Pattern-avoiding ascent sequences

Lara Pudwell Valparaiso University
faculty.valpo.edu/lpudwell

2015 Joint Mathematics Meetings
AMS Special Session on Enumerative Combinatorics
January 11, 2015

Ascent Sequences

Definition

An ascent sequence is a string $x_1 \cdots x_n$ of non-negative integers such that:

- $x_1 = 0$
- $x_n \leq 1 + \mathrm{asc}(x_1 \cdots x_{n-1})$ for $n \geq 2$

$\mathcal{A}_n$ is the set of ascent sequences of length n

$$\mathcal{A}_2 = \{00, 01\}$$
$$\mathcal{A}_3 = \{000, 001, 010, 011, 012\}$$

More examples: 01234, 01013

Non-example: 01024

Ascent Sequences

Valparaiso University

Definition

An ascent sequence is a string $x_1 \cdots x_n$ of non-negative integers such that:

- $x_1 = 0$
- $x_n \leq 1 + \operatorname{asc}(x_1 \cdots x_{n-1})$ for $n \geq 2$

$\mathcal{A}_n$ is the set of ascent sequences of length n

$$\mathcal{A}_2 = \{00, 01\}$$
$$\mathcal{A}_3 = \{000, 001, 010, 011, 012\}$$

More examples: 01234, 01013

Non-example: 01024

Theorem

$|\mathcal{A}_n|$ is the nth Fishburn number (OEIS A022493).

$$\sum_{n\geq 0} |\mathcal{A}_n| x^n = \sum_{n\geq 0} \prod_{i=1}^{n} (1-(1-x)^i)$$

Patterns

Valparaiso University

Definition

The reduction of $x = x_1 \cdots x_n$, $\mathrm{red}(x)$, is the string obtained by replacing the ith smallest digits of x with $i - 1$.

Example: $\mathrm{red}(273772) = 021220$

Patterns

Valparaiso University

Definition

The reduction of $x = x_1 \cdots x_n$, $\mathrm{red}(x)$, is the string obtained by replacing the ith smallest digits of x with $i - 1$.

Example: $\mathrm{red}(273772) = 021220$

Pattern containment/avoidance

$a = a_1 \cdots a_n$ contains $\sigma = \sigma_1 \cdots \sigma_m$ iff there exist $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that $\mathrm{red}(a_{i_1} a_{i_2} \cdots a_{i_m}) = \sigma$.

$a_B(n) = |\{a \in \mathcal{A}_n \mid a \text{ avoids } B\}|$

001010345 contains 012, 000, 1102; avoids 210.

Patterns

Valparaiso University

Definition

The reduction of $x = x_1 \cdots x_n$, $\mathrm{red}(x)$, is the string obtained by replacing the ith smallest digits of x with $i-1$.

Example: $\mathrm{red}(273772) = 021220$

Pattern containment/avoidance

$a = a_1 \cdots a_n$ contains $\sigma = \sigma_1 \cdots \sigma_m$ iff there exist $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that $\mathrm{red}(a_{i_1} a_{i_2} \cdots a_{i_m}) = \sigma$.

$a_B(n) = |\{a \in \mathcal{A}_n \mid a \text{ avoids } B\}|$

001010345 contains 012, 000, 1102; avoids 210.

Patterns

Valparaiso University

Definition

The reduction of $x = x_1 \cdots x_n$, $\mathrm{red}(x)$, is the string obtained by replacing the ith smallest digits of x with $i-1$.

Example: $\mathrm{red}(273772) = 021220$

Pattern containment/avoidance

$a = a_1 \cdots a_n$ contains $\sigma = \sigma_1 \cdots \sigma_m$ iff there exist $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that $\mathrm{red}(a_{i_1} a_{i_2} \cdots a_{i_m}) = \sigma$.

$a_B(n) = |\{a \in \mathcal{A}_n \mid a \text{ avoids } B\}|$

001010345 contains 012, 000, 1102; avoids 210.

