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A New Tail-Based Correlation Measure and Its Application in Global Equity Markets

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Abstract

The co-dependence between assets tends to increase when the market declines. This paper develops a correlation measure focusing on market declines using the expected shortfall (ES), referred to as the ES-implied correlation, to improve the existing value at risk (VaR)-implied correlation. Simulations which define period-by-period true correlations show that the ES-implied correlation is much closer to true correlations than is the VaR-implied correlation with respect to average bias and root-mean-square error. More importantly, this paper develops a series of test statistics to measure and test correlation asymmetries, as well as to evaluate the impact of weights on the VaR-implied correlation and the ES-implied correlation. The test statistics indicate that the linear correlation significantly underestimates correlations between the US and the other G7 countries during market downturns, and the choice of weights does not have significant impact on the VaR-implied correlation or the ES-implied correlation.

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A new tail-based correlation measure and its application in global equity markets

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1 Introduction

It is a core principle of portfolio theory that diversification reduces risk. Risk diversification depends on assets being less correlated, so that a fall in one investment can be offset by a rise in another investment. The correlation between assets is traditionally estimated by the linear correlation. However, a number of empirical studies have found that asset correlations increase when the market falls, and the linear correlation underestimates asset dependence during market declines.[1] It is exactly during market downturns that wealth decreases and diversification is most valuable. Underestimating the dependence leads investors to overestimate the benefits of risk diversification and cause unexpected losses when the market declines.

An alternative correlation measure is the value at risk (VaR)-implied correlation, which allows for correlations to vary based on market conditions. VaR is defined as the minimum value such that the probability of not exceeding this value at least equals a given confidence level. The VaR-implied correlation is the difference between the VaRs of a portfolio and its individual components, since the portfolio VaR is determined by VaRs of its individual components and their correlation. The VaR-implied correlation equals the linear correlation when asset returns follow multivariate normal distribution, but captures the increased correlation between assets during market downturns when asset returns are not from normal distribution. However, the VaR-implied correlation has a number of disadvantages. First, VaR is just a quantile and does not consider losses beyond it. Although the probability of events occurring in the tails is very small, these events cause large losses once they happen. Disregarding losses beyond VaR may cause tail risk, the risk that arises when the possibility of extreme losses is greater than expected. Yamai and Yoshiba (2005) illustrate several cases where VaR underestimates losses in the tails. Second, VaR is not a coherent risk measure. Coherence requires the risk of a combination of individual assets not exceeding the sum of the individual risks, i.e., risk can be reduced with diversification.

In this paper, I develop a novel tail-based correlation measure based on expected shortfall (ES) to address the shortcomings of the VaR-implied correlation. Expected shortfall is the average of asset losses when the asset value falls below the VaR threshold. Whereas VaR provides the threshold of losses that will not be exceeded, expected shortfall provides the expected value of losses when that threshold is breached. Thus the implied correlation based on ES explicitly accounts for tail losses. In addition, ES is a coherent measure (Artzner, Delbaen, Eber, and Heath (1999), Acerbi and Tasche (2002) and Tasche (2002)). [2]

[1] Linear correlation is only natural in the context of elliptical models since only elliptical models can be fully characterized by a mean vector and a covariance matrix (McNeil, Frey, & Embrechts, 2005).

[2] Inui and Kijima (2005) even showed that expected shortfall is a basic coherent measure because it gives the minimum value among the class of plausible coherent risk measures, and any coherent risk measure is a convex combination of expected shortfalls.

To determine the possible costs and benefits of using the ES-implied correlation, I examine how the ES-implied correlation performs under three scenarios. In scenarios 1 and 2, I design simulations where the linear correlation is appropriate and is used as a benchmark to test whether the ES-implied correlation, which allows extra generality, embodies a large sacrifice when correlation is constant. In particular, asset returns are drawn from the multivariate normal distribution in scenario 1, and multivariate T distribution in scenario 2. The latter is found to be a better fit of returns in reality. Scenario 3 illustrates what gains may be possible to use the ES-implied correlation when the linear correlation is not appropriate and assets exhibit dynamic correlation based on market conditions. Simulation results show that the ES-implied correlation does not cause significant sacrifice when the linear correlation is appropriate, but produces substantial gains when the linear correlation is not appropriate; and the ES-implied correlation is much closer to the true correlation than is the VaR-implied correlation.

To illustrate how the ES-implied correlation can be used in empirical analyses, I investigate the relation between equity returns of G7 countries. Previous studies have discovered that correlations between international equity returns increase in bear markets (Campbell, Koedijk, and Kofman (2002), Longin and Solnik (2001), and Garcia and Tsafack (2011)). The ES-implied correlation reaches the same conclusion. I further develop a series of test statistics based on the VaR- and ES-implied correlations to measure the degree and test the significance of correlations deviate from the linear correlation during market downturns. The test statistics using the ES-implied correlation indicate that most correlations increase significantly during market downturns, while the test statistics using the VaR-implied correlation do not provide a consistent result.

One concern when estimating the implied correlations is the weights used to construct the portfolio. Although Cotter and Longin (2011) found little difference in the VaR-implied correlations from using different weights by eyes, they did not provide a method to test the significance of the difference. In addition, their paper is limited to estimating correlations in a two-asset environment. According to whether to include a third asset in the portfolio, estimation could be based on a two-asset or multi-asset environment. By developing a series of test statistics, this paper shows that the choice of weights does not have significant impact on the VaR- or ES-implied correlation.

This paper contributes to the literature on dynamic correlation measures. There exist three types of tail-based dependence measures in the literature: the exceedence correlation, the tail dependence coefficient, and the VaR-implied correlation. The exceedance correlation was pioneered by Longin and Solnik (2001) and studied by Ang and Chen (2002) and Campbell, Forbes, Koedijk, and Kofman (2008). The exceedance correlation estimates the correlation between assets conditional on asset returns falling above or below a pre-specified level.[3] Although the exceedance correlation is easy to understand and simple to calculate, Ang and Chen (2002), Campbell et al. (2002), and Longin and Solnik (2001) show that the exceedance correlation has a conditioning bias. For example, no matter how strongly two assets following a multivariate normal distribution are correlated, the exceedence correlation in the tails equals 0. Because of the conditioning bias, the exceedance correlation needs to be adjusted before measuring correlation asymmetries.

The tail dependence coefficient calculates the asymptotic probability of one asset providing extremely small or large returns given another asset provides extreme returns.[4] See Garcia and Tsafack (2011), Patton (2006), and Fortin and Kuzmics (2002) for example. One advantage of the tail dependence coefficient is that it does not need to choose a threshold as other conditional correlations do. However, this also induces a drawback: as a measure of dependence under very extreme circumstances, the tail dependence coefficient is realized infrequently. In addition, the tail dependence coefficient does not provide information on the dependence during normal times. For example, the tail dependence coefficient equals 0 when returns follow multivariate normal distribution.

Different with the exceedance correlation and tail dependence coefficient, the VaR-implied correlation does not have a conditioning bias. The VaR-implied correlation is conditional not only on individual assets' returns falling beyond a given threshold, but also on returns of a portfolio composed of the assets falling beyond the threshold. The additional condition on the portfolio counteracts the conditioning bias from truncating individual asset returns and thus makes the VaR-implied correlation free of conditioning bias.

Campbell et al. (2002) proposed the VaR-implied correlation and showed that it outperformed the linear correlation using data of the US, the UK, France, and Germany. Cotter and Longin (2011) investigated the impact of portfolio weights, the type of position, the frequency of data and the probability level on VaR-implied correlations by using the equity indexes of the US and the UK. Mittnik (2014) extended the pairwise method used in these papers to joint estimation.

[3] A general form of the exceedance correlation between two variables X and Y at thresholds δ_1 and δ_2 is

$$\rho(\delta_1,\delta_2) = \begin{cases} Corr(X,Y|X \leq \delta_1, Y \leq \delta_2), \delta_1 < 0, \delta_2 < 0 \\ Corr(X,Y|X \geq \delta_1, Y \geq \delta_2), \delta_1 \geq 0, \delta_2 \geq 0 \end{cases}$$

[4] The coefficient of upper tail dependence is

$$\tau_U = \lim_{\alpha \to 0} Pr[F_X(x) \geq \alpha | F_Y(y) \geq \alpha]$$

and the coefficient of lower tail dependence is

$$\tau_L = \lim_{\alpha \to 0} Pr[F_X(x) \leq \alpha | F_Y(y) \leq \alpha].$$

This paper contributes to the literature by proposing a development of the third style of the tail-based correlation measure. Simulations and empirical studies show that the new measure outperforms the VaR-implied correlation and has practical applications in risk management and portfolio optimization. I present the estimation approach in two-asset and multi-asset environments, respectively. A series of test statistics are developed to test for deviations with the correlation during normal times and to evaluate the impact of weights on the VaR- and ES-implied correlations.

This paper is organized as follows. Section 2 presents the estimation of the ES-implied correlation in two-asset and multi-asset environments, as well as the construction of the test statistics. Section 3 reports the results of simulations, which are designed to evaluate the performance of the ES-implied correlation in comparison with the linear correlation and the VaR-implied correlation. Section 4 analyzes the dependence between the US and the other G7 countries conditional on different market situations and illustrates how to apply the ES-implied correlation in risk management and asset allocation. Section 5 concludes.

2 Method

2.1 Pairwise approach

This section presents a pairwise approach in a two-asset environment to calculate the ES-implied correlation. As a background to the discussion, I first introduce the VaR-implied correlation.

VaR is a function of losses. The loss L is usually given as the negative of returns. The VaR at confidence level α is defined as the minimum value such that the probability of not exceeding this value at least equals α. Formally,

$$VaR(L)_\alpha = \inf\{l | P(L \leq l) \geq \alpha\}, \tag{1}$$

In other words, VaR is the α-quantile of the loss distribution. To simplify the notation, q_α is used to denote VaR_α in the following.

Let r_1 and r_2 be the returns of two assets and r_p be the return of a portfolio which is composed of the two assets with weights w_1 and w_2, where $w_1 + w_2 = 1$. Assume that the loss distribution of asset i belongs to a location-scale family and is characterized by a location parameter μ_i, a scale parameter σ_i and a zero-location, unit-scale distribution F_{Z_i}, referred to as the standard distribution, then

$$L_i = \mu_i + \sigma_i Z_i, \tag{2}$$

where Z_i follows the standard distribution F_{Z_i}.

The VaR of asset i is

$$VaR_{i,\alpha} = \mu_i + \sigma_i VaR(Z_i)_\alpha \tag{3}$$

where $VaR(Z_i)_\alpha$is the VaR of the noise variable Z_i at α confidence level.

Substituting (3) into

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho\sigma_1\sigma_2, \tag{4}$$

leads to the standardized VaR-implied correlation

$$\rho_{VaR,\alpha} = \frac{(\frac{q_{p,\alpha}-\mu_p}{q(Z_p)_\alpha})^2 - w_1^2(\frac{q_{1,\alpha}-\mu_1}{q(Z_1)_\alpha})^2 - w_2^2(\frac{q_{2,\alpha}-\mu_2}{q(Z_2)_\alpha})^2}{2w_1w_2\frac{(q_{1,\alpha}-\mu_1)(q_{2,\alpha}-\mu_2)}{q(Z_1)_\alpha q(Z_2)_\alpha}}. \tag{5}$$

Campbell et al. (2002) removed the standard distributions in equation (5) and pioneered the VaR-implied correlation:

$$\rho_{VaR,\alpha} = \frac{(q_{p,\alpha}-\mu_p)^2 - w_1^2(q_{1,\alpha}-\mu_1)^2 - w_2^2(q_{2,\alpha}-\mu_2)^2}{2w_1w_2(q_{1,\alpha}-\mu_1)(q_{2,\alpha}-\mu_2)}. \tag{6}$$

Campbell et al. (2002) used the VaR-implied correlation to examine whether returns are from normal distribution or not.

VaR is well known for not considering losses beyond itself and not being coherent; however, ES remedies these problems. Therefore, this paper develops a correlation measure using ES. ES at confidence level α is defined as the average of losses beyond the VaR, i.e.,

$$ES(L)_\alpha = E(L|L \geq VaR(L)_\alpha) \tag{7}$$

when the distribution of L is continuous.[5]

Similarly, given the loss $L_i = \mu_i + \sigma_i Z_i$,

$$ES_{i,\alpha} = \mu_i + \sigma_i ES(Z_i)_\alpha, \tag{8}$$

where $ES(Z_i)_\alpha$ is the ES of the noise variable Z_i at α confidence level.

Substituting (8) into equation (4) leads to the standardized ES-implied correlation,

$$\rho_{ES,\alpha} = \frac{(\frac{ES_{p,\alpha}-\mu_p}{ES(Z_p)_\alpha})^2 - w_1^2(\frac{ES_{1,\alpha}-\mu_1}{ES(Z_1)_\alpha})^2 - w_2^2(\frac{ES_{2,\alpha}-\mu_2}{ES(Z_2)_\alpha})^2}{2w_1w_2\frac{(ES_{1,\alpha}-\mu_1)(ES_{2,\alpha}-\mu_2)}{ES(Z_1)_\alpha ES(Z_2)_\alpha}}, \tag{9}$$

[5] When the distribution is discontinuous, $ES(L)_\alpha = \frac{E(L;L\geq VaR(L)_\alpha)+VaR(L)_\alpha(1-\alpha-Pr(L\geq VaR(L)_\alpha))}{1-\alpha}$. See Acerbi and Tasche (2002) and McNeil et al. (2005).

