GENERALIZED DOUBLE BINOMIAL SUMS FAMILIES BY GENERATING FUNCTIONS

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ABSTRACT. We consider various double binomial sums related with certain second, third and fourth order recursions. Moreover a new binomial sums with complex coefficients related with a generalized second order recursion is derived. The generating function methods are used to prove them. Also some interesting examples are given.

1. Introduction, linear recurrence sequences and generating functions

Let $\{U_n(p,q)\}$, or briefly $\{U_n\}$, be the second order linear recurrence defined, for $n>1$, by

$$U_n = pU_{n-1} + qU_{n-2}, \tag{1.1}$$

where $U_0 = 0$ and $U_1 = 1$.

Let $\{V_n(p,q,r)\}$, or briefly $\{V_n\}$, be the third order recursion defined, for $n>1$, by

$$V_n = pV_{n-1} + qV_{n-2} + rV_{n-3}, \tag{1.2}$$

where $V_{-1} = V_0 = 0$ and $V_1 = 1$.

Let $\{W_n(p,q,r,s)\}$, or briefly $\{W_n\}$, be the fourth order recursion defined, for $n>1$, by

$$W_n = pW_{n-1} + qW_{n-2} + rW_{n-3} + sW_{n-4}, \tag{1.3}$$

where $W_{-2} = W_{-1} = W_0 = 0$ and $W_1 = 1$.

The generating functions of these sequences are respectively

$$\begin{aligned}\sum_{n=0}^{\infty} U_{n+1}x^n &= \frac{1}{1-px-qx^2},\\ \sum_{n=0}^{\infty} V_{n+1}x^n &= \frac{1}{1-px-qx^2-rx^3},\end{aligned}$$

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$$\sum_{n=0}^{\infty} W_{n+1}x^n = \frac{1}{1-px-qx^2-rx^3-sx^4}.$$

For some special cases of these number sequences, we recall the following double binomial sums from [1]:

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n}\binom{n-i}{j-1}\binom{n-j}{i-1} &= F_{2n}, \\
\sum_{0\le i\le j\le n}\binom{n-i}{j}\binom{j}{i} &= P_n, \\
\sum_{0\le j\le i\le n}\binom{n-i}{i-j}\binom{i-j}{j} &= T_n, \\
\sum_{0\le i_k\le\cdots\le i_1\le n}\binom{n-i_1}{i_1-i_2}\binom{i_1-i_2}{i_2-i_3}\cdots\binom{i_{k-1}-i_k}{i_k} &= f_n^{(k)},
\end{aligned}$$

where $F_n = U_n(1,1)$, $P_n = U_n(2,1)$, $T_n = V_n(1,1,1)$ and $f_n^{(k)}$ stand for the n-th Fibonacci, Pell, Tribonacci and generalized order-k Fibonacci number defined by

$$f_n^{(k)} = \sum_{i=1}^{k} f_{n-i}^{(k)} \quad (n>k),$$

with initial conditions $f_0^{(k)} = 0$, $f_j^{(k)} = 2^{j-1}$ for $1 \le j \le k$.

In this paper, we shall derive various new double binomial sums and a binomial sums with complex coefficients related with the sequences $\{U_n\}$, $\{V_n\}$ and $\{W_n\}$. We use generating function methods to prove our results.

We can refer to [3, 4] for using generating functions in deriving and proving certain combinatorial identities.

The main advantage of such an interpretation of binomial coefficients is, for example, that one can omit the use of exact limits in sums like $\sum_{i=0}^{u}\binom{u}{i}$ by simply writing $\sum_i\binom{u}{i}$ instead. In the sequel, for the sake of convenience, we exploit this kind of allowance. For example, for n and s integers $\sum_{i,j}\binom{n-i}{sj}\binom{n-sj}{i}$, the nonvanishing terms are those for which $0 \le i \le n$, $0 \le sj \le n$ and $0 \le i+sj \le n$.

We need the following Lemma.