Patterns

Valparaiso University

Definition

The reduction of $x = x_1 \cdots x_n$, $\mathrm{red}(x)$, is the string obtained by replacing the ith smallest digits of x with $i-1$.

Example: $\mathrm{red}(273772) = 021220$

Pattern containment/avoidance

$a = a_1 \cdots a_n$ contains $\sigma = \sigma_1 \cdots \sigma_m$ iff there exist $1 \le i_1 < i_2 < \cdots < i_m \le n$ such that $\mathrm{red}(a_{i_1} a_{i_2} \cdots a_{i_m}) = \sigma$.

$a_B(n) = |\{a \in \mathcal{A}_n \mid a \text{ avoids } B\}|$

001010345 contains 012, 000, 1102; avoids 210.

Goal

Determine $a_B(n)$ for many of choices of B.

Previous Work

Valparaiso University

- Duncan & Steingrímsson (2011)

Pattern b	$a_b(n)$	OEIS
001, 010 011, 012	2^{n-1}	A000079
102 0102, 0112	$(3^{n-1}+1)/2$	A007051
101, 021 0101	$\frac{1}{n+1}\binom{2n}{n}$	A000108

- Mansour and Shattuck (2014)
  Callan, Mansour and Shattuck (2014)

Patterns B	$a_B(n)$	OEIS
1012	$\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}C_\ell$	A007317
0123	ogf: $\frac{1-4x+3x^2}{1-5x+6x^2-x^3}$	A080937
8 pairs of length 4 patterns	$\frac{1}{n+1}\binom{2n}{n}$	A000108

Other sequences (Baxter & P.)

Valparaiso University

Patterns B	OEIS	$a_B(n)$
000,011	A000027	n
011,100	A000124	$\binom{n}{2}+1$
001,210	A000125	$\binom{n}{3}+n$
000,101	A001006	M_n
000,001	A000045	F_{n+1}
001,100	A000071	$F_{n+2}-1$
101,110	A001519	F_{2n-1}
100,101	A025242	(Generalized Catalan)
021,102	A116702	$\lvert\mathcal{S}_n(123,3241)\rvert$
102,120	A005183	$\lvert\mathcal{S}_n(132,4312)\rvert$
101,120	A116703	$\lvert\mathcal{S}_n(231,4123)\rvert$
201,210	A007317	$\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}C_\ell$

Binomial convolutions

Theorem (P.)

$a_{201,210}(n) = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} C_\ell.$

Theorem (Mansour & Shattuck)

$a_{1012}(n) = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} C_\ell.$

Conjecture (Duncan & Steingrímsson)

$a_{0021}(n) = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} C_\ell.$

Binomial convolutions

Valparaiso University

Theorem (P.)

$a_{201,210}(n) = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} C_\ell.$

Theorem (Mansour & Shattuck)

$a_{1012}(n) = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} C_\ell.$

~~Conjecture~~ (Duncan & Steingrímsson) / Theorem (P.)

$a_{0021}(n) = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} C_\ell.$

Wilf Equivalence

Definition

Patterns σ and ρ are Wilf-equivalent if $a_\sigma(n) = a_\rho(n)$ for $n \geq 1$. In this case, write: $\sigma \sim \rho$.

Example: $00 \sim 01$.
$a_{00}(n) = 1$ (the strictly increasing sequence)
$a_{01}(n) = 1$ (the all zeros sequence).

Wilf Equivalence

Valparaiso University

Definition

Patterns σ and ρ are Wilf-equivalent if $a_\sigma(n) = a_\rho(n)$ for $n \geq 1$. In this case, write: $\sigma \sim \rho$.

Example: $00 \sim 01$.
$a_{00}(n) = 1$ (the strictly increasing sequence)
$a_{01}(n) = 1$ (the all zeros sequence).

