p-Coloring Classes of Torus Knots

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Abstract

We classify by elementary methods the p-colorability of torus knots, and prove that every p-colorable torus knot has exactly one nontrivial p-coloring class. As a consequence, we note that the two-fold branched cyclic cover of a torus knot complement has cyclic first homology group.

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1 Introduction

Our first result is a theorem specifically determining the p-colorability of any (m, n) torus knot. It has been previously shown that a $(m, m-1)$ torus knot is always p-colorable for p equal to m or $m-1$ depending on which is odd (see [6] and [13]). Another proven result is that a $(2, n)$ torus knot is always p-colorable for p equal to n and a $(3, n)$ torus knot is always 3-colorable if n is even [13]. A result similar to ours was also stated as a lemma without proof in [3].

Our second result shows that any p-colorable (m, n) torus knot has only one nontrivial p-coloring class. A general result investigating colorings of torus knots by finite Alexander quandles appears in [2]. Our result is a special instance of this result; however, we present a proof using only elementary techniques. p-coloring classes have also previously been investigated in relationship to pretzel knots by [4]. An immediate corollary of this result is that any nontrivial p-coloring of the standard braid representation of $T_{m,n}$ must use all p colors. Distribution of colors in p-colorings of knots has been previously investigated with the Kauffman-Harary Conjecture, which examines the distribution of colors in a p-coloring of an alternating knot with prime determinant. Asaeda, Przytcki, and Sikora prove the Harary-Kauffman Conjecture is true for pretzel knots and Montesinos knots in [1]. Another immediate corollary is that the first homology groups of certain branched cyclic covers of torus knot complements (see [5], [11]) are cyclic.

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2 Notation

Given a prime $p > 2$ and a projection of a knot K with strands $s_1, s_2, \ldots, s_r$, a *p-coloring* is an assignment $c_1, c_2, \ldots c_r$ of elements of $\mathbb{Z}_p$ to the strands of the projection that satisfies the condition that at each crossing where s_i, s_j are the undercrossing strands and s_k is the overcrossing strand, $c_i + c_j - 2c_k = 0 \bmod p$. A p-coloring is said to be *nontrivial* if at least two distinct "colors" in $\mathbb{Z}_p$ are used. p-colorability is invariant under Reidemeister moves and is thus a knot invariant, so a knot K is p-colorable if its projections admit non-trivial p-colorings.

Equivalently, a knot K is p-colorable if there exists an onto homomorphism from the knot group $\pi_1(S^3 - K)$ of K to the dihedral group

$$D_{2p} = \langle a, b \mid a^2 = 1, b^p = 1, abab = 1\rangle.$$

The knot group G can be expressed via the *Wirtinger presentation*, as follows. Given a projection of K, define loops $x_1, x_2, \ldots, x_r$ around the strands $s_1, s_2, \ldots, s_r$, respectively, following the right hand rule. At each crossing where s_i terminates, s_j originates, and s_k is the overcrossing strand we have a relation R_j that is either of the form $x_j = x_k^{-1} x_i x_k$ or $x_j = x_k x_i x_k^{-1}$, depending on whether the sign of the crossing is positive or negative (see Section D of chapter 3 of [12]). With this notation, the Wirtinger presentation for the knot group of K is:

$$\pi_1(S^3 - K) = \langle x_1, x_2, \ldots, x_r \mid R_1, R_2, \ldots, R_r\rangle.$$

It is a simple exercise to prove that an assignment of colors $c_1, c_2, \ldots c_r \in \mathbb{Z}_p$ is a proper p-coloring of a projection of K if and only if the map $\theta \colon \pi_1(S^3 - K) \to D_{2p}$ defined by $\theta(x_i) = ab^{c(s_i)}$ is an onto homomorphism (see p.122 of [8]).

Two p-colorings $c_1, c_2, \ldots, c_r$ and $d_1, d_2, \ldots, d_r$ of a projection of K are said to be *equivalent*, or in the same *p-coloring class* if for all $1 \le i, j \le r$, $c_i = c_j$ if and only if $d_i = d_j$; in this case we say that the two p-colorings differ only by a permutation of the colors. This definition of p-coloring classes corresponds directly to the $\bmod p$ rank discussed in chapter 3 of [9].