Lemma 1. *Let α and β be integers such that $\beta \ge \alpha$. The following identity holds*

$$\sum_{n\ge 0}\binom{n+\alpha}{\beta} z^n = \frac{z^{\beta-\alpha}}{(1-z)^{\beta+1}} \tag{1.4}$$

Proof.

$$\sum_{n\geq 0}\binom{n+\alpha}{\beta}z^n=\left(\sum_{n\geq 0}\binom{n+\beta}{\beta}z^n\right)z^{\beta-\alpha}=\frac{z^{\beta-\alpha}}{(1-z)^{\beta+1}}.$$

□

2. Double Binomial Sums of first kind

This section is devoted to some combinatorial identities related to double sums of binomial coefficient with a given parameter. In the sequel, we generalize most of the identities given by Kılıç and Prodinger in [1].

Theorem 1. *Let n and s be positive integers, and t and u any complex numbers. The generating function of the sequence $\left\{A_n^{(s)}\right\}_n$ defined by*

$$A_n^{(s)}:=\sum_{i,j}\binom{n-i}{sj}\binom{n-sj}{i}t^iu^j$$

is

$$A(z):=\sum_{n\geq 0}A_n^{(s)}z^n=\frac{(1-(1+t)\,z)^{s-1}}{(1-(1+t)\,z)^s-uz^s\,(1-tz)^s}.$$

Proof. First, we replace i by $n-i$ and get

$$A_n^{(s)}=\sum_{i,j}\binom{i}{sj}\binom{n-sj}{i-sj}t^{n-i}u^j.$$

Now, we compute the generating function using Lemma 1:

$$\begin{aligned}A(z)&=\sum_{0\leq sj\leq i}\binom{i}{sj}z^iu^j\sum_{n\geq i}\binom{n-sj}{i-sj}(tz)^{n-i}\\&=\sum_{0\leq sj\leq i}\binom{i}{sj}\frac{z^iu^j}{(1-zt)^{i+1-sj}}\\&=\sum_{j\geq 0}\frac{(z^su(1-zt)^s)^j}{(1-(t+1)\,z)^{sj+1}}\\&=\frac{(1-(1+t)\,z)^{s-1}}{(1-(1+t)\,z)^s-uz^s\,(1-tz)^s}.\end{aligned}$$

□

Corollary 1. *For $n\geq 0$ and any complex numbers t, u,*

$$U_{n+1}(t+u+1,-ut)=\sum_{i,j}\binom{n-i}{j}\binom{n-j}{i}t^iu^j, \tag{2.1}$$

where $\{U_n\}$ *is defined by relation (1.1).*

Proof. It suffices to take $s = 1$. We obtain the generating function of the sequence $\{U_{n+1}(t + u + 1, -ut)\}$. □

We get for

(1) $t = u = 1$
$$\sum_{i,j} \binom{n-i}{j}\binom{n-j}{i} = F_{2n+2},$$
(2) $t = 4$ and $u = 1$
$$\sum_{i,j} \binom{n-i}{j}\binom{n-j}{i} 4^i = 2^n F_{2n+2}.$$

Corollary 2. *For $n \geq 0$ and any complex numbers t, u,*
$$W'_n\left(2(1+t), u-(t+1)^2, -2ut, ut^2\right) = \sum_{i,j} \binom{n-i}{2j}\binom{n-2j}{i} t^i u^j,$$
where $\{W'_n\}$ satisfies the relation (1.3) with the initials $W'_0 = 0, W'_1 = 1, W'_2 = 1+t, W'_3 = (t+1)^2 + u$.

Proof. It suffices to take $s = 2$. We obtain the generating function
$$\frac{1 - z(t+1)}{1 - 2(1+t)z + \left((t+1)^2 - u\right)z^2 + 2utz^3 - ut^2z^4}$$
of the sequence $\{W'_n(2+2t, u-(t+1)^2, -2ut, ut^2)\}$ with the initial values $W'_0 = 0, W'_1 = 1, W'_2 = 1+t, W'_3 = (t+1)^2 + u$. □

When $t = u = 1$ and $s = 2$, we obtain
$$\frac{F_{2n+2} + F_{n+1}}{2} = \sum_{0 \leq i,j \leq n} \binom{n-i}{2j}\binom{n-2j}{i},$$
which was also given in [1].