Known from Duncan/Steingrímsson: All possible Wilf equivalences of length at most 4 are:

$$00 \sim 01$$
$$10 \sim 001 \sim 010 \sim 011 \sim 012$$
$$102 \sim 0102 \sim 0112$$
$$101 \sim 021 \sim 0101 \sim 0012$$
$$0021 \sim 1012$$

(Duncan & Steingrímsson / Mansour & Shattuck / P.)

Binomial convolutions

Theorem (P.)

$\mathrm{a}_{201,210}(n) = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} C_\ell.$

Theorem (Mansour & Shattuck)

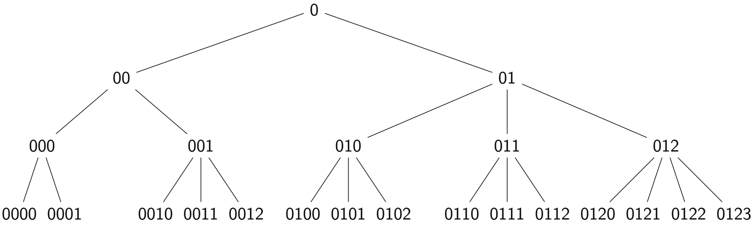
$\mathrm{a}_{1012}(n) = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} C_\ell.$

~~Conjecture~~ (Duncan & Steingrímsson) / Theorem (P.)

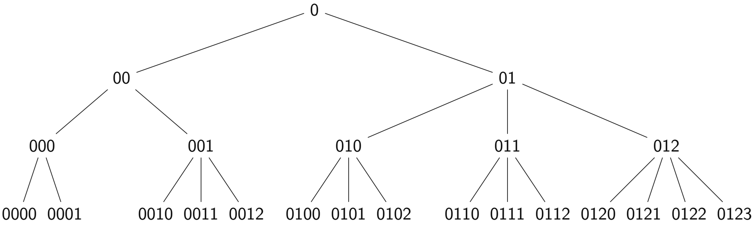
$\mathrm{a}_{0021}(n) = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} C_\ell.$

Proof scribble:
generating tree $\to$ recurrence $\to$ system of functional equations $\to$ experimental solution $\to$ plug in for catalytic variables

Generating Tree

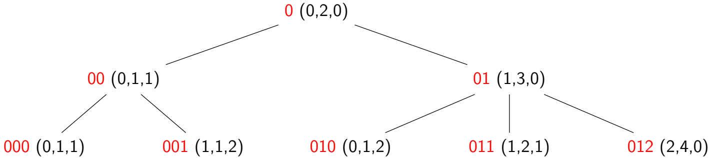


Generating Tree

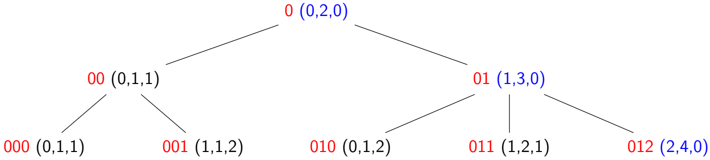


Idea: Replace *a* with an ordered triple of statistics on a.

Generating Tree

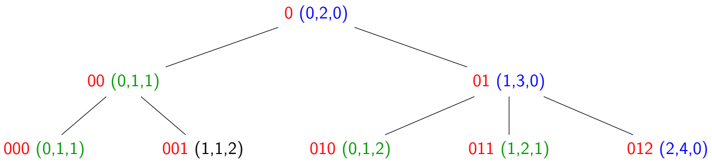


Generating Tree



- **root:** $(0, 2, 0)$
- **rules:**
  $(j-2, j, 0) \to (j-1, j+1, 0), (i, i+1, j-1-i)_{i=0}^{j-2}$

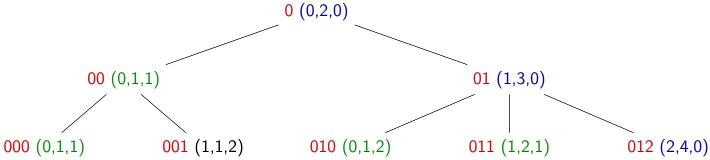
Generating Tree



- **root:** $(0,2,0)$
- **rules:**
  $(j-2,j,0) \to (j-1,j+1,0),(i,i+1,j-1-i)_{i=0}^{j-2}$