3 p-Colorability of Torus Knots

Let $T_{m,n}$ represent the torus knot characterized by the number of times m that it circles around the meridian of the torus and the number of times n that it circles around the longitude of the torus. $T_{m,n}$ has one component if and only if m and n are relatively prime. It is well-known that every knot is the closure of some braid (see chapter 3 of [9]). For example, the trefoil knot $T_{3,2}$ is the closure of the braid $(\sigma_1\sigma_2)^2$ shown in Figure **??**, where σ_i represents a crossing where string $i - 1$ crosses over string i. In general, the torus knot $T_{m,n}$ can be realized as the closure of the braid word $(\sigma_1\sigma_2 \cdots \sigma_{m-2}\sigma_{m-1})^n$.

Since $T_{m,n}$ is equivalent to $T_{n,m}$, the following theorem completely characterizes the p-colorability of torus knots. Note that if $T_{m,n}$ has one component then m and n cannot both be even. Results similar to those in Theorem 1 were

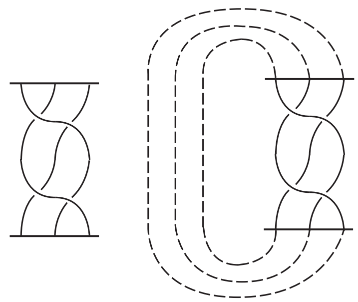

Figure 1: The braid $(\sigma_1\sigma_2)^2$ and its closure $T_{3,2}$.

stated without proof by Asami and Satoh in [3]. We present an elementary proof here.

Theorem 1. *Suppose $T_{m,n}$ is a torus knot and p is prime.*
i) If m and n are both odd, then $T_{m,n}$ is not p-colorable.
ii) If m is odd and n is even, then $T_{m,n}$ is p-colorable if and only if $p|m$.

Proof. If p is prime, then a knot K is p-colorable if and only if p divides $\det(K)$ (see chapter 3 of [9]). We will show that the determinant of $T_{m,n}$ is

$$\det(T_{m,n}) = \begin{cases} 1, & \text{if } m \text{ and } n \text{ are both odd} \\ m, & \text{if } m \text{ is odd and } n \text{ is even.} \end{cases}$$

By [10] we have $\det(K) = |\Delta_K(-1)|$, where $\Delta_K(t)$ is the Alexander polynomial of K. The Alexander polynomial for $T_{m,n}$ is given by (see part C of chapter 9 of [5])

$$\Delta_{T_{m,n}}(t) = \frac{(t^{mn}-1)(t-1)}{(t^m-1)(t^n-1)}.$$

Therefore if m and n are both odd, we have $\Delta_{T_{m,n}}(-1) = \frac{(-2)(-2)}{(-2)(-2)} = 1$. If m is odd and n is even, then by L'Hôpital's rule we have $\Delta_{T_{m,n}}(-1) = \frac{(mn+1)+mn-1}{(m+n)-m+n} = m$. □

4 p-Coloring Classes of Torus Knots

Our second theorem is a special case of a result found by Asami and Kuga in [2]. They prove that if a knot $T_{m,n}$ can be p-colored using a finite Alexander quandle, it has a total of p^2 trivial and non-trivial colorings. If $T_{m,n}$ cannot be colored by such a quandle, then it has only the p trivial colorings. It is important to note that Asami and Kuga only consider the total number of all p-colorings without distinguishing between equivalent colorings, while we consider equivalence classes of p-colorings, or p-coloring classes. Also note that the proof of Theorem 2 will show that every non-trivial p-coloring of the standard braid projection of $T_{m,n}$ must use all p colors.