Theorem 2. *Let n and s be positive integers and t, u any complex numbers. The generating function of the sequence $\left\{B_n^{(s)}\right\}_n$ defined by*
$$B_n^{(s)} := \sum_{i,j} \binom{n-i}{sj}\binom{n-j}{i} t^i u^j$$
is
$$B(z) := \sum_{n \geq 0} B_n^{(s)} z^n = \frac{(1-(1+t)z)^{s-1}}{(1-(1+t)z)^s - uz^s(1-tz)}.$$

Proof. First, we replace i by $n-i$ and get

$$B_n^{(s)} = \sum_{i,j} \binom{i}{sj}\binom{n-j}{n-i} t^{n-i}u^j.$$

Now, we compute the generating function using Lemma 1:

$$\begin{aligned} B(z) &= \sum_{0\le sj\le i} \binom{i}{sj} z^i u^j \sum_{n\ge i}\binom{n-j}{n-i}(tz)^{n-i} \\ &= \sum_{0\le sj\le i}\binom{i}{sj}\frac{z^i u^j}{(1-zt)^{i-j+1}} \\ &= \sum_{j\ge 0}\frac{(u(1-zt)z^s)^j}{(1-zt)^{sj+1}}\sum_{i\ge sj}\binom{i}{sj}\frac{z^{i-sj}}{(1-zt)^{i-sj}} \\ &= \sum_{j\ge 0}\frac{(z^s u(1-zt))^j}{(1-(t+1)z)^{sj+1}} \\ &= \frac{(1-(1+t)z)^{s-1}}{(1-(1+t)z)^s - uz^s(1-tz)}. \end{aligned}$$

□

Remark 1. *For $s=1$, we obtain Corollary 1.*

Corollary 3. *For $n \ge 0$ and any complex numbers t, u,*

$$V_n'(2+2t, u-(u+1)^2, -ut) = \sum_{i,j}\binom{n-i}{2j}\binom{n-j}{i} t^i u^j,$$

where $\{V_n'\}$ satisfies the relation (1.2) with the initials $V_0' = 0, V_1' = 1, V_2' = 1+t$.

Proof. It suffices to take $s=2$. We obtain the generating function

$$\frac{1-z(t+1)}{1-2(1+t)z+\left((t+1)^2-u\right)z^2+utz^3}$$

of the sequence $\{V_n'(2+2t, u-(u+1)^2, -ut)\}$ with the initial values $V_0' = 0, V_1' = 1, V_2' = 1+t$. □

Corollary 4. *For $n \ge 0$ and any complex numbers t, u,*

$$\sum_{i,j}\binom{n-i}{3j}\binom{n-j}{i} t^i u^j = W_n''(3(1+t), -3(t+1)^2, (t+1)^3+u, -ut),$$

where $\{W_n''\}_n$ satisfies the relation (1.3) with the initials $W_0'' = 0, W_1'' = 1, W_2'' = 1+t, W_3'' = 1+2t+t^2$.

Proof. It suffices to take $s = 3$. We obtain the generating function

$$\frac{1-2(t+1)z+(t+1)^2 z^2}{1-3(1+t)z+3(t+1)^2 z^2-\left((t+1)^3+u\right)z^3+utz^4}$$

of the sequence $\{W_n(3(1+t), -3(t+1)^2, (t+1)^3+u, -ut)\}$ with the initial values $W_0'' = 0, W_1'' = 1, W_2'' = 1+t, W_3'' = (t+1)^2$. □

Theorem 3. *Let n and s be positive integers and t, u any complex numbers. The generating function of the sequence $\left\{C_n^{(s)}\right\}_n$ defined by*

$$C_n^{(s)} := \sum_{i,j} \binom{n-i}{sj}\binom{n-j}{i-j} u^i t^j$$

is

$$C(z) := \sum_{n\geq 0} C_n^{(s)} z^n = \frac{(1-(1+u)z)^{s-1}}{(1-(1+u)z)^s - tuz^{s+1}}.$$