  $(j-1,j,k) \to$
  $(j-1,j,k),(i,i+1,j+k-1-i)_{i=1}^{j-2},(j,j,i)_{i=2}^{k+1}$

Generating Tree



- **root:** $(0,2,0)$
- **rules:**

$$(j-2,j,0) \to (j-1,j+1,0), (i,i+1,j-1-i)_{i=0}^{j-2}$$

$$(j-1,j,k) \to$$
$$(j-1,j,k), (i,i+1,j+k-1-i)_{i=1}^{j-2}, (j,j,i)_{i=2}^{k+1}$$

$$(j,j,k) \to (j,j,k), (i,i+1,j+k-1-i)_{i=0}^{j-1}, (j,j,i)_{i=2}^{k}$$

Counting Nodes

- **root:** $(0, 2, 0)$
- **rules:**
  $(j-2, j, 0) \to (j-1, j+1, 0), (i, i+1, j-1-i)_{i=0}^{j-2}$

  $(j-1, j, k) \to$
  $(j-1, j, k), (i, i+1, j+k-1-i)_{i=1}^{j-2}, (j, j, i)_{i=2}^{k+1}$

  $(j, j, k) \to (j, j, k), (i, i+1, j+k-1-i)_{i=0}^{j-1}, (j, j, i)_{i=2}^{k}$

Note: One blue node per level of tree.

Need to look at green and **black** nodes more closely.

Tree rules to functional equations

Define:

- $g0_{n,j,k}$ is number of (j,j,k) nodes at level n.
- $B(x,y,z) = \sum_{n\geq 3}\sum_{j\geq 1}\sum_{k\geq 2} g0_{n,j,k}x^jy^kz^n$.
- $g1_{n,j,k}$ is number of $(j-1,j,k)$ nodes at level n.
- $G(x,y,z) = \sum_{n\geq 2}\sum_{j\geq 1}\sum_{k\geq 1} g1_{n,j,k}x^jy^kz^n$.

From the rules, we obtain:

$$\begin{aligned}B(x,y,z) =& \frac{z(1-2y)}{1-y}B(x,y,z) + \frac{zy^2}{1-y}B(x,1,z) + \frac{xy^2z^2}{(1-z)(1-yz)(1-xz)}\\ &- \frac{zy^2}{1-y}\left(G(x,y,z) - \frac{xyz^2}{(1-xz)(1-yz)}\right)\\ &+ \frac{zy^2}{1-y}\left(G(x,1,z) - \frac{xz^2}{(1-xz)(1-z)}\right)\end{aligned}$$

$$\begin{aligned}G(x,y,z) =& \frac{xyz^2}{(1-xz)(1-yz)} + \frac{zx}{x-y}G(x,y,z)\\ &- \frac{zx}{x-y}G(y,y,z) + \frac{zx}{x-y}B(x,y,z) - \frac{zx}{x-y}B(y,y,z)\end{aligned}$$

Black node data

$A0_n$ is an $(n-2) \times (n-2)$ array with $g0_{n,j,k}$ in row j, column $k-1$.

$$A0_3 = \begin{bmatrix} 1 \end{bmatrix} \quad A0_4 = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix} \quad A0_5 = \begin{bmatrix} 14 & 6 & 1 \\ 4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A0_6 = \begin{bmatrix} 50 & 27 & 8 & 1 \\ 14 & 6 & 1 & 0 \\ 4 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad A0_7 = \begin{bmatrix} 187 & 113 & 44 & 10 & 1 \\ 50 & 27 & 8 & 1 & 0 \\ 14 & 6 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A0_8 = \begin{bmatrix} 730 & 468 & 212 & 65 & 12 & 1 \\ 187 & 113 & 44 & 10 & 1 & 0 \\ 50 & 27 & 8 & 1 & 0 & 0 \\ 14 & 6 & 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Black node data

