (3) without the λ term, and again has the perturbation solution $\Delta = 0$ and possibly other solutions.

A general covariant theory can be found by using the Feynman integral technique. This gives an explicit formula for the fermion Green's function

$$S(x', x) = \frac{\int S(x', x; \phi)\, e^{-iW(\phi)}\, e^{i\int L_M(\phi)\, d^4x}\, \delta\phi}{\int e^{-iW(\phi)}\, e^{i\int L_M(\phi)\, d^4x}\, \delta\phi}$$

Here $S(x'x\phi)$ is the fermion Green's function calculated in an external boson field ϕ (and without interacting bosons). $e^{-iW\,\phi}$ is the vacuum-vacuum S-matrix amplitude in an external field and $L_M(\phi)$ is the boson part of the Lagrangian. The integrations are carried out over all fields $\phi(x,t)$.

Let

$$\phi(x\ t) = \frac{1}{\Omega} \sum_k \phi_k\, e^{ikx}$$

where k is now a 4-vector and Ω a large space-time volume. To obtain the Bardeen-type solutions, first do all the integrations except that over ϕ_0 (this has $\underline{k} = \omega = 0$) and put $\phi_0 = \Omega\chi$ (χ finite). Then

$$S(x', x) = \frac{\int S(x', x; \chi)\, e^{-i\Omega F(\chi)}\, d\chi}{\int e^{-i\Omega F(\chi)}\, d\chi}$$

$$F(\chi) = w(\chi) + \frac{\mu_0^2}{2}\chi^2 + \frac{\lambda_0}{24}\chi^4$$

S is the one particle Green's function calculated in a constant external field $\phi(x) = \chi$ and including all the interacting boson degrees of freedom except $k = \omega = 0$. $e^{-i\Omega w}$ is the vacuum-vacuum amplitude calculated in the same way. The idea is to look for stationary points of $F(\chi)$ other than $\chi = 0$. If $F'(\chi_1) = 0$, then in the limit $\Omega \to \infty$, $S(x'x) = S(x'x\chi_1)$.

$-i\Omega w(\chi)$ is given by the sum of all connected vacuum diagrams. It is easy to see that

$$w(\chi) = \sum_{n=0}^{\infty} V_{2n} \frac{\chi^{2n}}{2n!}$$

where V_{2n} is the perturbation theory value of the 2n-boson vertex with all external momenta zero. For example $V_2 = \Pi(0)$. In perturbation theory, $V_{2n} = V_{2n}^{(r)}/Z^n$ where $\sqrt{Z}$ is the boson wave function renormalisation, and $V_{2n}^{(r)}$ is finite provided the terms μ_o^2 and λ_o are absorbed into V_2 and V_4. As before, the renormalisations are carried out at k=0. Thus if

$$\chi = \sqrt{Z}\,\chi^{(r)} \quad , \text{ then } \quad F(\chi^{(r)}) = \sum \frac{V_{2n}^{(r)}}{2n!}\,\chi^{(r)2n}$$

an expression from which all the divergences have been removed. It also follows that if $F'(\chi_1^{(r)}) = 0$, the new values for the vertex parameters V_n are given by

$$V_n' = \frac{\partial^n}{\partial \chi_1^{(r)n}} F(\chi_1^{(r)})$$

In particular the new boson mass is given by

$$\mu^2 = F''(\chi_1^{(r)})$$

(This is really the mass operator at k=0, not the mass). Thus one condition for the existence of these abnormal solutions is that $F(\chi)$ should have a minimum at some non-zero value of χ.

$F'(\chi)$ is in fact easier to evaluate than F. It is given by the sum of all diagrams with one external boson line. The previous approximations are recovered by putting $g_o\chi = \Delta$ and including only the diagrams in Fig. 5

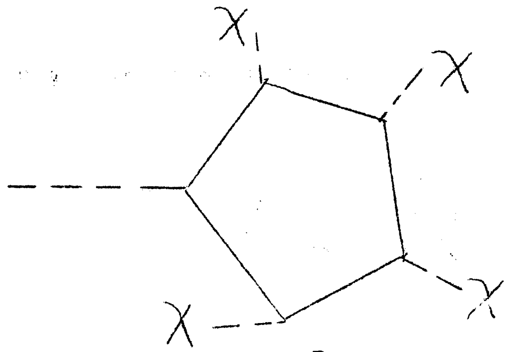

Fig. 5

This gives

$$F'(\chi) = \mu_0^2 \chi + \frac{\lambda_0}{6}\chi^3 + \frac{4g_0^2}{(2\pi)^4 i}\int \frac{\chi}{p^2 - m^2 - g_0^2\chi^2}\, d^4p$$