Proof. First, we replace i by $n-i$ and get

$$C_n^{(s)} = \sum_{i,j} \binom{i}{sj}\binom{n-j}{i} u^{n-i} t^j.$$

We compute the generating function using Lemma 1:

$$\begin{aligned}
C(z) &= \sum_{i,j} \binom{i}{sj} z^j t^j u^{j-i} \sum_{n\geq j} \binom{n-j}{i} (uz)^{n-j} \\
&= \sum_{j\geq 0} \frac{(tuz)^j z^{sj}}{(1-uz)^{sj+1}} \sum_{i\geq sj} \binom{i}{sj} \frac{z^{i-sj}}{(1-uz)^{i-sj}} \\
&= \sum_{j\geq 0} \frac{(tuz)^j z^{sj}}{(1-(1+u)z)^{sj+1}} = \frac{(1-(1+u)z)^{s-1}}{(1-(1+u)z)^s - tuz^{s+1}}.
\end{aligned}$$

□

Corollary 5. *Let n and s be positive integers and t, u any complex numbers. The generating function of the sequence $\left\{\widetilde{C}_n^{(s)}\right\}_n$ defined by*

$$\widetilde{C}_n^{(s)} := \sum_{i,j} \binom{n-i}{j}\binom{j}{si} t^i u^j$$

is

$$\widetilde{C}(z) := \sum_{n\geq 0} \widetilde{C}_n^{(s)} z^n = \frac{u(u-(1+u)z)^{s-1}}{(u-(1+u)z)^s - tu^{2s} z^{s+1}}.$$

Proof. It suffices to observe that $C^{(t,u)}(z) = \widetilde{C}^{(t,1/u)}(uz)$ and permute i and j. $\square$

Corollary 6. *For $n \geq 0$ and any complex numbers t, u,*

$$\sum_{i,j} \binom{n-i}{j} \binom{j}{i} t^i u^j = U_{n+1}(1+t, ut),$$

where $\{U_n\}$ is defined by relation (1.1).

(1) When $t = u = 1$ and $s = 1$, then $\{U_{n+1}(2,1)\}$ is the Pell sequence, given in [1].
(2) When $t = -u = -1/2$ and $s = 1$, then $\{U_{n+1}(-1/2, 1/2)\}$ is reduced to $\{F_{2n}/2^n\}$.

Corollary 7. *For $n \geq 0$ and any complex numbers t, u,*

$$V_n'(2(1+t), -(1+t)^2, ut^2) = \sum_{i,j} \binom{n-i}{j} \binom{j}{2i} t^i u^j,$$

where $\{V_n'\}$ satisfies the relation (1.2) with the initials $V_0' = 0, V_1' = 1, V_2' = 1+t$.

Proof. It suffices to take $s = 2$. We obtain the generating function

$$\frac{1-(t+1)z}{1-2(1+t)z+(1+t)^2 z^2 - ut^2 z^3}$$

of the sequence $\{V_n'(2(1+t), -(1+t)^2, ut^2)\}$ with the initial values $V_0' = 0, V_1' = 1, V_2' = 1+t$. $\square$

Theorem 4. *Let n and s be positive integers and t, u any complex numbers. The generating function of the sequence $\left\{D_n^{(s)}\right\}_n$ defined by*

$$D_n^{(s)} := \sum_{i,j} \binom{n}{i+j} \binom{i}{sj} t^i u^j$$

is

$$D(z) := \sum_{n \geq 0} D_n^{(s)} z^n = \frac{(1-(1+t)z)^{s-1}}{(1-(1+t)z)^s - ut^s(1-z)^{s-1} z^{s+1}}.$$

Proof. We compute the generating function using Lemma 1:

$$\begin{aligned} D(z) &= \sum_{i,j} \binom{i}{sj} z^{i+j} t^i u^j \sum_{n \geq i+j} \binom{n}{i+j} (z)^{n-i-j} \\ &= \sum_{0 \leq sj \leq i} \binom{i}{sj} \frac{(tz)^i (uz)^j}{(1-z)^{i+j+1}} \end{aligned}$$

$$= \frac{1}{1-z}\sum_{j\geq 0}\left(\frac{t^s u z^{s+1}}{(1-z)^{s+1}}\right)^j \sum_{i\geq sj}\binom{i}{sj}\frac{(tz)^{i-sj}}{(1-z)^{i-sj}}$$

