On Some Features of Symmetric Diagonal Latin Squares*

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Abstract. In this paper, we study the dependencies of the number of symmetric and doubly symmetric diagonal Latin squares on the order N. Using fast generator of diagonal Latin squares (augmented by symmetry checker), we determined these dependencies for order at most 8. We also found a number of doubly symmetric diagonal Latin squares of orders 12, 16 and 20.

Keywords: Latin square, symmetric Latin square, enumeration

1 Introduction

Latin square (LS) of order N is a square table $A = \|a_{ij}\|, i, j = \overline{1, N}$, which consists of elements from some set $U, |U| = N$ [4]. Further we will use U equal to $0, 1, \ldots, N-1$. In LS elements in each row and each column are distinct. Diagonal Latin square (DLS) is a LS, in which each symbol from U occurs precisely once in its main diagonal and main antidiagonal.

DLS is normalized, if elements of its first row are sorted in ascending order. It is easy to show, that any DLS can be normalized using bijective substitution (permutation) of its elements. Thus, we can define classification on the set of all possible DLSs of a particular order: all DLSs, which can be reduced to the same normalized DLS, form one equivalence class. This classification can be used in application to some combinatorial problems (enumeration of LSs and DLSs [6, 8, 7, 2, 3], search for pairs and triples of (partially) mutually orthogonal LS and DLS [10, 9, 5]). Squares from the same class have similar features (existence/nonexistence of an orthogonal mate, the number and values of transversals, the values of main diagonal and main antidiggonal, etc.), and this in turn leads to decrease of required computational resources in the corresponding experiments.

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2 Symmetric Latin squares

By symmetric LS we mean such LS, for which one-to-one correspondence occurs between all pairs of elements $(a_{ij}, a_{i,N-j+1}), i = \overline{1, N}, j = \overline{1, \lfloor \frac{N}{2} \rfloor}$. This symmetry occurs in the horizontal plane – with respect to the vertical line (across the central column for DLSs of odd order and between two central columns for DLSs of even order). By analogy, one can denote the symmetry in the vertical plane. For normalized DLS the condition of occurrence of symmetry in the horizontal plane can be rewritten in more simple form: $a_{ij} + a_{i,N-j+1} = N - 1$. Symmetric LSs have symmetric set of transversals, this can be employed for constructing pairs and triples of (partially) mutually orthogonal diagonal Latin squares. Examples of two symmetric DLSs (in the horizontal and vertical plane respectively) are depicted in Figure 1.

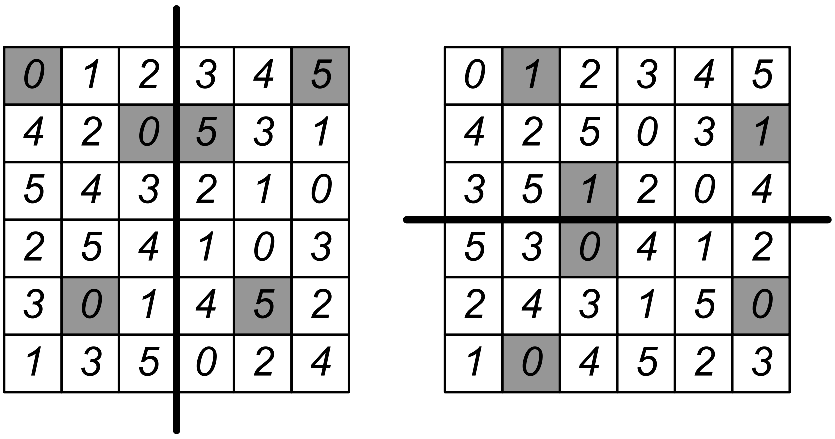

Fig. 1. Examples of symmetric diagonal Latin squares. Axis of symmetry are depicted by bold lines, several symmetric pairs of elements are marked with gray.

Dependence of the number of symmetric LSs with fixed diagonal on order N is represented by the sequence A003191 in Online Encyclopedia of Integer Sequences (OEIS) [1]. An interesting fact is that there is no symmetric LS of odd order (except the case where $N = 1$), that is why the sequence contains only the following cases: $N \in 2, 4, 6, 8, 10$.

3 Symmetric diagonal Latin squares

Dependence of the number of symmetric DLSs on the order N is not presented in OEIS at the present moment, that is why determining of this dependence is of great importance. The most natural way to solve this problem for a given order N is to generate all DLSs, and perform symmetry checking for each of them.

It is also possible to improve this approach by filling half of square's elements in combination with varying the order of filling elements based on principle of minimum feasibilities and the branch and bound method.

Generation of DLSs can be done by an effective generator, which was developed by authors of the present paper. This generator is based on some algorithmic features of the considered problem. The following techniques made it possible to achieve high efficiency of our generator (see [7]).

