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Integer partitions into arithmetic progressions

ABSTRACT. Every number not in the form 2^k can be partitioned into two or more consecutive parts. Thomas E. Mason has shown that the number of ways in which a number n may be partitioned into consecutive parts, including the case of a single term, is the number of odd divisors of n. This result is generalized by determining the number of partitions of n into arithmetic progressions with a common difference r, including the case of a single term.

KEY WORDS. Integer partitions, Arithmetic progression, divisors of an integer.

1 Introduction

Let n be an integer. A partition of n is an integral solution of the system:

$$\begin{cases} n = n_1 + \cdots + n_k, \\ 1 \leq n_1 \leq \cdots \leq n_k. \end{cases}$$

The positive integers $n_1, \ldots, n_k$ are called parts, and k is the length of the partition. In partition identities, we are often interested in the number of partitions that satisfy some conditions. For example, the number of partitions into odd parts, the number of partitions into even parts, the number of partitions into even and odd parts [3], the number of partitions into distinct parts, and so on. For more on integer partitions the reader is hereby invited to see for instance [1], [2], [4], [5], [6] and [8]. Thomas E. Mason [7] was interested in the number of ways in which a number n may be partitioned into consecutive parts, including the case of a single term, and he has shown that this number equals the number of odd divisors of n. Let $n = 2^\alpha \cdot p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where the $p's$ are distinct odd primes and the $\alpha' s$ are the powers for which they occur. Then the number of ways in which a number n may be partitioned into consecutive parts, including the case of a single term, equals $\prod_{i=1}^{r}(\alpha_i + 1)$, except in the case $n = 2^\alpha$ where the number of ways is 1. In this paper we treat the case of partitions of n into arithmetic progressions.

For $1 \leq r \leq n-2$, if $n_i - n_{i-1} = r$, $i = 2, \dots, k$, we say that we have a partition into an arithmetic progression with a common difference r.

Our problem can be formulated in terms of partitions: For a given positive integer n, what is the number of partitions of n into an arithmetic progression with a common difference r?

Partitioning n into an arithmetic progression with a common difference r means that there exist two positive integers $l \geq 1$ and $m \geq 0$, such that

$$n = l + (l + r) + (l + 2r) + \cdots + (l + mr). \tag{1}$$

Let us denote such a partition briefly by $\pi(l, m)$.

From (1),

$$n = (m+1)l + \frac{m(m+1)}{2}r.$$

Hence,

$$rm^2 + (2l + r)m + 2l - 2n = 0. \tag{2}$$

The solution of equation (2) in m yields

$$m = \frac{-(2l + r) + \sqrt{(2l - r)^2 + 8rn}}{2r}. \tag{3}$$

Since m is an integer, $(2l - r)^2 + 8rn$ is a perfect square, i.e.,

$$(2l - r)^2 + 8rn = u^2. \tag{4}$$

Then,

$$2rn = \frac{(u - (2l - r))}{2} \times \frac{(u + (2l - r))}{2}.$$

Putting,

$$A = \frac{u - (2l - r)}{2} \quad \text{and} \quad B = \frac{u + (2l - r)}{2},$$

we have

$$A + B = u, \quad \text{and} \tag{5}$$

$$B - A = 2l - r. \tag{6}$$

By (3), (5) and (6) we get

$$l = \frac{r + B - A}{2} \quad \text{and} \quad m = \frac{A - r}{r}. \tag{7}$$

Consequently, r divides A. Hence

$$2n = \frac{A}{r} \times B. \tag{8}$$

Now we discuss the parity of r.

1.1 Arithmetic progressions with an odd common difference r

By (4) u is odd if r is odd. Then in view of (5), A and B must have different parity.

Case 1. If A is even then B is odd and $n = \dfrac{A}{2r} \times B$. In this case we have

$$l = \frac{r + d - \dfrac{2rn}{d}}{2} \quad \text{and} \quad m = \frac{2n}{d} - 1,$$

where $d = B$, is an odd divisor of n.