The standardized ES-implied correlation only reflects the relation between assets in the second moment and is equivalent to the linear correlation.

Following Campbell et al. (2002), Cotter and Longin (2011), and Mittnik (2014), this paper defines the ES-implied correlation as the correlation by removing $ES(Z_i)_\alpha$, $i = 1, 2, p$ in equation (9). The ES-implied correlation is

$$\rho_{ES,\alpha} = \frac{(ES_{p,\alpha} - \mu_p)^2 - w_1^2(ES_{1,\alpha} - \mu_1)^2 - w_2^2(ES_{2,\alpha} - \mu_2)^2}{2w_1 w_2 (ES_{1,\alpha} - \mu_1)(ES_{2,\alpha} - \mu_2)}. \tag{10}$$

Under the assumption of individual assets and the portfolio having the same standard distribution, equation (10) is equivalent to equation (9). That is, when returns follow normal distribution, the standardized ES-implied correlation and the ES-implied correlation equal the linear correlation. When this assumption does not hold, contrary to the standardized ES-implied correlation, the ES-implied correlation also reflects the information in the standard distribution and thereby is referred to as non-standardized.

2.2 Modification of the ES-implied correlation

The traditional definition of the ES is the average of losses falling beyond the corresponding VaR. However, there are two problems with this definition. First, $ES_0 = \mu$, implying that the denominator is close to 0 in equations (9) and (10) when α is very small. Second, this definition only considers values beyond the given quantile level, making ES not consider the extreme positive values. Analyzing the dependence between positive values is also useful, as the literature found that asset correlation tends to decrease when the market rises. Thus, this paper modifies the definition of ES. When $\alpha \geq 0.5$, ES is defined the same as ES, dented by ES^-. When $\alpha < 0.5$, the ES, denoted by ES^+, is defined as

$$ES^+_\alpha = E(L|L < VaR(L)_\alpha). \tag{11}$$

The standardized ES-implied correlation is therefore modified to be

$$\rho_{ES,\alpha} = \begin{cases} \dfrac{(\frac{ES^+_{p,\alpha}-\mu_p}{ES(Z_p)^+_\alpha})^2 - w_1^2(\frac{ES^+_{1,\alpha}-\mu_1}{ES(Z_1)^+_\alpha})^2 - w_2^2(\frac{ES^+_{2,\alpha}-\mu_2}{ES(Z_2)^+_\alpha})^2}{2w_1 w_2 \frac{(ES^+_{1,\alpha}-\mu_1)(ES^+_{2,\alpha}-\mu_2)}{ES(Z_1)^+_\alpha ES(Z_2)^+_\alpha}}, \alpha < 0.5, \\ \\ \dfrac{(\frac{ES^-_{p,\alpha}-\mu_p}{ES(Z_p)^-_\alpha})^2 - w_1^2(\frac{ES^-_{1,\alpha}-\mu_1}{ES(Z_1)^-_\alpha})^2 - w_2^2(\frac{ES^-_{2,\alpha}-\mu_2}{ES(Z_2)^-_\alpha})^2}{2w_1 w_2 \frac{(ES^-_{1,\alpha}-\mu_1)(ES^-_{2,\alpha}-\mu_2)}{ES(Z_1)^-_\alpha ES(Z_2)^-_\alpha}}, \alpha \geq 0.5. \end{cases} \tag{12}$$

The non-standardized ES-implied correlation is modified to be

$$\rho_{ES,\alpha} = \begin{cases} \frac{(ES^+_{p,\alpha}-\mu_p)^2 - w_1^2(ES^+_{1,\alpha}-\mu_1)^2 - w_2^2(ES^+_{2,\alpha}-\mu_2)^2}{2w_1w_2(ES^+_{1,\alpha}-\mu_1)(ES^+_{2,\alpha}-\mu_2)}, \alpha < 0.5, \\ \\ \frac{(ES^-_{p,\alpha}-\mu_p)^2 - w_1^2(ES^-_{1,\alpha}-\mu_1)^2 - w_2^2(ES^-_{2,\alpha}-\mu_2)^2}{2w_1w_2(ES^-_{1,\alpha}-\mu_1)(ES^-_{2,\alpha}-\mu_2)}, \alpha \geq 0.5. \end{cases} \tag{13}$$

The following proves that the modified ES-implied correlations calculated from the left and the right are the same at $\alpha = 0.5$.

Proposition 1. *The standardized and non-standardized ES-implied correlations are continuous at* $\alpha = 0.5$.

Proof. From Corollary 3.3 in Acerbi and Tasche (2002), we know that ES is continuous. Thus $\lim_{\alpha \to 0.5^-} ES^+ = ES^+_{0.5}$. Since $ES^-_{0.5} + ES^+_{0.5} = \frac{\int_0^1 VaR_u du}{0.5} = 2\mu$,

$$ES^-_{0.5} - \mu = -(ES^+_{0.5} - \mu). \tag{14}$$

Substituting equation (14) into equations (12) and (13), we can see that $\lim_{\alpha \to 0.5^+} \rho_{ES,\alpha} = \lim_{\alpha \to 0.5^+} \rho_{ES,\alpha} = \rho_{ES,0.5}$ for both the standardized and non-standardized correlations. □

In the empirical analysis, the VaR-implied correlation violates the [-1,1] correlation interval frequently. However, due to the fact that the ES is coherent, the ES-implied correlation has a desirable characteristic as follows:

Proposition 2. *The ES-implied correlation does not exceed* 1 *when short selling is not allowed.*

Proof. Recall that a risk measure ζ is coherent if it is: 1) subadditive, meaning $\zeta(L_1 + L_2) \leq \zeta(L_1) + \zeta(L_2)$; 2) positive homogeneous, meaning $\zeta(wL) = w\zeta(L)$ for every $w > 0$; 3) monotonic, meaning $\zeta(L_1) \leq \zeta(L_2)$ for $L_1 \leq L_2$; and 4) translation invariant, meaning $\zeta(L + l) = \zeta(L) + l$ for every $l \in R$.

When short selling is not allowed,

$$\begin{aligned} ES_{p,\alpha} &= ES(w_1 L_1 + w_2 L_2)_\alpha \\ &\leq ES(w_1 L_1)_\alpha + ES(w_2 L_2)_\alpha \\ &= w_1 ES(L_1)_\alpha + w_2 ES(L_2)_\alpha \\ &= w_1 ES_{1,\alpha} + w_2 ES_{2,\alpha}, \end{aligned}$$

where the inequality holds because of subadditivity and the second equality holds due to positive homogeneity. No short selling guarantees positive homogeneity.

Since expected shortfall is a monotonic risk measure, $ES_\alpha \geq ES_0$, where the latter equals $\int_0^1 VaR_u du = \mu$. Hence,

$$0 \leq ES_{p,\alpha} - \mu \leq w_1(ES_{1,\alpha} - \mu) + w_2(ES_{2,\alpha} - \mu). \tag{15}$$

Thus under the assumption of no short selling, $\rho_{ES,\alpha} \leq 1$ for $\alpha \geq 0.5$.

When $\alpha < 0.5$, the definition of the modified ES is different from the traditional definition of the ES. To prove $\rho_{ES,\alpha} \leq 1$ when $\alpha < 0.5$, I express ES_α^+ as the traditional expected shortfall ES_α:

$$ES_\alpha^+ - \mu = \frac{\int_0^1 VaR_u du - \int_\alpha^1 VaR_u du}{\alpha} - \mu = -\frac{1-\alpha}{\alpha}(ES_\alpha - \mu). \tag{16}$$

The following holds after substituting equation (16) into equation (15):

$$0 \geq ES_{p,\alpha}^+ - \mu \geq w_1(ES_{1,\alpha}^+ - \mu) + w_2(ES_{2,\alpha}^+ - \mu). \tag{17}$$

Thus $\rho_{ES,\alpha} \leq 1$ also holds when $\alpha < 0.5$. □

2.3 Estimation and consistency

To estimate the risk measure-implied correlations, I need to compute the VaR and ES of the individual assets and the portfolio. There are three main methods to estimate the VaR and ES: the Gaussian approach, the extreme value theory (EVT) approach and the empirical approach.

The Gaussian approach assumes that returns follow normal distributions. In this case, the VaR and ES are functions of the mean and the standard deviation. See Castellacci and Siclari (2003) for an application of this approach. The VaR and ES are very easy to compute using this approach. However, many empirical studies have found that the assumption of normality is unrealistic. The VaR and ES computed with this approach are thus inaccurate. Moreover, the implied correlations estimated by the Gaussian approach equal the linear correlation.

The extreme value theory focuses on the study of the tail behaviour and is used widely to estimate the VaR and ES. See Fernandez (2010) for example. However, the EVT is accurate only in the tails. Thus Danielsson and De Vries (2000) used the EVT along with the empirical method. Under the assumption that returns follow extreme value distributions, VaR and ES are functions of the parameters of the extreme value distributions. Generally, there are two methods to estimate the extreme value distribution, block maxima method (BMM) and peak over threshold (POT). Fitting the extreme value distribution requires specifying either the size of the block or the threshold. An inappropriate choice of block size or threshold will lead the estimation to be inaccurate. Thus this paper does not employ this approach to estimate VaR or ES either.

The empirical approach uses the empirical distribution of the data to approximate the actual distribution. Fernandez (2010) and Danielsson and De Vries (2000) demonstrated that this approach generates smaller errors than the Gaussian approach. More importantly, the empirical approach is parameter free and easy to implement. Thus, this paper chooses the empirical approach to estimate VaR and ES.

The rest of this section presents the estimation and convergence of the implied correlations. I start from the estimation of the VaR-implied correlation. Let $L_{j:T}$ be the jth largest value in the historical losses $L_t, t = 1,2,...,T$, F be the cumulative distribution function of losses and F_T be the empirical distribution function. The empirical estimation of VaR at confidence level α is $L_{[\alpha T]:T}$, where $[\alpha T]$ is the integer part of αT.

The empirical estimate of the VaR-implied correlation is

$$\hat{\rho}_{VaR,\alpha} = \frac{(\hat{q}_{p,\alpha} - \hat{\mu}_p)^2 - w_1^2(\hat{q}_{1,\alpha} - \hat{\mu}_1)^2 - w_2^2(\hat{q}_{2,\alpha} - \hat{\mu}_2)^2}{2w_1 w_2(\hat{q}_{1,\alpha} - \hat{\mu}_1)(\hat{q}_{2,\alpha} - \hat{\mu}_2)},$$

where $\hat{q}_{i,\alpha} = L^i_{[\alpha T]:T}$ approximates the α-quantiles and the sample mean $\hat{\mu}_i$ approximates population mean. Shorack and Wellner (2009) proved that $L_{[\alpha T]:T}$ converges to $F^{-1}(\alpha)$ almost surely. Thus $\hat{\rho}_{VaR,\alpha}$ converges to $\rho_{VaR,\alpha}$ in probability when weak law of large number holds and almost surely when strong law holds. When returns are multivariate normally distributed or when they have the same standard distribution F_Z, the VaR-implied correlation equals linear correlation.

For the estimation of the ES-implied correlation, the ES at confidence level less than 0.5 can be estimated by $\frac{\sum_{j=[\alpha T]+1}^{T} L_{j:T}}{T-[\alpha T]}$ and the ES at confidence level exceeding 0.5 can be estimated by $\frac{\sum_{j=1}^{[\alpha T]} L_{j:T}}{[\alpha T]}$. Acerbi and Tasche (2002) proved that the estimate of the traditional ES converges to the actual expected shortfall almost surely. Similarly, the estimate of the modified ES also converges to the actual value of the modified expected shortfall. Therefore, the empirical estimate of the ES-implied correlation converges to the ES-implied correlation in probability when weak law of large number holds and almost surely when strong law holds.

2.4 Joint estimation of the ES-implied correlation

The above sections have presented the pairwise method of estimating the correlation between two assets. For the number of assets $n \geq 1$, there exist $C(n,2) = \frac{n(n-1)}{2}$ correlations, where $C(n,k)$ denotes the number of k combinations from n elements. The $C(n,2)$ correlations could either be estimated one by one, using the pairwise method, or be estimated together. Estimating correlations one by one may result in a loss of information since other assets in the portfolio may have an impact on the correlation. Mittnik (2014) found that assigning weights to other assets could improve efficiency and reduce the frequency of the VaR-implied correlation falling outside of the [-1,1] interval. Following Mittnik (2014), this section introduces a closed-form solution for estimating the $C(n,2)$ ES-implied correlations jointly.