$$= \frac{1}{1-(1+t)z}\sum_{j\geq 0}\left(\frac{ut^s z^{s+1}}{(1-(1+t)z)^s(1-z)}\right)^j$$

$$= \frac{(1-z)(1-(1+t)z)^{s-1}}{(1-z)(1-(1+t)z)^s - ut^s z^{s+1}}.$$

□

Corollary 8. *For $n \geq 0$ and any complex numbers t, u,*

$$\sum_{i,j}\binom{n}{i+j}\binom{i}{j}t^i u^j = U_{n+1}(1+t, ut),$$

where $\{U_n\}$ is defined by relation (1.1).

Proof. It suffices to take $s=1$. We obtain the generating function

$$\frac{1}{(1-z)(1-(1+t)z) - tuz^2} = \frac{1}{1-(2+t)z-(1+t-ut)z^2}$$

of the sequence $\{U_{n+1}(2+t, 1+t-ut)\}$. □

Corollary 9. *For $n \geq 0$ and any complex numbers t, u,*

$$\sum_{i,j}\binom{n}{i+j}\binom{i}{2j}t^i u^j = W'_n(2(1+t), -(1+t)^2, ut^2, -ut^2),$$

where $\{W'_n\}$ satisfies the relation (1.3) with the initials $W'_0 = 0, W'_1 = 1, W'_2 = 1+t, W'_3 = (t+1)^2$.

When $t = u$ and $s = 1$, the terms

$$U_{n+1}\left(t+2, t^2-t-1\right) = \sum_{0\leq i,j\leq n}\binom{n}{i+j}\binom{i}{j}t^{i+j},$$

are in the sequence *A094441* "Triangular array $T(n,u) = \binom{n}{u}F_{n-u+1}$" of OEIS.

For $t=1$, $\{U_{n+1}(3,-1)\}$ is the odd Fibonacci sequence $\{F_{2n+1}\}$.

Theorem 5. *Let n and s be positive integers and t, u any complex numbers. The generating function of the sequence $\left\{E_n^{(s)}\right\}_n$ defined by*

$$E_n^{(s)} := \sum_{i,j}\binom{n-i}{i-sj}\binom{i-sj}{j}t^i u^j$$

is

$$E(z) := \sum_{n\geq 0} E_n^{(s)} z^n = \frac{1}{1-z-tz^2-ut^{s+1}z^{s+2}}.$$

Proof. We compute the generating function using Lemma 1:

$$\begin{aligned} E(z) &= \sum_{0\leq sj\leq i} \binom{i-sj}{j} t^i z^{2i-sj} u^j \sum_{n\geq 2i-sj} \binom{n-i}{i-sj} z^{n-2i+sj} \\ &= \sum_{0\leq sj\leq i} \binom{i-sj}{j} t^i z^{2i-sj} u^j \frac{1}{(1-z)^{i-sj+1}} \\ &= \frac{1}{1-z} \sum_{j\geq 0} \frac{(tz)^{sj+j} (uz)^j}{(1-z)^j} \sum_{i\geq sj+j} \binom{i-sj}{j} \frac{\left(tz^2\right)^{i-sj-j}}{(1-z)^{i-sj-j}} \\ &= \frac{1}{1-z-tz^2-ut^{s+1}z^{s+2}}. \end{aligned}$$

□

Corollary 10. *For $n \geq 0$ and any complex numbers t, u,*

$$V_{n+1}\left(1,t,ut^2\right) = \sum_{i,j} \binom{n-i}{i-j} \binom{i-j}{j} t^i u^j,$$

where $\{V_n\}$ is defined by relation (1.2).

Proof. It suffices to take $s=1$. We obtain the generating function of the sequence $\left\{V_{n+1}\left(1,t,ut^2\right)\right\}$. □

When $t=u=1$ and $s=1$, we get the Tribonacci sequence $\{V_{n+1}(1,-1,1)\}$ (see [1]).