- Using the order of filling elements based on principle of minimal feasibilities.
- Using static data structures instead of dynamic ones.
- Taking into account the cardinality of the set of possible values for currently unfilled square's cells in combination with early clipping unpromising branches of a combinatorial tree in the case of finding square's cells without any possible elements.
- Applying auxiliary data structures (one-dimensional arrays) for fast construction of the set of possible elements.
- Employing bit arithmetic.

4 Computational experiments

In accordance with the first strategy (see Section 3), we developed a program implementation, which was used to determine the dependence of the number of normalized symmetric DLSs on the order N. The results of the computational experiments are shown in Table 1.

Table 1. The dependence of the number of (normalized) symmetric DLSs on the order N.

N	Normalized symmetric DLSs	Symmetric DLSs	Time
1	1	1	< 1 s
2	0	0	< 1 s
3	0	0	< 1 s
4	2	48	< 1 s
5	0	0	< 1 s
6	64	46 080	< 1 s
7	0	0	< 1 s
8	3 612 672	145 662 935 040	19 h

At the present moment, the number of symmetric normalized DLSs of order 10 can not be calculated in a reasonable time. Let us consider an approach, which consists in the generation of the corresponding DLSs with symmetry checking of each generated DLS. The speed of the corresponding sequential program (written in Delphi) is about 200 000 DLSs per second on i7 4770 CPU. If we assume, that there are about 10^{22} normalized DLSs of order 10, than it will take about $1.6 \cdot 10^9$ years for this program to process all of them. A supercomputer with the performance of 1 teraflops will take about 160 years to perform the same experiment.

Besides that, there are also DLSs, which are symmetric in horizontal and vertical plane at the same time (further we will call them *doubly symmetric*). An example of such DLS is depicted in Figure 2. DLSs of this type are quite rare combinatorial objects. The dependence of the number of doubly symmetric normalized DLSs on the order N is shown in Table 2. For each considered order it was obtained by generating all DLSs with checking doubly symmetry for each of them.

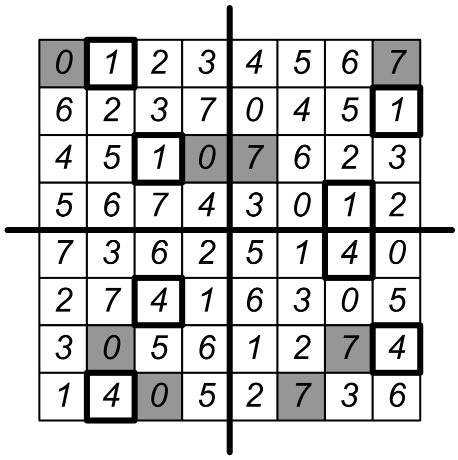

Fig. 2. An example of doubly symmetric normalized DLS.

Table 2. The dependence of the number of (normalized) doubly symmetric DLSs on the order N.

N	Normalized doubly symmetric DLSs	Doubly symmetric DLSs	Time
1	1	1	< 1 s
2	0	0	< 1 s
3	0	0	< 1 s
4	2	48	< 1 s
5	0	0	< 1 s
6	0	0	< 1 s
7	0	0	< 1 s
8	15 780	636 249 600	19 h 30 m

According to results of the experiment, we can conclude, that for $2 \leq N \leq 8$ there is are doubly symmetric DLSs for orders, which are not multiples of 4. For $9 \leq N \leq 20$ we also could find doubly symmetric DLSs only for $N = 4n, n \in \mathbb{N}$, i.e. for $N = 12, 16, 20$. However, in these additional experiments we did not use an exhaustive search – for each considered N its own 1-day random search based experiment was performed. We presume, that this feature holds true for all $N \geq 1$.

Symmetric and doubly symmetric DLSs have a number of peculiarities, which are outlined below.

- As a rule, they have a lot of transversals.
- They usually (but not always) have a lot of orthogonal diagonal mates.
- For $N = 4$ and $N = 8$ all known DLSs with maximal possible number of orthogonal mates, are doubly symmetric.

Let us show these doubly symmetric DLSs with maximal possible number of orthogonal mates.

$$\begin{pmatrix} 0&1&2&3 \\ 3&2&1&0 \\ 1&0&3&2 \\ 2&3&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&2&3&4&5&6&7 \\ 4&2&1&7&0&6&5&3 \\ 6&7&3&2&5&4&0&1 \\ 7&6&5&4&3&2&1&0 \\ 5&3&7&1&6&0&4&2 \\ 3&5&6&0&7&1&2&4 \\ 1&0&4&5&2&3&7&6 \\ 2&4&0&6&1&7&3&5 \end{pmatrix}.$$

By analogy, we tried to find additional symmetries in DLSs with respect to their diagonals, but it turned out, that there are no such symmetries for $N \leq 8$.

Thus, in the course of the experiments we found two integer sequences: $(0, 2, 64, 3\,612\,672)$ – the number of normalized symmetric DLSs of order $2n$ and $(0, 2, 0, 15\,780)$ – the number of normalized doubly symmetric DLSs of order $2n$. None of these sequences is presented in OEIS at the present moment. We have not found any other types of symmetries in DLSs.

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