$A0_n$ is an $(n-2)\times(n-2)$ array with $g0_{n,j,k}$ in row j, column $k-1$.

$$A0_8 = \begin{bmatrix} 730 & 468 & 212 & 65 & 12 & 1 \\ 187 & 113 & 44 & 10 & 1 & 0 \\ 50 & 27 & 8 & 1 & 0 & 0 \\ 14 & 6 & 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Let $f(z) = \frac{1-z-\sqrt{1-6z+5z^2}}{2z}$.
- Let $g(z) = \frac{16z^2(z-1)}{\left(1-z+\sqrt{1-6z+5z^2}\right)^3\left(-1+3z+\sqrt{1-6z+5z^2}\right)}$.
- Experimentally predict:
  Column i has generating function $\frac{f(z)-1}{1-z}g(z)^{i-1}$.
  $B(x,y,z) = \frac{2xy^2z^3}{(1-xz)((1-(y+1)z)\sqrt{5z^2-6z+1}+(1-(y+3)z)(1-z))}$

Green node data

$A1_n$ is an $(n-1)\times(n-1)$ array with $g1_{n,j,k}$ in row j column k.

$$A1_2 = \begin{bmatrix} 1 \end{bmatrix} \quad A1_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad A1_4 = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A1_5 = \begin{bmatrix} 1 & 8 & 5 & 1 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad A1_6 = \begin{bmatrix} 1 & 23 & 19 & 7 & 1 \\ 1 & 8 & 5 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A1_7 = \begin{bmatrix} 1 & 74 & 69 & 34 & 9 & 1 \\ 1 & 23 & 19 & 7 & 1 & 0 \\ 1 & 8 & 5 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad A1_8 = \begin{bmatrix} 1 & 262 & 256 & 147 & 53 & 11 & 1 \\ 1 & 74 & 69 & 34 & 9 & 1 & 0 \\ 1 & 23 & 19 & 7 & 1 & 0 & 0 \\ 1 & 8 & 5 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Green node data

$A1_n$ is an $(n-1)\times(n-1)$ array with $g1_{n,j,k}$ in row j column k.

$$A1_8 = \begin{bmatrix} 1 & 262 & 256 & 147 & 53 & 11 & 1 \\ 1 & 74 & 69 & 34 & 9 & 1 & 0 \\ 1 & 23 & 19 & 7 & 1 & 0 & 0 \\ 1 & 8 & 5 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A091698 Matrix inverse of triangle A063967.

1, -1, 1, 1, -3, 1, -1, 8, -5, 1, 1, -23, 19, -7, 1, -1, 74, -69, 34, -9, 1, 1, -262, 256, -147, 53, -11, 1, -1, 993, -986, 615, -265, 76, -13, 1, 1, -3943, 3935, -2571, 1235, -431, 103, -15, 1, -1, 16178, -16169, 10862, -5591, 2216, -653, 134, -17, 1, 1 (list; table; graph; refs; listen; history; text; internal format)

OFFSET 0,5

COMMENTS Riordan array (1/(1+x), (sqrt(1+6x+5x^2)-x-1)/(2(1+x))). The absolute value array is (1/(1-x),xc(x)/(1-xc(x))) where c(x) is the g.f. of A000108. It factorizes as (1/(1-x),x/(1-x))(1,xc(x)). - Paul Barry, Jun 10 2005

LINKS Table of n, a(n) for n=0..55.

EXAMPLE 1; -1,1; 1,-3,1; -1,8,-5,1; 1,-23,19,-7,1; ...