Since $l \geq 1$, d must verify $(2n - d)r \leq d(d-2)$, i.e., $d \geq \dfrac{2 - r + \sqrt{(r-2)^2 + 8rn}}{2}$.

Case 2. If A is odd then $\dfrac{A}{r}$ is odd as well and $n = \dfrac{A}{r} \times \dfrac{B}{2}$. In this case we have

$$l = \frac{r + \dfrac{2n}{d} - dr}{2} \quad \text{and} \quad m = d - 1,$$

where $d = \dfrac{A}{r}$ is an odd divisor of n.

Because $l \geq 1$, d must verify $d(d-1)r \leq 2(n-d)$, i.e., $d \leq \dfrac{r - 2 + \sqrt{(r-2)^2 + 8rn}}{2r}$.

For every such odd divisor of n there exists a partition into an arithmetic progression with an odd common difference r, and vice versa.

A moment's reflection will show that an odd divisor d of n cannot satisfy the inequalities $(2n - d)r \leq d(d-2)$, $d(d-1)r \leq 2(n-d)$ simultaneously, otherwise $(d-1)r \leq (d-2)$, a contradiction.

1.2 Arithmetic progressions with an even common difference r

By (4) u is even, if r is even. Then in view of (5), A and B must have the same parity.

Since r divides A, A and B are even. Hence from (8)

$$n = \frac{A}{r} \times \frac{B}{2}.$$

From (7) we get

$$l = \frac{r + 2d - \dfrac{nr}{d}}{2} \quad \text{and} \quad m = \frac{n}{d} - 1,$$

where $d = \dfrac{B}{2}$ is a divisor of n.

Since $l \geq 1$, d must verify $(n-d)r \leq 2d(d-1)$, i.e., $d \geq \dfrac{2-r+\sqrt{(r-2)^2+8rn}}{4}$.

Now we are able to formulate our results as follows:

Theorem 1 *Let $n \geq 3$ be a positive integer and let $1 \leq r \leq n-2$ be an odd integer. Then the number of partitions of n into an arithmetic progression with an odd common difference r, including the case of a single term, is the number of odd divisors d of n, satisfying $\big(2n-d\big)r \leq d(d-2)$ or $d(d-1)r \leq 2(n-d)$ and for every such odd divisor, the partition $\pi(l,m)$ is given by:*

$$\begin{cases} l = \dfrac{r+\frac{2n}{d}-dr}{2} \text{ and } m = d-1 & \text{if} \quad d(d-1)r \leq 2(n-d), \\ \\ l = \dfrac{r+d-\frac{2rn}{d}}{2} \text{ and } m = \dfrac{2n}{d}-1 & \text{if} \quad (2n-d)r \leq d(d-2). \end{cases}$$

Example 1 An example illustrating the above theorem is the following:

Let $n = 15$ and $r = 3$. The odd divisors of 15 satisfying $(2n-d)r \leq d(d-2)$ or $d(d-1)r \leq 2(n-d)$ are 1, 3 and 15, so 15 admits three partitions into an arithmetic progression of the common difference 3, each one is associated with one of these divisors and this can be shown as follows:

► $\boldsymbol{d=1}$ satisfies $d(d-1)r \leq 2(n-d)$. We have $l = \dfrac{r+\frac{2n}{d}-dr}{2} = 15$ and $m = d-1 = 0$. Hence $\pi(l,m) = \mathbf{15}$.

► $\boldsymbol{d=3}$ satisfies $d(d-1)r \leq 2(n-d)$. We have $l = \dfrac{r+\frac{2n}{d}-dr}{2} = 2$ and $m = d-1 = 2$. Hence $\pi(l,m) = \mathbf{2{+}5{+}8}$.