Given a portfolio composed of n assets with weights $w_i, i = 1,2,...,n, \sum_{i=1}^{n} w_i = 1$,

$$(ES_{\alpha,p})^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j (ES_{\alpha,i})(ES_{\alpha,j})\rho_{ES,\alpha,ij} \tag{18}$$

holds for demeaned ES.

Denoting the correlation matrix by R, equation (18) can be rewritten as

$$ES_p^2 = (ES \diamond w)' R (ES \diamond w),$$

where α is dropped in order to simplify notation, ES is a $n \times 1$ vector composed of expected shortfalls of all assets, w is a $n \times 1$ vector of weights and $\diamond$ is the Schur product, i.e., $ES \diamond w = \sum_{i=1}^{n} w_i ES_i$. Bring all the known $\rho_{ii} = 1, i = 1,2,...,n$ to the left, then

$$\tilde{ES}_p^2 = ES_p^2 - \sum_{j=1}^{n} w_j^2 ES_j^2 = (ES \diamond w)'(R - I)(ES \diamond w),$$

where I is the identity matrix and $\tilde{ES}_p^2$ represents excess squared quantiles.

Employing the formula $vec(ABC) = (C' \otimes A)vec(B)$, where $\otimes$ is the Kronecker product and vec is the conventional vectorization operator, the above equation equals

$$\tilde{ES}_p^2 = (ES \diamond w)' \otimes (ES \diamond w)' vec(R - I).$$

There exists a unique $n^2 \times \frac{n(n-1)}{2}$ matrix D composed of zeros and ones such that the vectorization of $R - I$ equals

$$\begin{aligned} vec(R-I) &= (0 \quad \rho_{12} \quad ... \quad \rho_{1n} \quad \rho_{12} \quad 0 \quad ... \quad \rho_{2n} \quad ... \quad \rho_{n-1,n} \quad 0)' \\ &= D \quad (\rho_{12} \quad ... \quad \rho_{1n} \quad \rho_{23} \quad ... \quad \rho_{2n} \quad ... \quad \rho_{n-1,n})'. \end{aligned}$$

The $\frac{n(n-1)}{2} \times 1$ vector $\rho = (\rho_{12} \quad ... \quad \rho_{1n} \quad \rho_{23} \quad ... \quad \rho_{2n} \quad ... \quad \rho_{n-1,n})'$ includes all the correlations that need to be estimated.

Therefore, given a weight vector w,

$$\tilde{ES}_p^2 = (ES \diamond w)' \otimes (ES \diamond w)' D\rho. \tag{19}$$

To estimate the correlations exactly, $n(n-1)/2$ equations are needed. When there are $m = \frac{n(n-1)}{2}$

vectors of weights $(w_1,...w_m)$,

$$\begin{pmatrix} \tilde{ES}^2_{p_1} \\ ... \\ \tilde{ES}^2_{p_m} \end{pmatrix} = \begin{bmatrix} (ES \diamond w_1)' \otimes (ES \diamond w_1)' \\ ... \\ (ES \diamond w_m)' \otimes (ES \diamond w_m)' \end{bmatrix} D\rho.$$

Denote $\tilde{Q} = (\tilde{ES}^2_{p_1}...\tilde{ES}^2_{p_n})'$ and $X = \begin{bmatrix} (ES \diamond w_1)' \otimes (ES \diamond w_1)' \\ ... \\ (ES \diamond w_m)' \otimes (ES \diamond w_m)' \end{bmatrix} D$, then the correlation vector is obtained by

$$\rho = X^{-1}\tilde{Q}. \tag{20}$$

Equation (20) is referred to as exact identification since every equation in the formula is satisfied exactly. When assigning weights only to every two assets, correlations calculated from the exact identification are equal to correlations calculated from the pairwise method in section 2.1.

When adding the number of weight vectors more than the number of unknown correlations, X would not be a square matrix any more. Estimation of the correlations can be obtained by the least squares:

$$\rho = (X'X)^{-1}X'\tilde{Q}. \tag{21}$$

Equation (21) is referred to as overidentification, where an error term exists so that $\tilde{Q} = X\rho + u$ instead of $\tilde{Q} = X\rho$ in the case of exact identification.

2.5 Quantitative measures of correlation asymmetries

Since correlations are found to be asymmetric, this section develops a series of statistics for measuring and testing the amount that correlation deviates from the linear correlation during market downturns and upturns. These statistics are referred to as H and AH following Ang and Chen (2002). To be specific, I develop the downside H and upside H statistics for measuring the maximum degree to which implied correlations deviate from the linear correlation, and the downside and upside AH statistics for evaluating the average of correlation deviations. Since the VaR-implied correlation is very unsteady when the probability level is around 0.5 and the linear correlation is only inappropriate in the tails, the downside and upside statistics are constructed to assess correlation asymmetries in intervals of (0.7, 1) and (0, 0.3), respectively.

The downside H statistic is defined as the supremum of deviations of the linear correlation from tail-based correlations. The downside H statistic using the VaR-implied correlation is

$$H^-_{VaR} = \sup_{\alpha\in(0.7,1)} (\hat{\rho}_{VaR,\alpha} - \hat{\rho}) \tag{22}$$

and the downside H statistic using the ES-implied correlation is

$$H^-_{ES} = \sup_{\alpha\in(0.7,1)} (\hat{\rho}_{ES,\alpha} - \hat{\rho}), \tag{23}$$

where $\hat{\rho}$ is the empirical linear correlation, $\hat{\rho}_{VaR,\alpha}$ denotes the empirical VaR-implied correlation, and $\hat{\rho}_{ES,\alpha}$ denotes the empirical ES-implied correlation.

The upside H statistic evaluates the highest degree to which correlation is overestimated by the linear correlation in the right tail of return distribution, i.e.,

$$H^+_{VaR} = \sup_{\alpha\in(0,0.3)} (\hat{\rho} - \hat{\rho}_{VaR,\alpha}) \tag{24}$$

and

$$H^+_{ES} = \sup_{\alpha\in(0,0.3)} (\hat{\rho} - \hat{\rho}_{ES,\alpha}) \tag{25}$$

The downside AH statistic measures the average of correlation asymmetries in the left tail. The downside AH statistic using the VaR-implied correlation is

$$AH^-_{VaR} = \frac{1}{0.3}\int_{0.7}^{1} (\hat{\rho}_{VaR,\alpha} - \hat{\rho}) d\alpha \tag{26}$$

and the downside AH statistic using the ES-implied correlation is

$$AH^-_{ES} = \frac{1}{0.3}\int_{0.7}^{1} (\hat{\rho}_{ES,\alpha} - \hat{\rho}) d\alpha. \tag{27}$$

The upside AH statistic measures average upside correlation asymmetries. The upside AH statistic using the VaR-implied correlation is

$$AH^+_{VaR} = \frac{1}{0.3}\int_{0}^{0.3} (\hat{\rho} - \hat{\rho}_{VaR,\alpha}) d\alpha \tag{28}$$

and the upside AH statistic using the ES-implied correlation is

$$AH^+_{ES} = \frac{1}{0.3}\int_{0}^{0.3} (\hat{\rho} - \hat{\rho}_{ES,\alpha}) d\alpha. \tag{29}$$

The probability level α is assumed to be uniformly distributed between 0 and 1. When α follows other distributions, the difference between implied correlations and linear correlations is assigned different weights at different probability levels. Ang and Chen (2002), for example, chose weights proportional to the number of observations.

Under the null hypothesis that returns follow multivariate normal distribution, the linear correlation is appropriate and the theoretical implied correlations equal the theoretical linear correlation. Since the sample mean, the empirical estimates of VaR and ES are consistent estimators of their corresponding theoretical values, the estimated implied correlations converge to the theoretical implied correlations, which equal the linear correlation under H_0. The alternative hypothesis is that the linear correlation underestimates (overestimates) correlations during market downturns (upturns). The null hypothesis is rejected when the test statistic is too large.

I use bootstrap to test the significance of the test statistics. The bootstrap technique provides a simple way to test statistics whose distribution is unknown, but can be simulated.[6]

Take the H statistics for example. I use the following procedure to make statistical inference. Given two assets with sample size n,

Step A: Calculate the test statistic $\hat{H}$, sample means, standard deviations of each individual asset, and their correlation using the empirical data.

Step B: Draw n pairs of data from the bivariate normal distribution using the sample mean, standard deviation, and correlation calculated in Step A.

Step C: Compute the test statistic, named $H^{(1)}$ for H statistic from the simulated data.

Step D: Repeat step B and step C M times and get a sequence of test statistics, $H^{(1)}, ..., H^{(M)}$.

Step E: Calculate the p-values, $\hat{p}_H = \frac{1}{M+1} \sum_{m=1}^{M} (I(H^{(m)} \geq \hat{H}) + 1)$, where $I(\cdot)$ is known as the indicator function. The hypothesis is rejected at level α if the p-value is less than or equal to α.

Notice that the H statistics in this paper differ from the statistic in Ang and Chen (2002) in several ways. First, the thresholds considered in the paper are continuous, while the thresholds in the paper of Ang and Chen (2002) are a number of discrete points. Second, while Ang and Chen (2002) used the quadratic deviation and the sum of deviation between the linear correlation and the exceedance correlation, this paper measures the maximum and the average of the deviation of the linear correlation from implied correlations. It is common to construct statistics using maximum. See Hansen (1996), and Davies (1977, 1987) for example. Third, this paper uses the risk measure-implied correlations, which are free of conditioning bias, to measure correlation asymmetries, while Ang and Chen (2002) have to adjust conditioning bias of the exceedance correlation. Fourth, the significance of the test statistic is examined by bootstrap in this paper, while Ang and Chen (2002) used the generalized method of moments (GMM) and the delta method to get the standard deviation of test statistic first and then calculate the p-value.

[6] The asymptotic distribution of the test statistics is very complex. See Appendix A.

The H and AH statistics also can evaluate the impact of weights on risk measure-implied correlations. The sign of the difference is not important any more since the interest here is to test whether using different weights could lead to different implied correlations. Thus I use the absolute value of the difference to construct statistics. The new H statistics do not distinguish between downside and upside, and take the supremum across all probability levels except the interval $(0.3, 0.7)$ to avoid the unstable VaR-implied correlations. That is,

$$H_{VaR} = \sup_{\alpha \in (0,0.3)\cup(0.7,1)} |\hat{\rho}^{(1)}_{VaR,\alpha} - \hat{\rho}^{(2)}_{VaR,\alpha}| \tag{30}$$

and

$$H_{ES} = \sup_{\alpha \in (0,0.3)\cup(0.7,1)} |\hat{\rho}^{(1)}_{ES,\alpha} - \hat{\rho}^{(2)}_{ES,\alpha}|, \tag{31}$$

where $\hat{\rho}^{(1)}_{VaR,\alpha}$ and $\hat{\rho}^{(1)}_{ES,\alpha}$ are implied correlations for a given choice of weights, while $\hat{\rho}^{(2)}_{VaR,\alpha}$ and $\hat{\rho}^{(2)}_{ES,\alpha}$ are implied correlations for another choice of weights. The corresponding AH are

$$AH_{VaR} = \frac{1}{0.6}\left(\int_0^{0.3} |\hat{\rho}^{(1)}_{VaR,\alpha} - \hat{\rho}^{(2)}_{VaR,\alpha}| d\alpha + \int_{0.7}^{1} |\hat{\rho}^{(1)}_{VaR,\alpha} - \hat{\rho}^{(2)}_{VaR,\alpha}| d\alpha\right) \tag{32}$$

and

$$AH_{ES} = \frac{1}{0.6}\left(\int_0^{0.3} |\hat{\rho}^{(1)}_{ES,\alpha} - \hat{\rho}^{(2)}_{ES,\alpha}| d\alpha + \int_{0.7}^{1} |\hat{\rho}^{(1)}_{ES,\alpha} - \hat{\rho}^{(2)}_{ES,\alpha}| d\alpha\right). \tag{33}$$

Since the VaR- and ES-implied correlations are invariant with respect to weights for elliptical distributions,[7] the difference in implied correlations from choosing different weights should be insignificant when data are from the multivariate normal distribution. Thus this paper simulates data from the multivariate normal distribution and calculates the statistics using simulated data to test the significance of the statistics from the empirical data. The process is similar to the test of correlation asymmetries and thus is not repeated.

3 Simulation

In order to study the possible sacrifice and gains of using the ES-implied correlation, I design three cases in the simulation. In the first two cases, correlation is constant and the linear correlation is appropriate in order to judge whether allowing extra generality embodies a large sacrifice; in the

[7] See Mittnik (2014) and Campbell et al. (2002)

third case, correlation changes in the tails and the linear correlation is inappropriate in order to judge what gains may be possible by using the expected shortfall-based measure.

I calculate the average bias and root-mean-square error (RMSE) of the estimated correlations to compare the performance of the ES-implied correlation with the linear correlation and the VaR-implied correlation.

The average bias is the average of the bias across m simulations, i.e.,

$$\frac{1}{m}\sum_{j=1}^{m}(\hat{\rho}_j - \rho),$$

and the RMSE is

$$\sqrt{\frac{1}{m}\sum_{j=1}^{m}(\hat{\rho}_j - \rho)^2}.$$

The estimator which makes these values close to 0 is regarded as a good indicator.