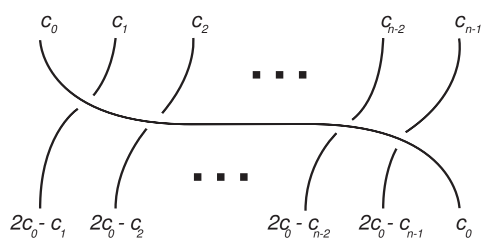

Figure 2: The action of ϕ on the j^{th} color array of $T_{m,n}$.

Theorem 2. *If p is prime and $T_{m,n}$ is p-colorable, then $T_{m,n}$ has only one nontrivial p-coloring class.*

Proof. If $T_{m,n}$ is p-colorable, then by Theorem 1 can assume without loss of generality that we have m odd, n even, and $p|m$. Given a p-coloring of $T_{m,n}$ in the standard m-stand braid projection, let its j^{th} *color array* be the element of $(\mathbb{Z}_p)^m$ whose i^{th} component is the color of the i^{th} strand of the braid representation of $T_{m,n}$ after j cycles. The map $\phi : (\mathbb{Z}_p)^m \to (\mathbb{Z}_p)^m$ defined by

$$\phi(c_0, c_1, \ldots, c_{m-1}) = (2c_0 - c_1, 2c_0 - c_2, \ldots, 2c_0 - c_{m-1}, c_0)$$

describes the transistion from the j^{th} to the $(j+1)^{\text{st}}$ color array of α according the rules of p-colorability, as seen in Figure 1. Note that a p-coloring of $T_{m,n}$ is entirely determined by its initial color array, and that to have a proper p-coloring it is necessary and sufficient that ϕ^n fixes this initial color array.

Now since p divides the number m of braid strands in our projection of $T_{m,n}$ we can consider the 0^{th} color array that consists of the colors $0, 1, \ldots, p-1$ listed in order $\frac{m}{p}$ times:

$$C_0 = (0, 1, \ldots, p-2, p-1,\ 0, 1, \ldots, p-2, p-1,\ \ldots, p-2, p-1).$$

Under the action of ϕ, the 1^{st} color array is clearly

$$\phi(C_0) = (p-1, p-2, \ldots, 1, 0,\ p-1, p-2, \ldots, 1, 0,\ \ldots, 1, 0),$$

and the 2^{nd} color array is

$$\phi^2(C_0) = (0, 1, \ldots, p-2, p-1,\ 0, 1, \ldots, p-2, p-1,\ \ldots, p-2, p-1).$$

Since ϕ^2 fixes C_0 and n is even, we know that ϕ^n fixes C_0 and thus the initial color array C_0 induces a nontrivial p-coloring.

Note that the p-coloring constructed above has the property that its initial color array $c_0, c_1, c_2, \ldots, c_{m-1}$ has *constant variance* of 1, since $c_{j+1} - c_j = 1$ for all $0 \leq j \leq m$ (with indices mod m). It can be shown by elementary, but tedious, methods that if $T_{m,n}$ is a one-component link then any p-coloring of its standard m-strand braid projection will have constant variance (not necessarily equal to 1), and that all such p-colorings are equivalent to the p-coloring constructed above. Thankfully, the reviewer for this paper suggested a much more elegant method of proving that up to equivalence there can be no more than one nontrivial p-coloring of $T_{m,n}$, as follows.

Seeking a contradiction, suppose that there are two non-equivalent nontrivial p-colorings $c_1, c_2, \ldots, c_r$, $d_1, d_2, \ldots, d_r$ of a projection of $T_{m,n}$ with strands $s_1, s_2, \ldots, s_r$. We will show that these colorings induce what we can think of as a $\mathbb{Z}_p \oplus \mathbb{Z}_p$-coloring $(c_1, d_1), \ldots (c_r, d_r)$ of the strands of the projection in the sense that we have an onto homomorphism from the knot group θ from $\pi_1(S^3 - K)$ to the generalized Dihedral group $\mathcal{D} = \langle a, b \mid a^2 = 1, b \in \mathbb{Z}_p \oplus \mathbb{Z}_p, abab = 1\rangle$.