► $\boldsymbol{d=15}$ satisfies $\big(2n-d\big)r \leq d(d-2)$. We have $l = \dfrac{r+d-\frac{2rn}{d}}{2} = 6$ and $m = \dfrac{2n}{d}-1 = 1$. Hence $\pi(l,m) = \mathbf{6{+}9}$.

Theorem 2 *Let $n \geq 3$ be a positive integer and let $1 \leq r \leq n-2$ be an even integer. Then the number of partitions of n into an arithmetic progression with an even common difference r, including the case of a single term, is the number of divisors d of n, satisfying $(n-d)r \leq 2d(d-1)$ and for every such divisor, the partition $\pi(l,m)$ is given by:*

$$l = \frac{r+2d-\frac{nr}{d}}{2} \quad \text{and} \quad m = \frac{n}{d}-1.$$

Example 2 An example illustrating the above theorem is the following:

Let $n = 30$ and $r = 4$. The divisors of 30 satisfying $(n-d)r \leq 2d(d-1)$ are 10, 15 and 30, so 30 admits three partitions into an arithmetic progression of the common difference 4, each one is associated with one of these divisors and this can be shown as follows:

▶ $\boldsymbol{d = 10} \Rightarrow l = \dfrac{r + 2d - \frac{nr}{d}}{2} = 6$ and $m = \dfrac{n}{d} - 1 = 2$, hence $\pi(l,m) = \mathbf{6{+}10{+}14}$.

▶ $\boldsymbol{d = 15} \Rightarrow l = \dfrac{r + 2d - \frac{nr}{d}}{2} = 13$ and $m = \dfrac{n}{d} - 1 = 1$, hence $\pi(l,m) = \mathbf{13{+}17}$.

▶ $\boldsymbol{d = 30} \Rightarrow l = \dfrac{r + 2d - \frac{nr}{d}}{2} = 30$ and $m = \dfrac{n}{d} - 1 = 0$, hence $\pi(l,m) = \mathbf{30}$.

2 Why does Theorem 1 generalize Mason's theorem ?

Theorem 1 is an extension of Mason's theorem [7]. Indeed, for $r = 1$ and for every odd divisor d of n, one and only one of the two inequalities $d(d-1)r \leq 2(n-d)$, $\big(2n-d\big)r \leq d(d-2)$ necessarily holds. In fact, $r = 1$, $d = 2p+1$ and $n = dk$, imply

$$d(d-1) \leq 2(n-d) \Leftrightarrow p \leq k-1,$$

and

$$2n - d \leq d(d-2) \Leftrightarrow p \geq k.$$

Hence, if $p \leq k-1$, we get

$$l = \frac{1 - d + 2k}{2} \quad \text{and} \quad m = d - 1,$$

otherwise

$$l = \frac{1 + d - 2k}{2} \quad \text{and} \quad m = 2k - 1.$$

For example, $n = 15$ has four odd divisors: 1, 3, 5 and 15:

▶ $\boldsymbol{d = 1} \Rightarrow p = 0$ and $k = 15$, then $l = \dfrac{1-d+2k}{2} = 15$ and $m = d - 1 = 0$, hence $\pi(l,m) = \mathbf{15}$.

▶ $\boldsymbol{d = 3} \Rightarrow p = 1$ and $k = 5$, then $l = \dfrac{1-d+2k}{2} = 4$ and $m = d - 1 = 2$, hence $\pi(l,m) = \mathbf{4{+}5{+}6}$.

▶ $\boldsymbol{d = 5} \Rightarrow p = 2$ and $k = 3$, then $l = \dfrac{1-d+2k}{2} = 1$ and $m = d - 1 = 4$, hence $\pi(l,m) = \mathbf{1{+}2{+}3{+}4{+}5}$.

▶ $\boldsymbol{d = 15} \Rightarrow p = 7$ and $k = 1$, then $l = \dfrac{1+d-2k}{2} = 7$ and $m = 2k - 1 = 1$, hence $\pi(l,m) = \mathbf{7{+}8}$.

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