I use the pairwise method and the weight vector $(0.5, 0.5)$ to calculate the correlations in this section. The effect of the choice of weight vectors on estimated implied correlations will be studied in the empirical analysis.

In case 1, correlation is constant $\rho = 0.5$ and the data are from a multivariate normal distribution with mean 0 and covariance matrix $\Sigma = \begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{pmatrix}$. The estimated linear correlation is the maximum-likelihood estimation (MLE), and therefore has the desirable asymptotic properties of maximum likelihood, including consistency and efficiency. See Fisher (1915) and Gayen (1951) for the distribution of the estimated linear correlation.

Figure 1 plots the average bias and root-mean-square error of the estimated correlations from the multivariate normal distribution with sample size $T = 10^4$ across $m = 10^4$ replications. The black, red and green points correspond to the estimated linear correlation, VaR-implied correlation and ES-implied correlation, respectively.

Consistent with the theory, the estimated linear correlation coefficient is consistent and efficient in case 1. The estimated ES-implied correlation is as good as the estimated linear correlation. The estimated VaR-implied correlation deviates from the actual correlation in the center of the distribution because the denominator in the formula of the VaR-implied correlation is close to 0 around the center. Even in the tails, it is not as close to the actual correlation as the ES-implied correlation does.

In case 2, correlation is still constant, but the data follow a multivariate T distribution with degrees of freedom 3 and covariance matrix $\frac{T-2}{T}\Sigma$, where T is the sample size. Figure 2 plots the results of the data simulated from the multivariate T distribution with the actual correlation $\rho = 0.5$, and sample size $T = 10^4$. Again, the process is repeated $m = 10^4$ times. In this case, the estimated linear correlation is no longer MLE. The estimated ES-implied correlation has both the smallest bias and the smallest root-mean-square error among the three correlation estimates.

In case 3, correlation is non-constant and changes at some designed probability levels. Since the correlation is non-constant, better performance from methods which allow for the change in the correlation is expected.

Assume there are two breaking points, which divide the whole space into 3 regions. The first region $(-\infty, -1] \times (-\infty, -1]$ mimics the bear market with the actual correlation of 0.75. The second region $(-1, 1] \times (-1, 1]$ mimics the moderate market situation, where the correlation is 0.58. The third region $(1, \infty) \times (1, \infty)$ mimics the bull market with the correlation of 0.52.

I use truncation to generate the data. However, Ang and Bekaert (2002) and Campbell et al. (2008) documented that truncation changes the correlation between assets. Given the correlation after truncation, one needs to decide the correlation before truncation. Employing the MC technique in a wide search, I find that the data for regions 1, 2, and 3 can be generated by truncating a bivariate normal distribution with marginal distributions $N(0, 1)$ and correlation coefficients $\rho^1_{before} = 0.93$, $\rho^2_{before} = 0.86$ and $\rho^3_{before} = 0.83$, respectively.

To be exact, the following steps are used to determine correlations before truncation.

Step A: Choose a possible value for the correlation before truncation, and generate 10^5 random samples from a bivariate normal distribution with the marginal distribution $N(0, 1)$ and this correlation.

Step B: Truncate the simulated data to the region that we concern and calculate the correlation in that region.

Step C: Repeat this process 10^4 times and record the correlation every time.

Step D: Calculate the average of the 10^4 truncated correlations. If the average correlation equals the targeted correlation, the correlation chosen in step A is the right correlation before truncation. If the average is greater than the targeted correlation, reduce the correlation and repeat steps A, B and C till the average of the correlations across 10^4 replications is equal to the targeted correlation. If the average is smaller than the targeted correlation, increase the correlation and repeat steps A, B and C till the average of the correlations and the targeted correlation are the same.

Suppose that $p_1 = 20\%$ of the data locating in region 1, $p_2 = 60\%$ in region 2 and $p_3 = 20\%$ in region 3. The actual correlation then is

$$\rho_\alpha = \begin{cases} 0.75, & \alpha \leq 20\%, \\ 0.58, & 20\% < \alpha \leq 80\%, \\ 0.52, & \alpha > 80\%. \end{cases}$$

The data is simulated by combing a proportion of p_1 of random variables truncating from a bivariate normal distribution with correlation ρ^1_{before} for region 1, p_2 of random variables truncating from a bivariate normal distribution with correlation ρ^2_{before} for region 2, and p_3 of random variables truncating from a bivariate normal distribution with correlation ρ^3_{before} for region 3.

Figure 3 plots the average estimated correlations across $m = 10^3$ simulations with sample size $T = 10^4$. In comparison with the large deviation of the estimated linear correlation from the actual linear correlation, the estimated implied correlations are more trustworthy. The estimated VaR-implied correlation is still unstable around the center of each region.

We are naturally interested in how far a deviation needs to be from constancy in order that the estimated ES-implied correlation provides an improvement. Table 1 reports the summary statistics of the RMSEs in four situations: $p_1 = 0, p_2 = 1, p_3 = 0$; $p_1 = 2\%, p_2 = 96\%, p_3 = 2\%$; $p_1 = 12\%, p_2 = 82\%, p_3 = 6\%$; and $p_1 = 12\%, p_2 = 76\%, p_3 = 12\%$. Situation 1 assumes no break point, in which case the estimated linear correlation is the best unbiased estimator. Situation 2 assumes that 2% of the data are from another multivariate normal distributions in the left tail and right tails. Situation 3 increases this proportion and makes the proportion different in the left and right tails in order to evaluate the impact of asymmetry. Situation 4 then increases the proportion in the right tail to the same level as the proportion in the left tail.

The estimated linear correlation generates the least RMSE in situation 1. However, even when only 2% of the data from other distributions are included in the tails, the RMSE of the estimated linear correlation increases sharply. The RMSE keeps growing when the proportion increases in the tails, but not that much from situation 3 to situation 4.

Compared to the estimated linear correlation, the estimated risk measure-implied correlation is less affected by the change of the proportion in the tails. The RMSE of estimated ES-implied correlations is less than the RMSE of estimated linear correlations almost at every probability level in the last three situations, where data in the tails are assumed to follow a different distribution. The estimated VaR-implied correlation is still untrustworthy around the center, leading to the mean and standard deviation of its RMSE very large at some probability levels.

4 Empirical analysis

In the empirical analysis, I investigate correlations between equity returns of G7 countries for risk diversification in the global financial market. The G7 countries are the United States (US), Canada (CA), the United Kingdom (UK), Italy (IT), Germany (DE), France (FR) and Japan (JP).

4.1 Data

This paper uses equity return indexes of the G7 countries from January 1, 1973 to December 31, 2015. To avoid time difference, I use weekly returns. The weekly frequency avoids market microstructure of using daily returns, yet provides sufficient observations in the tails. The data is acquired from Datastream and includes 2313 observations.

Panel A of Table 2 presents the summary statistics of returns for the whole period. The mean and standard deviation of returns are annualized by multiplying returns by 52. The average returns of all the countries, except Japan, are around 10%. Japan has an average return of only about 7%. Standard deviations of returns vary from 1.14 to 1.69. The row labeled "Skewness" reports the results of the D'Agostino test of skewness (D'Agostino, 1970). It shows that the equity returns in all countries, except the UK and Italy, are skewed to left at 5% significance level and higher. While the equity returns of the UK are significantly skewed to right, the returns of Italy do not exhibit significant skewness. The row labeled "Kurtosis" reports the results of the Anscombe-Glynn test (Anscombe & Glynn, 1983) and implies that all equity returns have acute peaks and tend to be heavy-tailed. The results of the Anderson-Darling test (Stephens (1986) and Thode (2002)) and the Shapiro-Wilk test (Shapiro and Wilk (1965) and Royston (1982)), reported in rows labeled "AD test" and "Shapiro test", indicate that the equity returns do not follow the normal distribution.

To see how the 2008 financial crisis affects the financial market, I divide the whole sample period into two subperiods. The first period includes the first 30 years from January 1, 1973 to December 31, 2002. The second period includes the 2008 financial crisis and extends from January 1, 2003 to December 31, 2015. Panel B and Panel C in Table 2 report the summary statistics for the two subperiods. Except Germany and Japan, the average return of all the countries decreases in the second period. In the first subperiod, returns of most countries, except the UK, Italy, and Japan, are skewed to the left, while in the second period, returns of all countries are skewed to the left, suggesting the possibility of having large negative returns in the second subperiod. Results from the kurtosis, Anderson-Darling test and Shapiro-Wilk test imply that equity returns of all the countries in the two subperiods are not normally distributed.

4.2 Empirical results

In total, there are $C(7,2) = \frac{7\times 6}{2} = 21$ correlation coefficients between the seven countries. To save space, only correlations between the US and the other G7 countries are reported. The weight vector used to compose the portfolio is $(0.5, 0.5)$ in this section.

Figure 4 plots the performance of the estimated linear correlation, VaR-implied correlation and ES-implied correlation in the whole sample period, where low quantiles correspond to bad market situations and high quantiles correspond to good market situations.

The figure implies that the estimated VaR-implied correlation is very unstable around the center and goes beyond 1 frequently. The estimated ES-implied correlation is higher than the estimated linear correlation in the left tail and lower than the estimated linear correlation in the right tail for all the countries, consistent with the empirical findings that dependence increases during market downturns and decreases during market upturns. Among all countries, the tail dependence between the US and Canada and Japan increases least. It is also noteworthy that although the US exhibits a higher linear correlation with Canada during normal times, the downside dependence does not increase much. For example, the tail dependence between the US and the UK is even higher than the tail dependence between the US and Canada, emphasizing the importance of estimating tail dependence.

Table 3 reports the H statistics and AH statistics for measuring correlation asymmetries. Panel A reports the results of the H statistics. The H statistics between the US and other countries are positive for VaR-implied and ES-implied correlations, implying that the linear correlation underestimates the dependence in the left tail and overestimates the dependence in the right tail. Rows 1 and 2, labeled "H^-_{VaR}" and "H^-_{ES}", respectively, report the value of downside H statistics using the VaR-implied correlation and the ES-implied correlation, respectively. The results show that the VaR-implied correlation tends to produce higher but less significant correlation asymmetries than ES-implied correlation. In particular, the VaR-implied correlation indicates that relations between the US and Italy, France, and Japan in the left tail increase significantly at 5% level, and the ES-implied correlation implies significant relations between the US and the UK, Italy, Germany, and France. Compared to rows 1 and 2, rows 3 and 4, labeled "H^+_{VaR}" and "H^+_{ES}", reveal that the degree that correlation decreases during market upturns is generally less than the degree that correlation increases during market downturns.

Panel B reports the results of the AH statistics. Signs of the AH statistics using the VaR-implied correlation are incongruous, but only the positive statistics are significant. The AH statistics using the ES-implied correlation are all positive and significant at 10% level and higher, implying that correlations are generally higher during market downturns and lower during market upturns than the linear correlation. The downside AH statistics using the VaR-implied correlation are significant at 5% level only for the relation between the US and Canada and the relation between the US and Japan. The downside AH statistics using the ES-implied correlation imply that correlations between the US and Canada, the UK, Italy, and France during market downturns are significantly underestimated by the linear correlation at 1% level, and correlations between the US and Germany and Japan during market downturns are underestimated by the linear correlation at 10% level. The upside AH statistics using the VaR-implied correlation demonstrates a positive and significant relation between the US and Germany, and the upside AH statistics using the ES-implied correlation show that all correlations decrease significantly at 10% and higher.

I then estimate the linear correlation and implied correlations for the subperiods. Figure 5 plots the results. The black points, red points, and green points represent the results of the estimated linear correlation, VaR-implied correlation, and ES-implied correlation for the first subperiod from January 1973 to December 2002, respectively. The grey points, pink points, and blue points correspond to the estimated linear correlation, VaR-implied correlation, and ES-implied correlation for the second subperiod from January 2003 to December 2015. Correlations increase in the second period, as predicted by the fact that market situations get worse in the second period and dependence increases when the market situation worsens. Correlation between the US and Canada increases least among all correlations, consistent with the results in Figure 4 and Table 3.

4.3 Effect of weights

This section tests the impact of portfolio weights on risk measure-implied correlations by two approaches. In the first approach, I use two sets of weights and estimate the difference in the implied correlations. In the second approach, I compare the difference in the estimated correlations from exact identification and overidentification in order to check whether giving weight vectors more than the number of correlations could improve efficiency, since Mittnik (2014) documented that assigning weights to other assets in the portfolio provides more information and reduces the problem of the VaR-implied correlation locating outside the $[-1,1]$ interval.

4.3.1 Result of changing the value of weights

Instead of using the weight vector $(0.5,0.5)$, this section uses the weight vector $(0.2,0.8)$, i.e., investing 20% in the US and 80% in the other market. Figure 6 plots the difference in the correlations from using the weight vectors $(0.5,0.5)$ and $(0.2,0.8)$. The difference in the estimated VaR-implied correlations varies a little in the tails, but differs a lot around the center. The estimated ES-implied correlation does not show discernible difference across probability levels.