Hence

$$F'(\chi^{(r)}) = \mu_1^2 \chi^{(r)} + \frac{\lambda_1}{6}\chi^{(r)3} + \frac{4g_1^2}{(2\pi)^4 i}\int \frac{\chi^{(r)} g_1^4 \chi^{(r)4}}{(p^2-m^2)^2(p^2-m^2-\Delta^2)}\, d^4p$$

Thus Δ is given by

$$0 = \frac{\mu_1^2}{g_1^2} + \frac{\lambda_1}{6g_1^4}\Delta^2 - \frac{m^2}{4\pi^2}\left\{\left(1+\frac{\Delta^2}{m^2}\right)\log\left(1+\frac{\Delta^2}{m^2}\right) - \frac{\Delta^2}{m^2}\right\}$$

Let

$$\frac{4\pi^2\mu_1^2}{m^2 g_1^2} = A\ , \quad \frac{2\pi^2}{3}\frac{\lambda_1}{g_1^4} = B\ , \quad \frac{\Delta^2}{m^2} = x$$

Then

$$A + Bx = (1+x)\log(1+x) - x = h(x)$$

The new boson mass is given by

$$\mu^2 = F''(\chi) = \frac{m^2 g_1^2}{2\pi^2}\, x\left\{B - h'(x)\right\}$$

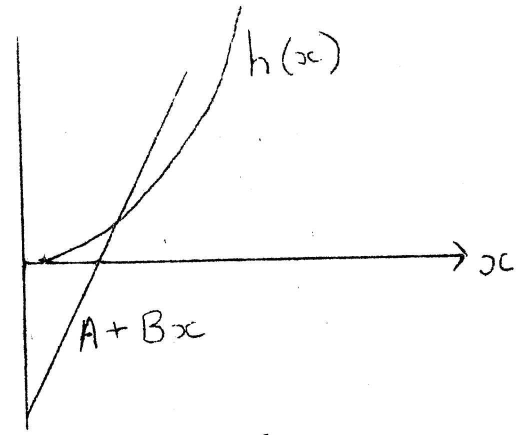

Fig. 6

It can be seen from Fig. 6 that there will be roots with $\mu^2 > 0$ only when $\mu_1^2 < 0$ and $B > B_{crit}$ where B_{crit} is given by $h'(x) = B$. Thus the abnormal solutions exist when

$$B > 0 \qquad 0 > A > -(e^B - 1 - B)$$

This calculation is exact in the limit $g_1 \to 0$ keeping $\frac{\mu_1^2}{g_1^2}$, $\frac{\lambda_1}{g_1^4}$ and $g_1 \chi^{(r)}$ finite. Thus in at least one case a solution of the required type can be established as plausibly as the more usual perturbation theory solutions.

Symmetry properties and a simple model.

It is now necessary to discuss the principal peculiar feature of this type of solution. The original Lagrangian had a reflection symmetry. From this it follows that $F(\chi)$ must be an even function. Thus if $\chi = \chi_1$ is one solution of $F'(\chi_1) = 0$, $\chi = -\chi_1$ is another. By choosing one solution, the reflection symmetry is effectively destroyed. It is possible to make a very simple model which shows this kind of behaviour, and also demonstrates that so long as there is a boson field in the theory to start with, the essential features of the abnormal solutions have very little to do with fermion pairs.

Consider the theory of a single neutral pseudoscalar boson interacting with itself,

$$L = \frac{1}{2}\left(\frac{\partial \phi}{\partial x_\mu}\frac{\partial \phi}{\partial x^\mu} - \mu_0^2 \phi^2\right) - \frac{\lambda_0}{24}\phi^4$$

Normally this theory is quantised by letting each mode of oscillation of the classical field correspond to a quantum oscillator whose quantum number gives the number of particles. When $\mu_0^2 < 0$, this approach will not work. However, if $\lambda_0 > 0$, the function

$$\frac{\mu_0^2}{2}\phi^2 + \frac{\lambda_0}{24}\phi^4$$

is as shown in Fig. 7.