Corollary 11. *For $n \geq 0$ and any complex numbers t, u,*

$$W_{n+1}\left(1,t,0,ut^3\right) = \sum_{i,j} \binom{n-i}{i-2j} \binom{i-2j}{j} t^i u^j,$$

where $\{W_n\}$ is defined by relation (1.3).

Proof. It suffices to take $s=2$. We obtain the generating function

$$\frac{1}{1-z-tz^2-ut^3z^4}$$

of the sequence $\{W_{n+1}\left(1,t,0,ut^3\right)\}$. □

Theorem 6. *Let n and s be positive integers and t, u any complex numbers. The generating function of the sequence $\left\{G_n^{(s)}\right\}_n$*

$$G_n^{(s)} := \sum_{i,j} \binom{n-i}{i-j}\binom{i}{sj} t^i u^j$$

is

$$G(z) := \sum_{n\geq 0} G_n^{(s)} z^n = \frac{\left(1-z-tz^2\right)^{s-1}}{\left(1-z-tz^2\right)^s - ut^s z^{2s-1} + ut^s z^{2s}}.$$

Proof. We use the same approach as used in Theorem 5. □

Corollary 12. *For $n \geq 0$ and any complex numbers t, u*

$$U_{n+1}(1+tu, t-tu) = \sum_{i,j} \binom{n-i}{i-j}\binom{i}{j} t^i u^j$$

where $\{U_n\}$ defined by the relation (1.1).

Corollary 13. *For $n \geq 0$ and any complex numbers t, u,*

$$W'_{n+1}\left(2, -1+2t, -t(2-tu), -t^2(u+1)\right) = \sum_{i,j} \binom{n-i}{i-j}\binom{i}{2j} t^i u^j,$$

where $\{W'_n\}$ satisfies the relation (1.3) with the initials $W'_0 = 0, W'_1 = 1, W'_2 = 1, W'_3 = t+1$.

Proof. It suffices to take $s = 2$, we obtain the generating function

$$\frac{1-z-tz^2}{1-2z+(1-2t)z^2+t(2-tu)z^3+t^2(u+1)z^4}$$

of the sequence $\left\{W'_n\left(2, -1+2t, -t(2-tu), -t^2(u+1)\right)\right\}$, with the initials $W'_0 = 0, W'_1 = 1, W'_2 = 1, W'_3 = t+1$. □

3. Double Binomial Sums of the second kind (of binomial coefficient)

Theorem 7. *Let n and s be positive integers and t, u any complex numbers. The generating function of the sequence $\left\{H_n^{(s)}\right\}_n$ defined by*

$$H_n^{(s)} := \sum_{i,j} \binom{n-i}{i-sj} t^i u^j$$

is

$$H(z) := \sum_{n\geq 0} H_n^{(s)} z^n = \frac{1}{1-z-tz^2}\frac{1}{1-ut^s z^s}.$$

Proof. We compute the generating function using Lemma 1:

$$\begin{aligned} H(z) &= \sum_{0\le sj\le i} t^i z^{2i-sj} u^j \sum_{n\ge 2i-sj} \binom{n-i}{i-sj} z^{n-2i+sj} \\ &= \frac{1}{1-z} \sum_{j\ge 0} (tz)^{sj} u^j \sum_{i\ge sj} \frac{\left(tz^2\right)^{i-sj}}{(1-z)^{i-sj}} \\ &= \frac{1}{1-z-tz^2} \frac{1}{1-ut^s z^s}. \end{aligned}$$

□

Corollary 14. *For $n \ge 0$ and any complex numbers t, u,*

$$V_{n+1}\left(ut+1, t(1-u), -ut^2\right) = \sum_{0\le i,j\le n} \binom{n-i}{i-j} t^i u^j,$$

where $\{V_n\}$ is defined by relation (1.2).

Proof. It suffices to take $s = 1$. We obtain the generating function

$$\frac{1}{1-(1+ut)z - t(1-u)z^2 + ut^2z^3}$$

of the sequence $\left\{V_{n+1}\left(ut+1, t(1-u), -ut^2\right)\right\}$. □

When $t = u = 1$ and $s = 1$, the sequence $\{V_{n+1}(2, 0, -1)\}$ is the sequence $\{F_n - 1\}$, where F_n stands for the n-th Fibonacci number.