Panel A of Table 4 reports the test statistics measuring the difference in correlations from using different weights. Rows 1 and 2 show that for correlations between the US and the other G7 countries, the H statistics based on the ES-implied correlation are smaller than the H statistics based on the VaR-implied correlation, suggesting that the ES-implied correlation is less affected by the choice of weights than the VaR-implied correlation. Rows 3 and 4 reporting the AH statistics reach the same conclusion. In addition, all the test statistics are insignificant, except the AH statistic based on the VaR-implied correlation between the US and Japan.

4.3.2 Difference in estimated correlations from exact identification and overidentification

Figure 7 reports the difference in estimated correlations from exact identification and overidentification. The weights for exact identification are obtained by assigning every two assets an equal weight $(0.5, 0.5)$. The weights for overidentification are acquired by drawing every k assets and assigning them an equal weights $1/k$, where k is an integer from 2 to n, and n is the number of total assets. This finally generates $C(n,2) + ... + C(n,n) = 2^n - n - 1$ vectors of weights. Figure 7 indicates that overall, the difference in estimated correlations from exact identification and overidentification is very small and negligible compared to the values of estimated implied correlations. The figure shows that the VaR-implied correlation is affected more by estimation methods than is the ES-implied correlation.

Panel B of Table 4 reports the H and AH statistics measuring the difference in implied correlations from using exact identification and overidentification. All of the statistics are insignificant, implying no significant impact of estimation methods on implied correlations. The statistics from ES-implied correlations are still smaller than the statistics from VaR-implied correlations, implying that ES-implied correlations are less affected by using different estimation methods than VaR-implied correlations.

4.4 Simple illustrations for potential applications

Since asset correlations increase during market downturns, investors who diversify risk according to the linear correlation may underestimate the risk in the tails. Considering that investors care more about losses than gains in reality, the ES-implied correlation has important applications in risk management and asset allocation. A direct application is using the ES-implied correlation to assess the risk of a portfolio. For example, the volatility of a portfolio composed of 50% of the US equity index and 50% of the Canadian equity index is 1.22 using the linear correlation, but increases to 1.26 using the ES-implied correlation at 5%-quantile.

The ES-implied correlation also compensates the linear correlation in asset allocation. Since correlation increases when the market falls, the protection from risk diversification also erodes. For investors who care about risk diversification during market downturns, they can use the ES-implied correlation to construct portfolios. The solid line and dashed line in Figure 8 plot the classic efficient frontier, which uses the linear correlation, and the efficient frontier using the ES-implied correlation at 5%-quantile, respectively. It appears that investors using the ES-implied correlation are more risk sensitive: they demand higher expected returns for one percentage increase in volatility than their mean-variance counterparts.

One concern of applying the ES-implied correlation in asset allocation is that the ES-implied correlation requires pre-knowledge of weights. The following empirical analysis shows that weights for computing the ES-implied correlation do not have a significant impact on the results. Consider the efficient frontier between US and Canada. The ES-implied correlation equals 0.7920 using equal weights 50% and 50% at 5%-quantile. Assuming the annual risk-less rate is 3%, the tangency weights, which provide the highest Sharpe ratio, are 55.81% in the US equity index and 44.19% in the Canadian equity index. We can update this result. Using this newly updated tangency weights, the ES-implied correlation is 0.7924. The difference in the ES-implied correlation, 0.0004, is trivial. This new ES-implied correlation then leads to a new efficient frontier and a new set of tangency weights. I repeat this process until the difference in the ES-implied correlation is less than 10^{-7}. It appears that the ES-implied correlation converges after three iterations. The final tangency weight set is (55.28%,44.72%) and the corresponding ES-implied correlation is 0.7923. In this case, the starting weight vector (50%, 50%) is very close to the final optimal tangency weights. Even if the starting weight vector is (20%, 80%), the tangency weights still converge to the optimal weights after three iterations. Thus, not using the optimal weights to calculate the ES-implied correlation affects little of finding the optimal weights. Similar to section 4.3, I employ the H and AH statistics to test the impact on the ES-implied correlation between using and not using the optimal weights. Table 5 reports the difference in the implied correlation from using the optimal tangency weights and using equal weights. There is a significant difference in the VaR-implied correlation between the US and Germany, and between the US and Japan at 10% significance level but no discernible difference in the ES-implied correlation.

5 Conclusion

Tail-based dependence measures play a central role in risk management and asset allocation. It is well known that the linear correlation provides a poor indicator of the co-movement between financial assets under extreme circumstances, particularly during market crashes. Other dependence measures, the exceedance correlation and tail dependence coefficient, have conditioning biases and can not be used to measure correlation asymmetry directly. An alternative measure, the VaR-implied correlation, does not have a conditioning bias; however, it has a number of disadvantages, including the fact that the VaR is not coherent, disregards the data beyond it, and the VaR-implied correlation does not work around the center of distributions.

A development of the VaR-implied correlation, the ES-implied correlation, can address the shortcomings of the VaR-implied correlation. Simulations indicate that the estimated ES-implied correlation is as accurate as the estimated linear correlation when the estimated linear correlation is appropriate, but is much more accurate than the estimated linear correlation when the estimated linear correlation is inappropriate with respect to average bias, standard deviation and root-mean-square errors. The VaR-implied correlation is much less stable than the ES-implied correlation in all simulations.

In the empirical analysis of international equity indexes, the VaR-implied correlation violates the $[-1, 1]$ interval frequently. However, the ES-implied correlation is much more steady. More importantly, the ES-implied correlation clearly demonstrates that the linear correlation underestimates the correlation during market downturns and overestimates the dependence during market upturns. Thus using the linear correlation may underestimate risk and cause large losses when the market declines.

A series of test statistics are developed for measuring and testing correlation asymmetries. The test statistics involving the ES-implied correlation clearly demonstrate that correlations between the US and the other G7 countries are significantly underestimated by the linear correlation during market downturns.

The test statistics can also be used to measure and examine the impact of the choice of weights on the VaR-implied correlation and the ES-implied correlation. The test statistics imply that the implied correlations are overall independent of the choice of weights, suggesting the possibility of applying the risk measure-implied correlations in asset allocation.

In addition to risk management and asset allocation, the ES-implied correlation has another two potential applications. First, the standardized ES-implied correlation can be used to test or find the distribution of returns. If the assumption of the distribution is correct, the standardized ES-implied correlation should be close to the linear correlation. Thus a statistic measuring the difference between the linear correlation and the standardized ES-implied correlation can evaluate the accuracy of the hypothesis on the distribution of assets. By varying the hypothesis, one can find the true distribution of returns.

Second, the ES-implied correlation can be applied to measure the dependence of distributions of which the second moment does not exist, for example, the stable distribution. The linear correlation is no longer accessible in this situation, but the expected shortfall is accessible as long as the first moment exists. Thus we can estimate the correlation between the assets if we obtain the relation between the portfolio ES and the individual ES.

Appendix A Asymptotic properties of the implied correlations

This section shows the asymptotic property of the implied correlations. I start from the VaR-implied correlation.

Denote returns of asset i by $r_{i1},...,r_{iT}$. Assume that

1) $r_{i1},...,r_{iT}$ are i.i.d. random variables with the cumulative distribution function F_i and probability density function f_i. To have a simplified formula for the asymptotic distribution of the test statistics, I further assume that F_i is the normal distribution with mean 0 and the standard deviation σ_i, and the joint distribution between different assets follows the multivariate normal distribution. Notice that this assumption is not mandatory.

2) for $\alpha \in (0,1)$, $F_i(x)$ is differentiable at $F_i^{-1}(\alpha)$ and $F_i'(F_i^{-1}(\alpha)) = f_i(F_i^{-1}(\alpha)) > 0$.

I first derive the distribution of quantiles from the empirical distribution function. Notice that $\hat{F}_i(x) = \frac{1}{T}\sum_{t=1}^{T} I(r_{it} \le x)$ has the expectation of $F_i(x)$ and the variance of

$$\sigma_i^2(x) = \frac{F_i(x)(1-F_i(x))}{T}$$

The covariance between $\hat{F}_i(x)$ and $\hat{F}_j(x)$ is

$$\sigma_{ij}^2(x) = \frac{F_{ij}(x,x) - F_1(x)F_2(x)}{T}$$

where $F_{ij}(x,x)$ is the multivariate cumulative distribution function between r_i and r_j. It is obvious that $F_i(x)(1-F_i(x))$ and $F_{ij}(x,x) - F_i(x)F_j(x)$ are finite. By the multivariate central limit theorem, the empirical distributions of assets 1, 2 and the portfolio composed of them have the following multivariate distribution:

$$\sqrt{T}\begin{pmatrix} \hat{F}_1(x) - F_1(x) \\ \hat{F}_2(x) - F_2(x) \\ \hat{F}_p(x) - F_p(x) \end{pmatrix} \overset{a}{\sim} N(\mathbf{0'}, \Sigma).$$

where **0'** is a vector of three 0s, the (i,i)th element of Σ is $F_i(x)(1-F_i(x))$, and the (i,j)th element for $i \neq j$ is $F_{ij}(x,x) - F_i(x)F_j(x)$.

Now consider the α-quantile of asset i's cdf $q_{i,\alpha} = F_i^{-1}(\alpha)$. Notice that $F_i(q_{i,\alpha}) = \alpha$. Taking the derivative, we have $f_i(q_{i,\alpha})dq_{i,\alpha} = d\alpha$. Therefore, $\frac{dq_{i,\alpha}}{d\alpha} = \frac{1}{f_i(q_{i,\alpha})} = \frac{1}{f_i(F_i^{-1}(\alpha))}$. The empirical quantiles of assets 1, 2 and the portfolio p thus converge to the following distribution:

$$\sqrt{T}\left[\hat{q}_{1,\alpha} - q_{1,\alpha} \quad \hat{q}_{2,\alpha} - q_{2,\alpha} \quad \hat{q}_{p,\alpha} - q_{p,\alpha}\right]' \overset{a}{\sim} N\left(\mathbf{0'}, g'\Sigma_q g\right),$$

where g is a 3×3 diagonal matrix with the (i,i)th element being $\frac{1}{f_i(q_{i,\alpha})}$, and Σ_q is a matrix with the (i,i)th element being $\alpha(1-\alpha)$, and the (i,j)th element, where $i \neq j$, being $F_{ij}(q_{i,\alpha}, q_{i,\alpha}) - \alpha^2$.

Babu and Rao (1988) reached the same result using a representation of the sample quantiles from Bahadur (1966).

Since $q_{i,\alpha} = \sigma_i q_{\Phi,\alpha}$, where $q_{\Phi,\alpha}$ is the α-quantile of the standard normal distribution, the (i,i)th element of g thus can be simplified to $\frac{\sigma_i}{\phi(q_{\Phi,\alpha})}$, where ϕ is the pdf of the standard normal distribution. The estimate of the VaR-implied correlation is $\hat{\rho}_{VaR,\alpha} = \frac{\hat{q}_{p,\alpha}^2 - w_1^2\hat{q}_{1,\alpha}^2 - w_2^2\hat{q}_{2,\alpha}^2}{2w_1w_2\hat{q}_{1,\alpha}\hat{q}_{2,\alpha}}$ for α not around the center of the distribution such that $\hat{q}_{1,\alpha} \neq 0$ and $\hat{q}_{2,\alpha} \neq 0$. Using the Delta method, the estimate of the VaR-implied correlation follows

$$\sqrt{T}(\hat{\rho}_{VaR,\alpha} - \rho_{VaR,\alpha}) \overset{a}{\sim} N\Big(0, h'g'\Sigma_q gh\Big),$$

where $h = \begin{bmatrix} \frac{d\rho_{VaR,\alpha}}{dq_{1,\alpha}} & \frac{d\rho_{VaR,\alpha}}{dq_{2,\alpha}} & \frac{d\rho_{VaR,\alpha}}{dq_{p,\alpha}} \end{bmatrix}'$. The first element of h is $\frac{d\rho_{VaR,\alpha}}{dq_{1,\alpha}} = -\frac{w_1\sigma_1 + \rho w_2\sigma_2}{w_2\sigma_1\sigma_2 q_{\Phi,\alpha}}$, the second $\frac{d\rho_{VaR,\alpha}}{dq_{2,\alpha}} = -\frac{w_2\sigma_2 + \rho w_1\sigma_1}{w_1\sigma_2\sigma_1 q_{\Phi,\alpha}}$, and the third $\frac{d\rho_{VaR,\alpha}}{dq_{p,\alpha}} = \frac{\sigma_p}{w_1w_2\sigma_1\sigma_2 q_{\Phi,\alpha}}$.