In the notation above, and writing $b = (b_1, b_2)$, define $\theta(x_i) = a(b_1^{c_i}, b_2^{d_i})$. We will show that θ is onto by showing that a, $(b_1, 1)$, and $(1, b_2)$ are in its image. By non-equivalence there must exist some i, j such that either $c_i = c = j$ but $d_i \neq d_j$, or $d_i = d_j$ but $c_i \neq c_j$. Without loss of generality we will assume the former. With this i, j it is a simple exercise to show that $\phi(x_i x_j) = (1, b_2^{d_j - d_i})$. Since $d_j - d_i \neq 0$ and p is prime, some power of this element is $(1, b_2)$. Now since $c_1, c_2, \ldots, c_r$ is a nontrivial p-coloring there must exist some k such that $c_i \neq c_k$, and for this i, k we have $\phi(x_i x_k) = (b_1^{c_k - c_i}, b_2^{d_k - d_i})$. The product of this element with $(1, b_2)^{d_i - d_k}$ is $(b_1^{c_k - c_i}, 1)$, and again since $c_k - c_i \neq 0$ and p is prime, some power of this is $(b_1, 1)$. We now see immediately that a is in the image of θ since $\theta(x_i)(b_1^{-c_i}, b_2^{-d_i}) = a$.

The existence of this onto map θ provides a contradiction to there being two non-equivalent p-colorings, as follows. It is well-known (see p.58 of [5]) that $\pi_1(S^3 - T_{m,n}) = \langle x, y \mid x^m = y^n\rangle$, and that the center Z of this group is generated by x^m. Since θ is onto and thus carries centers into centers, $\theta(Z)$ is contained in the center of $\mathcal{D}$, which is trivial since p is odd. Therefore $Z \in \ker\theta$, and thus the map θ factors through the group $\langle x, y \mid x^m = 1, y^m = 1\rangle$. This induces a map $\beta\colon \langle x, y \mid x^m = 1, y^m = 1\rangle \to \mathcal{D}$, which must be onto since θ is onto. But there can be no onto homomorphism from a free product of two cyclic groups to a group whose presentation requires at least three generators. Therefore there can be only one nontrivial p-coloring class for $T_{m,n}$, namely the one we constructed above. □

Notice that the proof of Theorem 2 shows that any p-coloring of the standard minimal projection of a torus knot must use all p colors. In particular, this gives another proof that torus knots of the form $T_{p,2}$ for p an odd prime satisfy the Kauffman-Harary conjecture (6.2 in [7]); such torus knots are alternating with determinant p, and the least number of colors needed to nontrivially color a minimal projection of $T_{p,2}$ will be equal to the crossing number p.

The reviewer for this paper pointed out to the authors that another immediate consequence of our elementary result in Theorem 2 is that the first homology groups of certain q-fold branched cyclic covers of torus knot complements are cyclic. The requirement that this homology group be cyclic in the two-fold case has been suggested as a weaker hypothesis for the Kauffman-Harary conjecture (p.7 of [1]). Given a torus knot $T_{m,n}$, let $\widehat{C}^q_{m,n}$ denote the q-fold branched cyclic cover of $S^3 - T_{m,n}$ (see 8.18 in [5]).

Corollary 3. *If $q = 2+kmn$ for some nonnegative integer k, then the homology group $H_1(\widehat{C}^q_{m,n})$ is cyclic.*

Proof. By 14.8 in [5], $T_{m,n}$ is p-colorable for some prime p if and only if p divides $|H_1(\widehat{C}^2_{m,n})|$. For each such prime p, Theorem 2 shows that there is only one nontrivial p-coloring class, which in turn guarantees that the 2-fold branched cyclic cover of the knot complement contains only one subgroup of order p (see Section 3 of [11]). Since this is true for all primes p that divide $|H_1(\widehat{C}^2_{m,n})|$, the result follows in the $q = 2$ (i.e. $k = 0$) case. The general result now follows from the fact that the q-fold coverings have period mn (see 6.15 in [5]), i.e. $\widehat{C}^q_{m,n} \cong \widehat{C}^{q+kmn}_{m,n}$ for any nonnegative integer k. □

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