Corollary 15. *For $n \ge 0$ and any complex numbers t, u,*

$$W_{n+1}\left(1, t(1+ut), -ut^2, ut^3\right) = \sum_{i,j} \binom{n-i}{i-2j} t^i u^j,$$

where $\{W_n\}$ is defined by relation (1.3).

Proof. It suffices to take $s = 2$. We obtain the generating function

$$\frac{1}{1-z+(-ut^2-t)z^2 + t^2uz^3 + t^3uz^4}$$

of the sequence $\{W_{n+1}\left(1, t(1+ut), -ut^2, ut^3\right)\}$. □

Theorem 8. *Let n and s be positive integers and t, u any complex numbers such that $ut^s \ne 1$. The generating function of the sequence $\left\{I_n^{(s)}\right\}_n$ defined by*

$$I_n^{(s)} := \sum_{i,j} \binom{n-i+sj}{i-sj} t^i u^j$$

is

$$I(z) := \sum_{n\geq 0} I_n^{(s)} z^n = \frac{1}{1-z-tz^2} \frac{1}{\left(1-\left(tz+(tz)^2\right)^s u\right)}.$$

Proof. First we replace i by $n-i$ and get

$$I_n^{(s)} := \sum_{i,j} \binom{i+sj}{n} t^{n-i} u^j.$$

Then using generating function approach we obtain

$$\begin{aligned} I(z) &= \sum_{i,j\geq 0} \sum_{n\geq 0} \binom{i+sj}{n} t^{n+sj} u^j z^{n+i+sj} = \sum_{i,j\geq 0} t^{sj} u^j z^{sj+i} (1+tz)^{i+sj} \\ &= \sum_{i\geq 0} (z+tz)^i \sum_{j\geq 0} \left((1+tz)^s t^s z^s u\right)^j \\ &= \frac{1}{1-z-tz^2} \frac{1}{\left(1-\left(tz+(tz)^2\right)^s u\right)}. \end{aligned}$$

□

Corollary 16. *For $n \geq 0$ and any complex numbers t, u*

$$W_{n+1}(1,t) = \sum_{i,j} \binom{n-i+j}{i-j} t^i u^j,$$

where $\left\{W_{n+1}\left(1+ut, t-ut+ut^2, -2ut^2, -ut^3\right)\right\}$ is defined by relation (1.3).

Theorem 9. *Let n be positive integer and t, u any complex numbers. The generating function of the sequence $\{J_n\}$ defined by*

$$J_n := \sum_{i,j} \binom{2n-i-j}{i-j} t^i u^j$$

is the third order sequence defined by

$$J_n = (2t+ut+1) J_{n-1} - \left(ut + t^2(1+2u)\right) J_{n-2} + \left(ut^2\right) J_{n-3}$$

with initials $J_{-1} = 0, J_0 = 1$ and $J_1 = 1+t+ut$.

Proof. First, we replace i by $n-i$ and get

$$J_n = \sum_{i,j} \binom{n+i-j}{n-i-j}.$$

Now we compute the generating function

$$J(z) = \sum_{n\geq 0} \sum_{0\leq i,j\leq n} \binom{n+i-j}{n-i-j} t^{n-i} u^j z^n$$

$$= \sum_{i,j\geq 0} \frac{z^i\left(ztu\right)^j}{\left(1-zt\right)^{2i+1}} = \frac{1-tz}{\left(1-utz\right)\left(1-\left(2t+1\right)z+t^2z^2\right)}$$

which is the generating function of the sequence $\{J_n\}$. □

(1) When $k=t=1$, the sequence $\{J_n\}$ is reduced to the Fibonacci sequence $\{F_{2n+2}\}$.
(2) When $t=-k=1$, the sequence $\{J_n\}$ is reduced to the sequence of powers of the Fibonacci numbers, $\left\{F_{n+1}^2\right\}$.