Therefore, $gh = \frac{-1}{w_1w_2\sigma_1\sigma_2 q_{\Phi,\alpha}\phi(q_{\Phi,\alpha})}\begin{bmatrix} w_1\sigma_1(w_1\sigma_1 + \rho w_2\sigma_2) & w_2\sigma_2(w_2\sigma_2 + \rho w_1\sigma_1) & -\sigma_p^2 \end{bmatrix}'$, denoted by w. The estimate of the VaR-implied correlation then converges to the linear correlation ρ and has the following distribution

$$\sqrt{T}(\hat{\rho}_{VaR,\alpha} - \rho) \overset{a}{\sim} N\Big(0, w'\Sigma_q w\Big).$$

To deduce the asymptotic property of the H statistics related to the expected shortfall, I use Theorem 1 stated in Chen (2008). In addition to assumption 1), I assume:

3) There exists a $\tau \in (0,1)$ such that the α-mixing coefficient[8] $\alpha(k) \leq C\tau^k$ for all $k \geq 1$ and a positive constant C.[9]

4) The cdf of asset i, F_i, is absolutely continuous and the probability density f_i has continuous second derivatives in the neighborhood of the α-quantile; $E(|r_{it}|^{2+\delta}) \leq C$ for some $\delta > 0$ and a positive constant C.

Chen (2008) proved that the difference between the empirical estimate and the actual expected shortfall follows

$$\hat{ES}_{i,\alpha} - ES_{i,\alpha} = \left(\frac{1}{T\alpha}\sum_{t=1}^{T}(r_{it} - q_{i,\alpha})I(r_{it} \geq q_{i,\alpha}) - (ES_{i,\alpha} - q_{i,\alpha})\right) + o(T^{-3/4+k}),$$

where k is an arbitrary positive number. Therefore, the asymptotic variance of the empirical estimate equals the variance of the first term. The multivariate distribution of the empirical estimate

[8] Let $\mathcal{F}^{k,l}$ be the σ-algebra of events generated by $r_t, k \leq t \leq l$ for $k \leq l$. The α-mixing coefficient is $\alpha(k) = \sup\{|P(AB) - P(A)P(B)| : -\infty < j < \infty, A \in \mathcal{F}^{1,j}, B \in \mathcal{F}^{j+k,\infty}\}$.

[9] This assumption holds for many commonly used financial time series such as the ARMA, ARCH, the stochastic volatility and diffusion models.

of expected shortfalls of assets 1, 2, and the portfolio follows

$$\sqrt{T}\left[\hat{E}S_1(\alpha)-ES_1(\alpha) \quad \hat{E}S_2(\alpha)-ES_2(\alpha) \quad \hat{E}S_p(\alpha)-ES_p(\alpha)\right]' \overset{a}{\sim} N(\mathbf{0}', \Sigma_{ES})$$

where the (i, j)th element of Σ_{ES} is $\frac{1}{\alpha}cov\Big((r_{i1}-q_{i,\alpha})I(r_{i1} \geq q_{i,\alpha}), (r_{j1}-q_{j,\alpha})I(r_{j1} \geq q_{j,\alpha})\Big)$.

Using the Delta method, I obtain the asymptotic distribution of the ES-implied correlation as follows:

$$\sqrt{T}(\hat{\rho}_{ES,\alpha}-\rho) \overset{a}{\sim} N(0, m'\Sigma_{ES}m),$$

where $m = \frac{1}{w_1 w_2 \sigma_1 \sigma_2 ES_{\Phi,\alpha}}\left[-w_1\sigma_1(w_1+\rho w_2\sigma_2) \quad -w_2(w_2\sigma_2+\rho w_1\sigma_1) \quad \sigma_p\right]'$.

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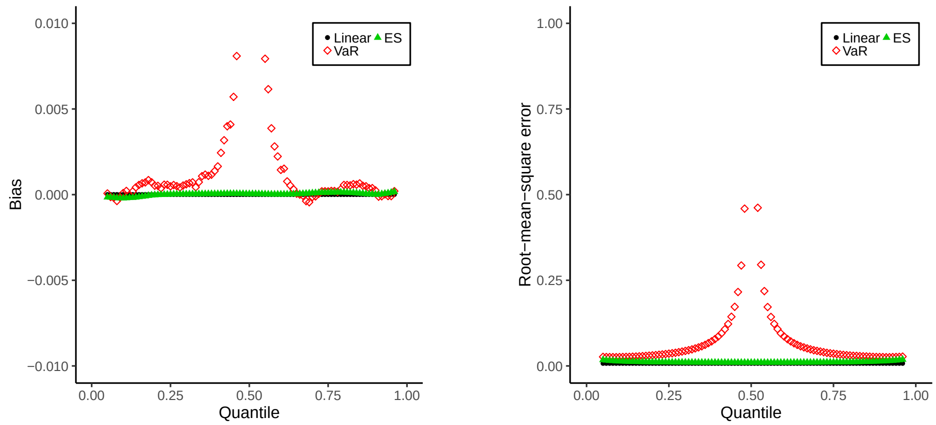
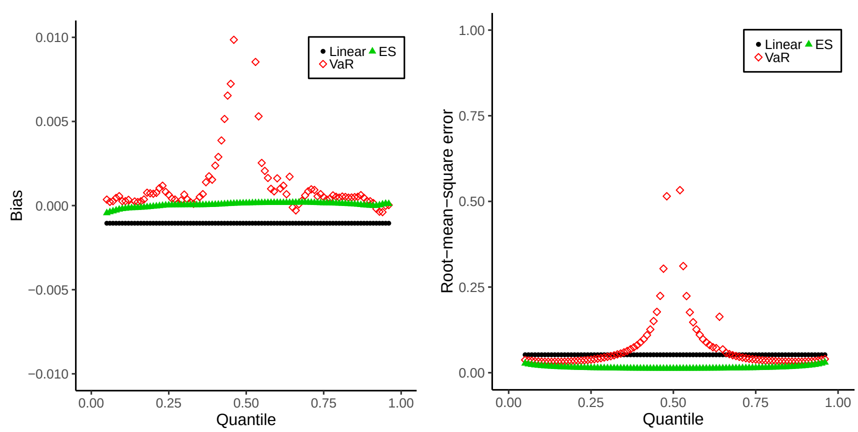


(a) Average bias, $T = 10^4$ (b) RMSE, $T = 10^4$

Figure 1: Average bias and RMSE in the multivariate normal distribution

This figure plots average bias (left panel) and RMSE (right panel) of the empirical linear correlation (black solid dots), VaR-implied correlation (red hollow rhombuses) and ES-implied correlation (green solid triangles) over 10^4 repetitions in the simulations of multivariate normal distribution with actual correlation $\rho = 0.5$ and sample size 10^4.



(a) Average bias, $T = 10^4$ (b) RMSE, $T = 10^4$

Figure 2: Average bias and RMSE in the multivariate T distribution

This figure plots average bias (left panel) and RMSE (right panel) of the empirical linear correlation (black solid dots), VaR-implied correlation (red hollow rhombuses) and ES-implied correlation (green solid triangles) over 10^4 repetitions in the simulations of the multivariate T distribution with the actual correlation $\rho = 0.5$, degrees of freedom 3 and sample size $T = 10^4$.

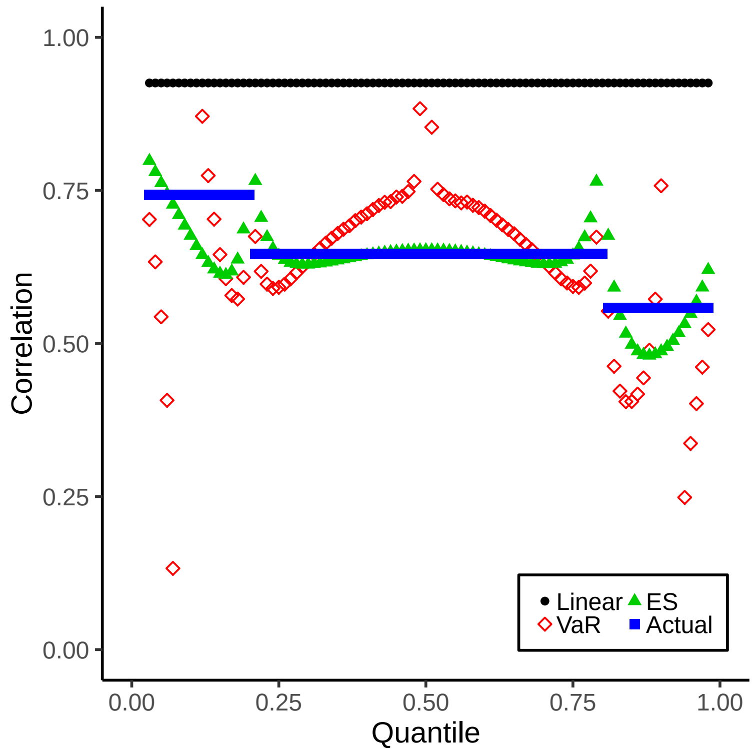
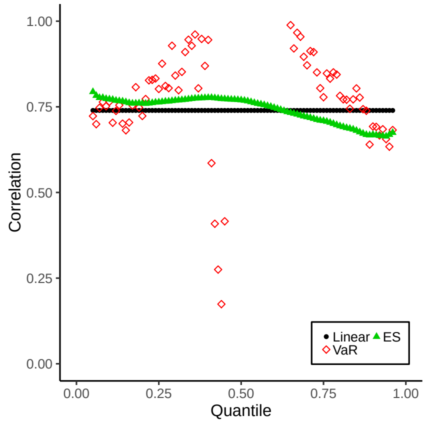
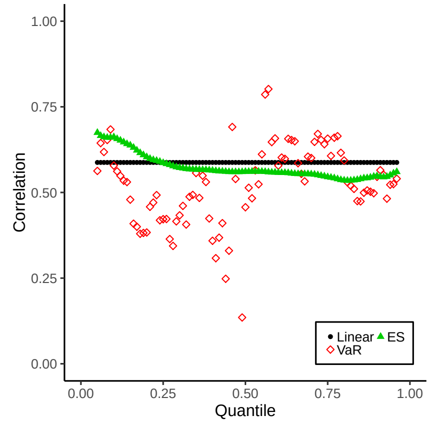


Figure 3: Estimated correlations in simulations of the non-constant model
This figure plots the average of the empirical linear correlation (black solid dots), VaR-implied correlation (red hollow rhombuses) and ES-implied correlation (green solid triangles) over 10^3 repetitions in the simulations of a non-constant model with sample size of 10^4. The solid squares plot the actual correlation, which varies at 20%-quantile and 80%-quantile, i.e.,

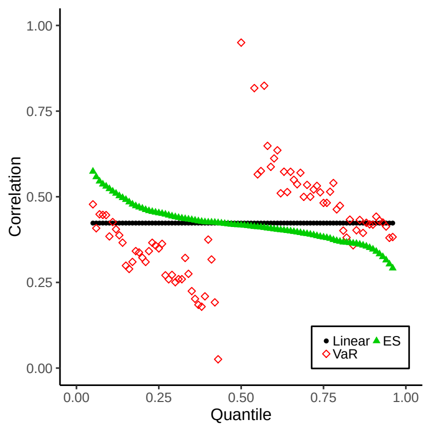
$$\rho_\alpha = \begin{cases} 0.75, & \alpha \leq 20\%, \\ 0.58, & 20\% < \alpha \leq 80\%, \\ 0.52, & \alpha > 80\%. \end{cases}$$