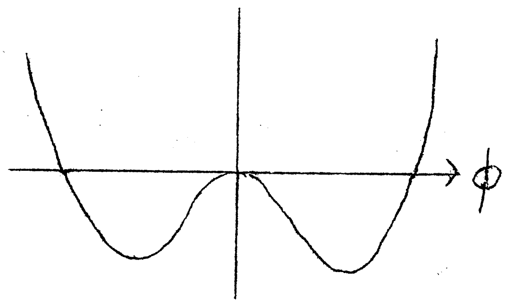

Fig. 7

The classical equations

$$(\Box^2 + \mu_0^2)\phi + \frac{\lambda_0}{6}\phi^3 = 0$$

now have solutions $\phi = \pm\sqrt{\frac{-6\mu_0^2}{\lambda_0}}$ corresponding to the minima of this curve. Infinitesimal oscillations round one of these minima obey the equation

$$(\Box^2 - 2\mu_0^2)\,\delta\phi = 0$$

These can now be quantised to represent particles of mass $\sqrt{-2\mu_0^2}$. This is simply done by making the transformation $\phi = \phi' + \chi$

$$\chi^2 = -\frac{6\mu_0^2}{\lambda_0}$$

Then

$$L = \frac{1}{2}\left(\frac{\partial\phi'}{\partial x_\mu}\frac{\partial\phi'}{\partial x^\mu} + 2\mu_0^2\phi'^2\right) - \frac{\lambda_0}{24}\phi'^4 - \frac{\lambda_0\chi}{6}\phi'^3 + \frac{3}{2}\frac{\mu_0^2}{\lambda_0}$$

This new Lagrangian can be treated by the canonical methods.

In any state with a finite number of particles, the expectation value of ϕ' is infinitesimally different from the vacuum expectation value. Thus the eigenstates corresponding to oscillations round $\phi = \chi$ are all orthogonal to the usual states corresponding to oscillations round $\phi = 0$, and also to the eigenstates round $\phi = -\chi$. This means that the theory has two vacuum states, with a complete set of particle states built on each vacuum, but that there is a superselection rule between these two sets so that it is only necessary to consider one of them. The symmetry $\phi \rightarrow -\phi$ has disappeared. Of course it can be restored by introducing linear combinations of states in the two sets but because of the superselection rule this is a highly artificial procedure.

Now consider the case when the symmetry group of the Lagrangian is continuous instead of discrete. A simple example is that of a complex boson field, $\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$

$$L = \frac{\partial \phi^*}{\partial x_\mu} \frac{\partial \phi}{\partial x^\mu} - \mu_0^2 \phi^* \phi - \frac{\lambda_0}{6} (\phi^* \phi)^2$$

The symmetry is $\phi \rightarrow e^{i\alpha} \phi$. The canonical transformation is $\phi = \phi' + \chi$

$$|\chi|^2 = -\frac{3\mu_0^2}{\lambda_0}$$

The phase of χ is not determined. Fixing it destroys the symmetry. With χ real the new Lagrangian is

$$L = \frac{1}{2}\left(\frac{\partial \phi_1'}{\partial x_\mu} \frac{\partial \phi_1'}{\partial x^\mu} + 2\mu_0^2 \phi_1'^2\right) + \frac{1}{2} \frac{\partial \phi_2'}{\partial x_\mu} \frac{\partial \phi_2'}{\partial x^\mu}$$

$$- \frac{\lambda_0}{6} \chi \phi_1' (\phi_1'^2 + \phi_2'^2) - \frac{\lambda_0}{24} (\phi_1'^2 + \phi_2'^2)^2$$

The particle corresponding to the ϕ_2' field has zero mass. This is true even when the interaction is included, and is the new way the original symmetry expresses itself.

A simple picture can be made for this theory by thinking of the two dimensional vector ϕ at each point of space. In the vacuum state the vectors have magnitude χ and are all lined up (apart from the quantum fluctuations). The massive particles ϕ_1' correspond to oscillations in the direction of χ. The massless particles ϕ_2' correspond to "spin-wave" excitations in which only the direction of ϕ makes infinitesimal oscillations. The mass must be zero, because when all the $\phi(x)$ rotate in phase there is no gain in energy because of the symmetry.

This time there are infinitely many vacuum states. A state can be specified by giving the phase of χ and then the numbers of particles in the two different oscillation modes. There is now a superselection rule on the phase of χ. States with a definite charge can only be constructed artificially by superposing states with different phases.

Conclusion.

This result is completely general. Whenever the original Lagrangian has a continuous symmetry group, the new solutions have a reduced symmetry and contain massless bosons. One consequence is that this kind of theory cannot be applied to a vector particle without losing Lorentz invariance. A method of losing symmetry is of course highly desirable in elementary particle theory but these theories will not do this without introducing non-existent massless bosons (unless discrete symmetry groups can be used). Skyrme [3)] has hoped that one set of fields could have excitations both of the usual type and of the "spin-wave" type, thus for example obtaining the π-mesons as collective oscillations of the four K-meson fields, but this does not seem possible in this type of theory. Thus if any use is to be made of these solutions, something more complicated than the simple models considered in this paper will be necessary.

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