4. A Binomial Sum

In the webpage of R. RAM [2], one can find the following formula for the Fibonacci numbers:

$$F_n = \sum_{k=1}^{n} \binom{2n-k}{k-1} \mathbf{i}^{k-1}\left(1-2\mathbf{i}\right)^{n-k}, \quad \mathbf{i} = \sqrt{-1}$$

In this section, motivated by the above result, we generalize the formula and then derive new binomial sums with complex coefficients for the terms of a general second order recursion:

Theorem 10. *For $n>0$ and any complex numbers p, q,*

$$U_n\left(p,q\right) = \sum_{k=0}^{n} \binom{2n-k-1}{k} \left(\mathbf{i}\sqrt{q}\right)^k \left(p-2\mathbf{i}\sqrt{q}\right)^{n-k-1},$$

where $\{U_n\}$ is defined by relation (1.1).

Proof. Consider

$$\begin{aligned}
&\sum_{k=0}^{n} \left(\mathbf{i}\sqrt{q}\right)^k \left(p-2\mathbf{i}\sqrt{q}\right)^{n-k-1} \binom{2n-k-1}{k} \\
= \quad &\left(p-2\mathbf{i}\sqrt{q}\right)^{n-1} \sum_{k=0}^{n} \left(\frac{\sqrt{q}\left(\mathbf{i}p-2\sqrt{q}\right)}{p^2+4q}\right)^k \binom{2n-k-1}{k}
\end{aligned}$$

which, by writing $n-k$ instead of k, becomes

$$\begin{aligned}
&\left(\frac{\sqrt{q}\left(\mathbf{i}p-2\sqrt{q}\right)}{p^2+4q}\right)^n \left(p-2\mathbf{i}\sqrt{q}\right)^{n-1} \\
&\times \sum_{k=0}^{n} \left(\frac{\sqrt{q}\left(\mathbf{i}p-2\sqrt{q}\right)}{p^2+4q}\right)^{-k} \binom{n+k-1}{n-k} \\
= \quad &\frac{\left(\mathbf{i}\sqrt{q}\right)^n}{\left(p-2\mathbf{i}\sqrt{q}\right)} \sum_{k=0}^{n} t^k \binom{n+k-1}{2k-1}
\end{aligned}$$

$$= \frac{\left(\mathbf{i}\sqrt{q}\right)^n}{\left(p-2\mathbf{i}\sqrt{q}\right)}\sum_{k=0}^{n} t^{k+1}\binom{n+k}{2k+1} = \left(\mathbf{i}\sqrt{q}\right)^{n-1}\sum_{k=0}^{n} t^{k}\binom{n+k}{2k+1},$$

where $t = -\left(2\sqrt{q}+\mathbf{i}p\right)/\sqrt{q}$. We need to compute the sum on right hand side of the equation above. Now we compute its generating function;

$$\sum_{n\geq 0} z^n \sum_{k=0}^{n} t^k \binom{n+k}{2k+1} z^n = \frac{z}{1-z\left(2+t\right)z+z^2} = \sum_{n\geq 0} z^n U_n\left(t+2,-1\right),$$

where $U_n\left(a,b\right)$ and t are defined as before. Thus our sum takes the form:

$$\left(\mathbf{i}\sqrt{q}\right)^{n-1}\sum_{k=0}^{n} t^{k}\binom{n+k}{2k+1} = \left(\mathbf{i}\sqrt{q}\right)^{n-1} U_n\left(-\frac{2\sqrt{q}+\mathbf{i}p}{\sqrt{q}},-1\right),$$

which, by using the Binet formula of $\{U_n\}$, gives

$$\left(\mathbf{i}\sqrt{q}\right)^{n-1} U_n\left(-\frac{2\sqrt{q}+\mathbf{i}p}{\sqrt{q}},-1\right) = U_n\left(p,q\right),$$

as claimed. □

When $p=q=1$, and when $p=2,\ q=1$,

$$F_n = \sum_{k=0}^{n}\binom{2n-k-1}{k}\mathbf{i}^k\left(1-2\mathbf{i}\right)^{n-k-1}$$

and

$$P_n = \sum_{k=0}^{n}\binom{2n-k-1}{k}\mathbf{i}^k\left(2-2\mathbf{i}\right)^{n-k-1},$$

where F_n and P_n are the n^{th} Fibonacci and Pell numbers, respectively.

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