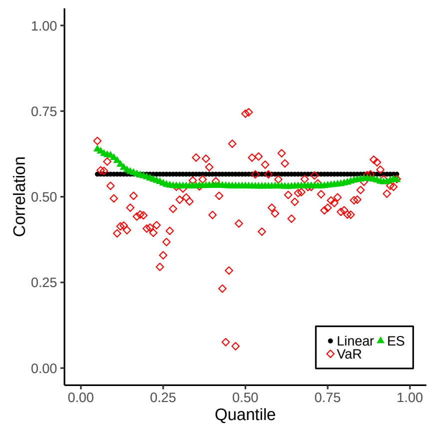
(a) US and CA



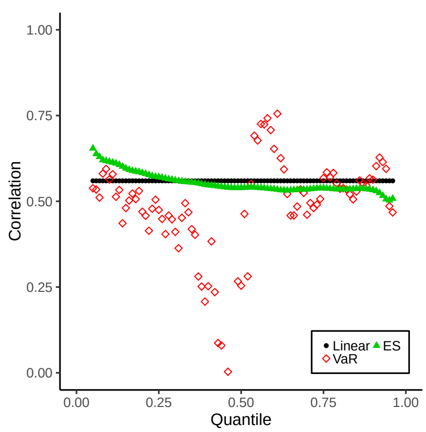
(b) US and UK



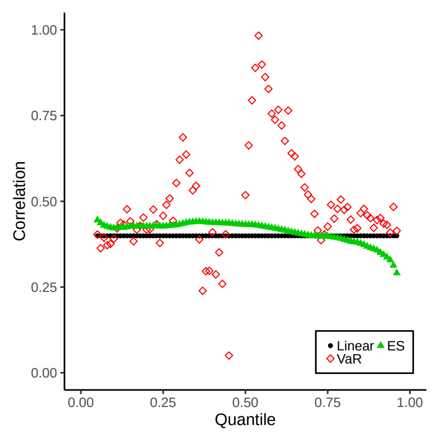
(c) US and IT



(d) US and DE

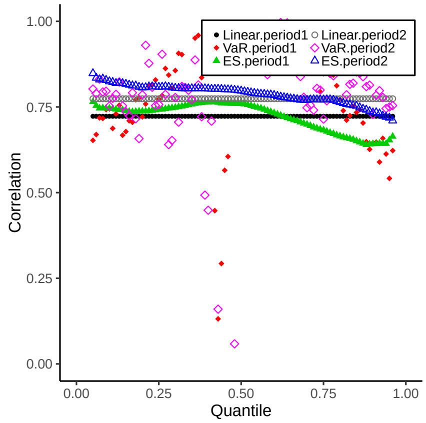


(e) US and FR

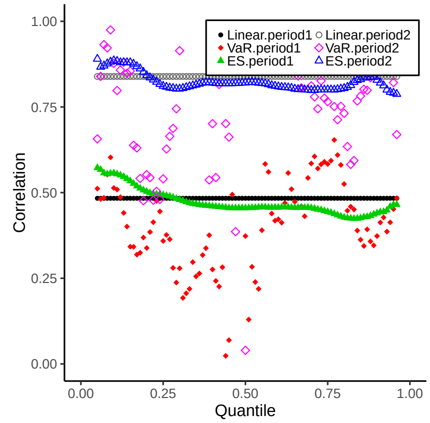


(f) US and JP

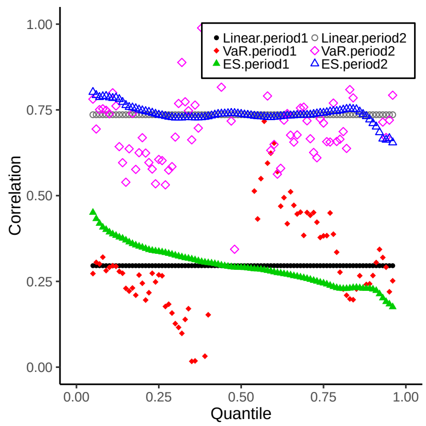
Figure 4: Estimated correlations between the US and the other G7 countries in the whole period
This figure plots the linear correlation (black solid dots), VaR-implied correlation (red hollow rhombuses) and ES-implied correlation (green solid triangles) between the US and Canada, the UK, Italy, Germany, France, and Japan in Panels (a) to (f), respectively. Correlations are computed using the weekly equity return indexes of G7 countries from January 1973 to December 2015.



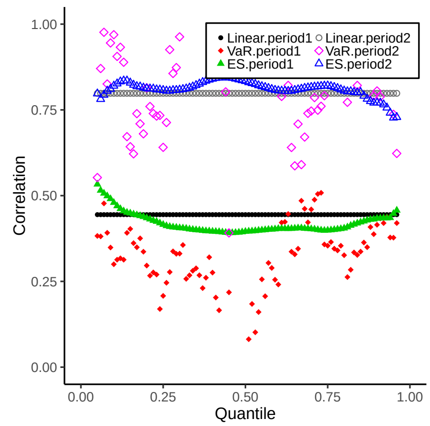
(a) US and CA



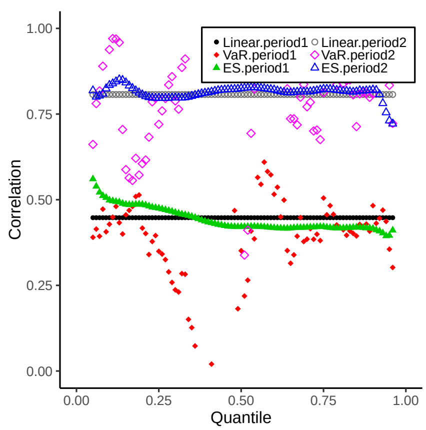
(b) US and UK



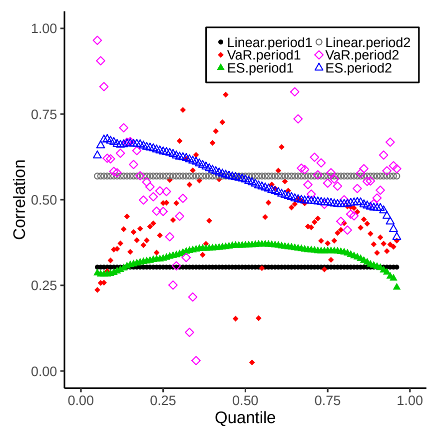
(c) US and IT



(d) US and DE

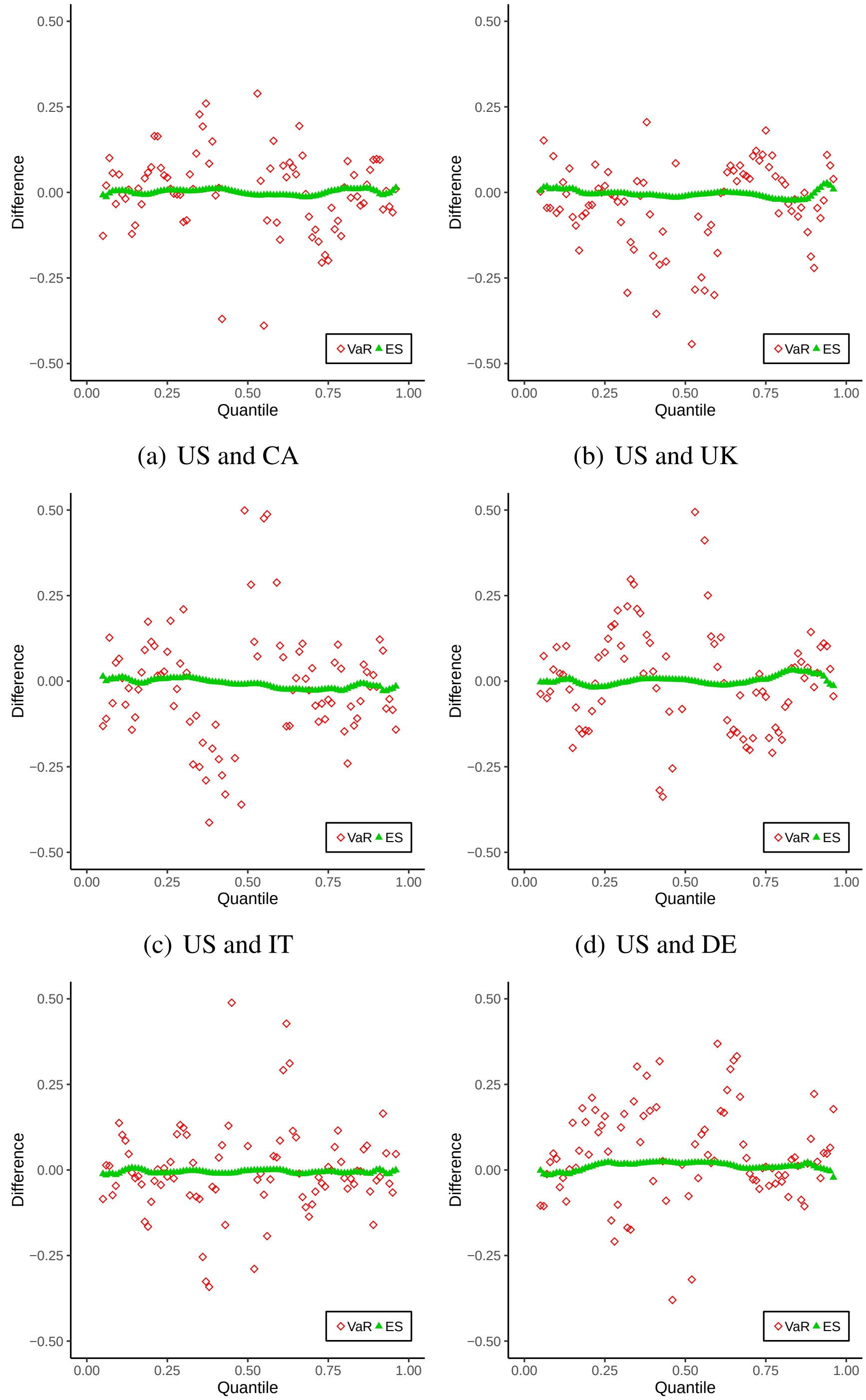


(e) US and FR



(f) US and JP

Figure 5: Estimated correlations between the US and the other G7 countries in subperiods
This figure plots the estimated linear correlation, VaR-implied correlation and ES-implied correlation between the US and the other G7 countries in two subperiods. The black solid dots, red hollow rhombuses and green solid triangles represent the results of the estimated linear correlation, VaR-implied correlation, and ES-implied correlation in the first subperiod from January 1973 to December 2002, respectively. The grey hollow dots, pink hollow rhombuses, and blue hollow triangles correspond to the estimated linear correlation, VaR-implied correlation, and ES-implied correlation in the second subperiod from January 2003 to December 2015.

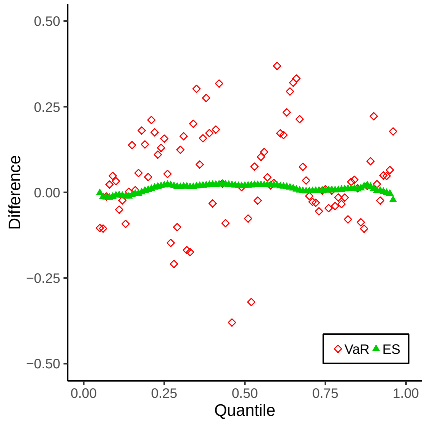


(a) US and CA (b) US and UK

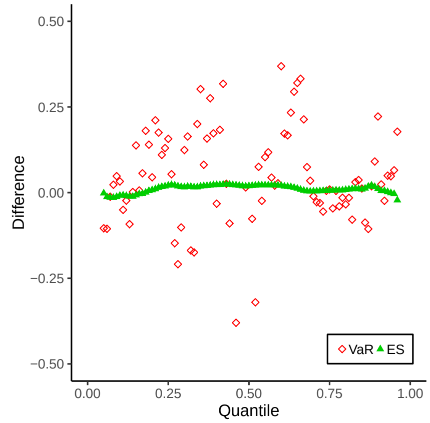
(c) US and IT (d) US and DE

(e) US and FR (f) US and JP

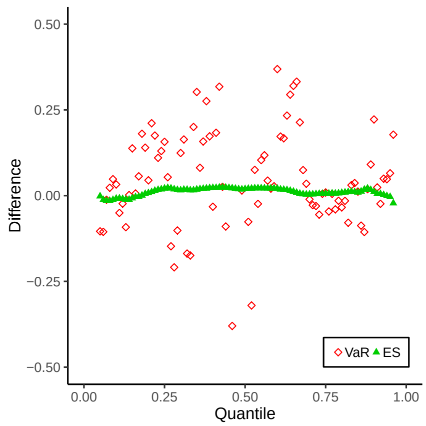
Figure 6: Difference in estimated correlations between the US and the other G7 countries from using different values of weights
This figure plots the difference in the VaR-implied correlation (red hollow rhombuses) and ES-implied correlation (green solid triangles) between the US and the other G7 countries from assigning equal weights and assigning 20% to the US and 80% to the other country. Correlations are computed using the weekly equity return indexes of G7 countries from January 1973 to December 2015.



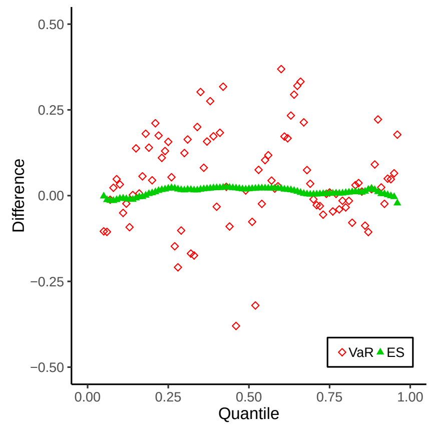
(a) US and CA



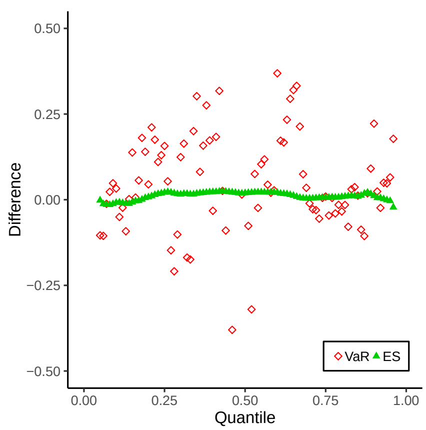
(b) US and UK



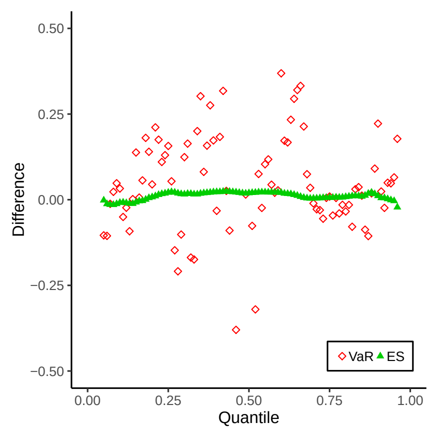
(c) US and IT



(d) US and DE



(e) US and FR



(f) US and JP

Figure 7: Difference in estimated correlations between the US and the other G7 countries from exact identification and overidentification

This figure plots the difference in the VaR-implied correlation (red hollow rhombuses) and the ES-implied correlation (green solid triangles) between the US and the other G7 countries from exact identification and overidentification. Correlations are computed using the weekly equity return indexes of G7 countries from January 1973 to December 2015.