Contribution from Paul Barry, Apr 15 2010: (Start)

Triangle begins

1,

-1, 1,

1, -3, 1,

-1, 8, -5, 1,

1, -23, 19, -7, 1,

-1, 74, -69, 34, -9, 1,

1, -262, 256, -147, 53, -11, 1,

-1, 993, -986, 615, -265, 76, -13, 1,

1, -3943, 3935, -2571, 1235, -431, 103, -15, 1

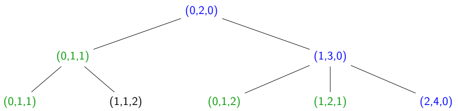
$$G(x,y,z) = \frac{2xyz^2}{(1-xz)(y\sqrt{5z^2-6z+1}+yz-2z-y+2)}$$

Recap

Goal

$a_{0021}(n) = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} C_\ell$

Technique: Count blue, black, and green nodes in the following generating tree:



governed by:

- **root:** $(0,2,0)$
- **rules:**
  $(j-2,j,0) \to (j-1,j+1,0), (i,i+1,j-1-i)_{i=0}^{j-2}$

  $(j-1,j,k) \to (j-1,j,k), (i,i+1,j+k-1-i)_{i=1}^{j-2}, (j,j,i)_{i=2}^{k+1}$

  $(j,j,k) \to (j,j,k), (i,i+1,j+k-1-i)_{i=0}^{j-1}, (j,j,i)_{i=2}^{k}$

Final Enumeration

Valparaiso University

- blue nodes on level n

  Generating function: $\frac{z}{1-z}$

- black nodes of type (j, j, k) on level n

  Generating function:
  $$B(x, y, z) = \frac{2xy^2z^3}{(1-xz)((1-(y+1)z)\sqrt{5z^2-6z+1}+(1-(y+3)z)(1-z))}$$

- green nodes of type $(j-1, j, k)$ on level n

  Generating function:
  $$G(x, y, z) = \frac{2xyz^2}{(1-xz)(y\sqrt{5z^2-6z+1}+yz-2z-y+2)}$$

- total nodes at level n:

  $$\frac{z}{1-z} + B(1, 1, z) + G(1, 1, z) = \frac{1-z-\sqrt{5z^2-6z+1}}{2(1-z)}.$$

Summary

- There are a plethora of nice enumeration questions for pattern-avoiding ascent sequences.
- Computing $\mathrm{a}_{0021}(n)$ completes Wilf-equivalence for patterns of length 4.
- Open: find a statistic $\mathrm{st} : \mathcal{A}_{0021}(n) \to \ell$ so that

$$|\{a \in \mathcal{A}_{0021}(n) \mid \mathrm{st}(a) = \ell\}| = \binom{n-1}{\ell} C_\ell.$$

References

- A. Baxter and L. Pudwell, Ascent sequences avoiding pairs of patterns, `arXiv:1406.4100`, submitted.
- D. Callan, T. Mansour, and M. Shattuck, Restricted ascent sequences and Catalan numbers, *Appl. Anal. Discrete Math.* **8** (2014), 288–303.
- P. Duncan and E. Steingrímsson, Pattern avoidance in ascent sequences, *Electron. J. Combin.* **18**(1) (2011), #P226 (17pp).
- T. Mansour and M. Shattuck, Some enumerative results related to ascent sequences, *Discrete Math.* **315-316** (2014), 29–41.
- L. Pudwell, Ascent sequences and the binomial convolution of Catalan numbers, `arXiv:1408.6823`, to appear in *Australas. J. Combin.*.

References

- A. Baxter and L. Pudwell, Ascent sequences avoiding pairs of patterns, arXiv:1406.4100, submitted.
- D. Callan, T. Mansour, and M. Shattuck, Restricted ascent sequences and Catalan numbers, *Appl. Anal. Discrete Math.* **8** (2014), 288–303.
- P. Duncan and E. Steingrímsson, Pattern avoidance in ascent sequences, *Electron. J. Combin.* **18**(1) (2011), #P226 (17pp).
- T. Mansour and M. Shattuck, Some enumerative results related to ascent sequences, *Discrete Math.* **315-316** (2014), 29–41.
- L. Pudwell, Ascent sequences and the binomial convolution of Catalan numbers, arXiv:1408.6823, to appear in *Australas. J. Combin.*.

Thanks for listening!