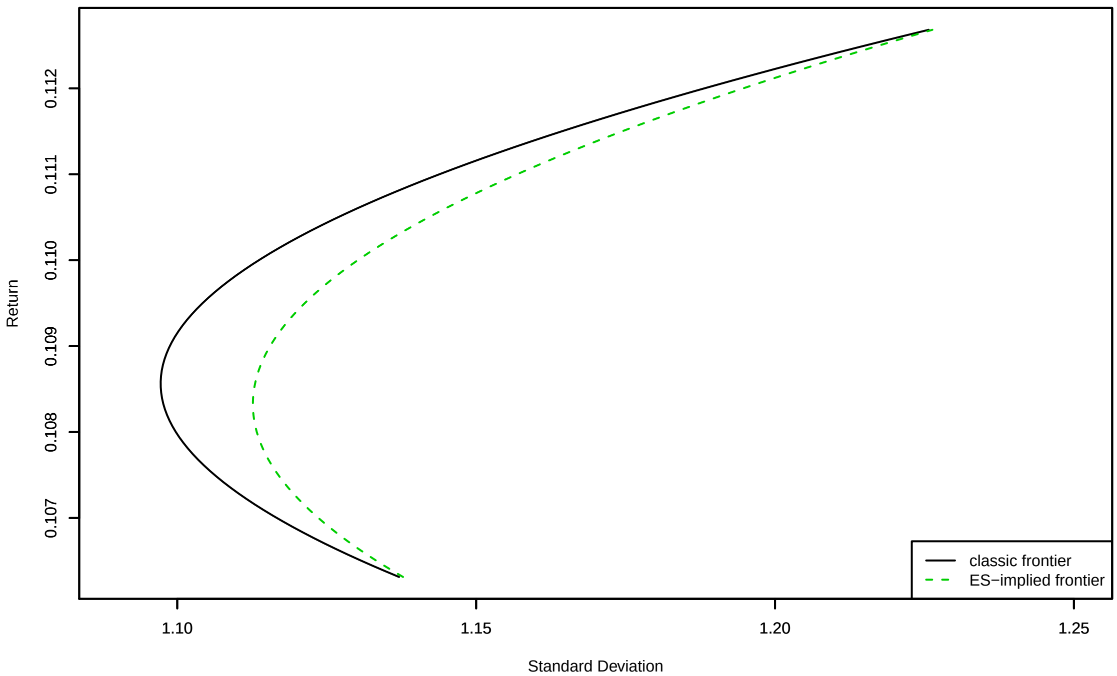


Figure 8: The classic mean-variance efficient frontier and the efficient frontier adjusted by the ES-implied correlation at 5%-quantile.
This figure plots the classic mean-variance efficient frontier (black solid line) and the efficient frontier using the ES-implied correlation at 5%-quantile (green dashed line) as the portfolio moves from the US equity index to the Canadian equity index. The data used are weekly equity return indexes of the two countries from January 1973 to December 2015.

Table 1: Summary statistics of the RMSE in the simulation of case 3

This table presents the summary statistics of the RMSE of the estimated correlations in the simulation of case 3. $100p_1\%$ of data falls into $(-\infty,-1]\times(-\infty,-1]$ with the correlation of 0.77, $100p_2\%$ of data falls into $(-1,1]\times(-1,1]$ with the correlation of 0.74, and $100p_3\%$ of data falls into $(1,\infty)\times(1,\infty)$ with the correlation of 0.66. Four sets of $p_1, p_2, \text{and} p_3$ are considered: $p_1 = 0, p_2 = 1, p_3 = 0$; $p_1 = 2\%, p_2 = 96\%, p_3 = 2\%$; $p_1 = 12\%, p_2 = 82\%, p_3 = 6\%$; and $p_1 = 12\%, p_2 = 76\%, p_3 = 12\%$. The quantiles of RMSE at probability level $\alpha = 1\%, 5\%, 10\%, 90\%, 95\%$ and 99% are also reported.

Statistics	Linear	VaR	ES	Statistics	Linear	VaR	ES
Situation 1: p_1=0, p_2=1, p_3=0				*Situation 2: p_1=2%, p_2=96%,p_3=2%*			
Mean	0.005	2.045	0.021	Mean	0.092	2.416	0.022
Std	0	19.100	0.013	Std	0.014	21.800	0.016
Min	0.005	0.022	0.008	Min	0.005	0.023	0.008
Max	0.005	190.200	0.094	Max	0.158	217	0.098
$\alpha = 1\%$	0.005	0.023	0.008	$\alpha = 1\%$	0.093	0.023	0.008
$\alpha = 5\%$	0.005	0.024	0.008	$\alpha = 5\%$	0.093	0.025	0.008
$\alpha = 10\%$	0.005	0.0273	0.009	$\alpha = 10\%$	0.093	0.030	0.009
$\alpha = 90\%$	0.005	0.225	0.032	$\alpha = 90\%$	0.093	0.283	0.033
$\alpha = 95\%$	0.005	0.353	0.033	$\alpha = 95\%$	0.093	0.551	0.042
$\alpha = 99\%$	0.005	4.926	0.093	$\alpha = 99\%$	0.093	12.300	0.093
Situation 3: p_1=12%, p_2=82%, p_3=6%				*Situation 4: p_1=12%, p_2=76%, p_3=12%*			
Mean	0.201	61.770	0.031	Mean	0.226	29.920	0.033
Std	0.034	59.49	0.029	Std	0.038	212.900	0.030
Min	0.119	0.021	0.009	Min	0.140	0.022	0.009
Max	0.274	5919	0.139	Max	0.295	2019	0.138
$\alpha = 1\%$	0.119	0.022	0.009	$\alpha = 1\%$	0.140	0.022	0.009
$\alpha = 5\%$	0.119	0.025	0.009	$\alpha = 5\%$	0.140	0.027	0.010
$\alpha = 10\%$	0.119	0.033	0.010	$\alpha = 10\%$	0.140	0.036	0.011
$\alpha = 90\%$	0.209	0.391	0.080	$\alpha = 90\%$	0.295	0.439	0.080
$\alpha = 95\%$	0.216	0.722	0.096	$\alpha = 95\%$	0.295	0.753	0.133
$\alpha = 99\%$	0.274	295.700	0.134	$\alpha = 99\%$	0.295	601.100	0.133

Table 2: Summary statistics of G7 countries

This table presents the summary statistics of annualized returns of G7 equity indexes. Panel A reports statistics of returns in the whole period from January 1973 to December 2015. Panel B reports statistics of returns in the first subperiod from January 1973 to December 2002 and Panel C reports statistics of the second period from January 2003 to December 2015. The superscripts *, ** and *** represent significance at 10%, 5%, and 1%, respectively.

Statistics	US	CA	UK	IT	DE	FR	JP
Panel A: Summary Statistics in whole period							
Mean	0.113	0.106	0.128	0.126	0.096	0.132	0.068
SD	1.230	1.139	1.325	1.689	1.264	1.433	1.363
Min	-7.808	-8.425	-8.461	-8.693	-7.482	-9.368	-10.080
Max	7.628	7.111	12.546	10.388	6.511	7.727	8.666
Skewness	-0.295***	-0.602***	0.227***	-0.020	-0.440***	-0.371***	-0.226***
Kurtosis	7.315***	8.482***	11.141***	5.579***	6.102***	6.033***	7.441***
AD test	14.640***	18.476***	19.608***	12.121***	15.201***	12.225***	17.574***
Shapiro test	0.957***	0.945***	0.934***	0.974***	0.964***	0.969***	0.958***
Panel B: Summary Statistics in period I							
Mean	0.117	0.111	0.143	0.150	0.086	0.144	0.059
SD	1.223	1.102	1.365	1.762	1.217	1.449	1.258
Min	-7.808	-6.725	-8.461	-8.693	-7.482	-9.368	-6.005
Max	6.627	5.293	12.546	10.388	6.317	6.718	8.305
Skewness	-0.360***	-0.495***	0.366***	0.054	-0.497***	-0.423***	0.044
Kurtosis	6.776***	6.309***	11.997***	5.540***	6.395***	6.062***	5.926***
AD test	7.732***	7.544***	12.916***	7.685***	9.135***	7.264***	12.928***
Shapiro test	0.964***	0.968***	0.930***	0.975***	0.963***	0.970***	0.967***
Panel C: Summary Statistics in period II							
Mean	0.103	0.096	0.094	0.072	0.118	0.102	0.090
SD	1.245	1.222	1.226	1.509	1.367	1.397	1.580
Min	-7.519	-8.425	-5.409	-6.590	-5.993	-5.999	-10.080
Max	7.628	7.111	8.090	6.957	6.511	7.727	8.666
Skewness	-0.151	-0.773***	-0.239**	-0.333***	-0.353***	-0.241**	-0.546***
Kurtosis	8.475***	11.584***	7.587***	5.181***	5.496***	5.953***	8.185***
AD test	8.009***	13.009***	7.231***	5.486***	6.140***	5.600***	4.596***
Shapiro test	0.937***	0.892***	0.946***	0.969***	0.964***	0.962***	0.949***

Table 3: *H* statistics measuring correlation asymmetries

This table reports the *H* and *AH* statistics. Panel A reports downside and upside *H* statistics based on the VaR-implied and the ES-implied correlations, where downside and upside *H* statistics are the supremums of the deviations of the linear correlation from implied correlations in the left tail and right tail, respectively. Panel B reports downside and upside *AH* statistics, which correspond to the average of deviations in the left tail and right tail, respectively. Correlations are computed using the weekly equity return indexes of G7 countries from January 1973 to December 2015. The superscripts *, ** and *** represent significance at 10%, 5%, and 1%, respectively.

	CA	UK	IT	DE	FR	JP
			Panel A: *H* statistics			
H^-_{VaR}	0.167	0.134	0.335***	0.186*	0.295***	0.219**
H^-_{ES}	0.054	0.212***	0.221***	0.143***	0.105**	0.048
H^+_{VaR}	0.111	0.157	0.181**	0.119	0.111	0.185**
H^+_{ES}	0.106**	0.092*	0.215***	0.036	0.056	0.247***
			Panel B: *AH* statistics			
AH^-_{VaR}	0.033**	-0.084	-0.036	-0.092	-0.029	0.045**
AH^-_{ES}	0.030***	0.050***	0.080***	0.022*	0.040***	0.027*
AH^+_{VaR}	-0.025	0.027	-0.012	0.048***	0.021	-0.043
AH^+_{ES}	0.057***	0.046***	0.080***	0.022*	0.030**	0.045***

Table 4: H statistics measuring impact of weights

This table reports H and AH statistics measuring the impact of weights. The H statistics are the supremums of the absolute difference between using different choices of weights in the tails. AH is the average of the absolute difference across the tails. Panel A reports the difference in implied correlations from choosing different values for portfolio weights. Panel B reports the difference in implied correlations from using exact identification and overidientification. The superscripts *, ** and *** represent significance at 10%, 5%, and 1%, respectively. Correlations are computed using the weekly equity return indexes of G7 countries from January 1973 to December 2015.

	CA	UK	IT	DE	FR	JP
Panel A: Difference from different values of weights						
H_{VaR}	0.130	0.213	0.192	0.140	0.199	0.187
H_{ES}	0.026	0.080	0.056	0.061	0.037	0.077
AH_{VaR}	0.046	0.063	0.058	0.057	0.049	0.080**
AH_{ES}	0.012	0.009	0.010	0.009	0.007	0.020
Panel B: Difference from exact and overidentification						
H_{VaR}	0.098	0.063	0.100	0.064	0.090	0.083
H_{ES}	0.045	0.025	0.033	0.013	0.018	0.036
AH_{VaR}	0.018	0.023	0.026	0.019	0.021	0.031
AH_{ES}	0.007	0.004	0.004	0.004	0.002	0.005

Table 5: H statistics measuring the impact of using optimal weights

This table reports H and AH statistics measuring the difference in implied correlations from using equal weights and using the optimal tangency weights in the mean-variance framework with variance constructed by implied correlations. The superscripts *, ** and *** represent significance at 10%, 5%, and 1%, respectively. Correlations are computed using the weekly equity return indexes of G7 countries from January 1973 to December 2015.

	CA	UK	IT	DE	FR	JP
H_{VaR}	0.045	0.108	0.253	0.346*	0.101	0.482*
H_{ES}	0.002	0.027	0.040	0.042	0.033	0.060
AH_{VaR}	0.009	0.031	0.083	0.104*	0.032	0.176*
AH_{ES}	0.001	0.008	0.013	0.014	0.006	0.018