Towards an automation of the circle method

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ABSTRACT. The derivation of the Hardy-Ramanujan-Rademacher formula for the number of partitions of n is reviewed. Next, the steps for finding analogous formulas for certain restricted classes of partitions or overpartitions is examined, bearing in mind how these calculations can be automated in a CAS. Finally, a number of new formulas of this type which were conjectured with the aid of *Mathematica* are presented along with results of a test for their numerical accuracy.

1. Introduction

A *partition* of an integer n is a representation of n as a sum of positive integers, where the order of the summands (called *parts*) is considered irrelevant. For example, there are seven partitions of the integer 5, namely 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$, and $1+1+1+1+1$. Euler [6] was the first to systematically study partitions. He showed that

$$(1.1) \qquad \sum_{n=0}^{\infty} p(n)q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^m},$$

where $p(n)$ denotes the number of partitions of n and we follow the convention that $p(0) = 1$.

The series and infinite product in (1.1) converge absolutely when $|q| < 1$. Hardy and Ramanujan were the first to study $p(n)$ analytically and showed that [21, p. 79, Eq. (1.41)]

$$(1.2) \qquad p(n) \sim \frac{\exp(\pi\sqrt{2/3})}{4n\sqrt{3}} \qquad \text{as } n \to \infty.$$

Noting that the value of $p(200)$ estimated by (1.2) was surprisingly close to the true value of $p(200)$ as computed by P.A. MacMahon, Hardy and Ramanujan were encouraged to push their analysis of $p(n)$ further.

Ultimately, they produced the formula [21, p. 85, Eq. (1.75)]

$$(1.3) \quad p(n) = \frac{1}{2\pi\sqrt{2}} \sum_{k=1}^{\alpha\sqrt{n}} \sqrt{k} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \omega(h,k) e^{-2\pi i h n/k} \frac{d}{dn} \left(\frac{\exp\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}} \right)$$
$$+ O(n^{-1/4}),$$

with α an arbitrary constant and $\omega(h,k)$ a certain complex $24k$th root of unity[1] that arises frequently in the theory of modular forms and is defined below in (2.3). Also, here and throughout (h,k) denotes the gcd of h and k.

While (1.3) is an asymptotic formula, it is incredibly accurate. For the case $n = 200$, summing k from 1 to 8 results in a value only 0.004 higher than the true value of 3 972 999 029 338.

Later, it was shown by D. H. Lehmer [26] that if the sum on k in (1.3) is extended to ∞, the resulting series diverges.

In [30], Rademacher made a slight change in Hardy and Ramanujan's analysis which led him to finding a convergent series representation for $p(n)$, very similar in form to that of (1.3). This result is presented below as Theorem 2.1. In a later paper, Rademacher altered the path of integration and as a result was able to give a simpler proof for the correctness of his series [31]. This latter technique is also described in books by Rademacher [32, Ch. 14] and Apostol [3, Ch. 5], while the former may be found in the text of Andrews [1, Ch. 5].

The technique of deriving the formula for $p(n)$ via integration of a certain function (see (2.8) below) which has singularities at every point of the unit circle in the complex plane has come to be known as the "circle method." The circle method has proven to be applicable to many problems and as such is one of the most important and useful tools in analytic number theory. There are far too many papers which have used the circle method to even begin to mention them here, but a subset of the literature which employs the circle method to find formulas for certain restricted classes of partitions includes Grosswald [9, 10], Haberzetle [11], Hagis [12, 13, 14, 15, 16, 17, 18, 19, 20], Hua [22], Iseki [23, 24, 25], Lehner [27], Livingood [28], Niven [29], and Subramanyasastri [37]. Recently, Bringmann and Ono [4] have given exact formulas for the coefficients of all harmonic Maas forms of weight $\le \frac{1}{2}$. Thus, all of the exact formulas for restricted partition and overpartition functions presented here could be derived from the general theorem in [4].

A main theme of this paper is that while the application of the circle method to find $p(n)$ or a given restricted partition formula may be complicated, it is essentially a *calculation*. As such, many of the steps involved are ripe for automation. Furthermore, a good number of the steps involve showing that a given integral approaches zero. As long as we can reliably predict when this will be the case, we can produce reasonable conjectures for formulas without worrying about the estimates that are required when a rigorous proof is desired.

We shall outline a derivation of $p(n)$, and then consider how the circle method applies to restricted partition and overpartition formulas, bearing in mind how to automate these calculations.

Finally, we shall present some new restricted formulas conjectured with the aid of *Mathematica*.

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[1]Apostol [2] showed that it is also a $12k$th root of unity.

2. An Overview of the Derivation of the Hardy-Ramanujan-Rademacher Formula for $p(n)$

2.1. Preliminaries.

2.1.1. *The Dedekind η-function.* Let $\mathcal{H} := \{\tau \in \mathbb{C} \mid \Im\tau > 0\}$, the upper half of the complex plane.

The Dedekind eta function is defined by

$$(2.1) \qquad \eta(\tau) := e^{\pi i \tau/12} \prod_{m=1}^{\infty} (1 - e^{2\pi i m \tau})$$

where $\tau \in \mathcal{H}$.

For a, b, c, $d \in \mathbb{Z}$ with $ad - bc = 1$, and $c > 0$, $\eta(\tau)$ satisfies the functional equation

$$(2.2) \qquad \eta\left(\frac{a\tau + b}{c\tau + d}\right) = \omega(-d, c) \exp\left(\pi i \left(\frac{a + d}{12c}\right)\right) \sqrt{-i(c\tau + d)} \, \eta(\tau),$$

where
$$(2.3)$$
$$\omega(h, k) = \begin{cases} \left(\frac{-k}{h}\right) \exp\left(-\pi i \left\{\frac{1}{4}(2 - hk - h) + \frac{1}{12}(k - \frac{1}{k})(2h - H + h^2 H)\right\}\right), & \text{if } 2 \nmid h \\ \left(\frac{-h}{k}\right) \exp\left(-\pi i \left\{\frac{1}{4}(k - 1) + \frac{1}{12}(k - \frac{1}{k})(2h - H + h^2 H)\right\}\right), & \text{if } 2 \nmid k \end{cases}$$

$\left(\frac{a}{b}\right)$ is the Legendre-Jacobi symbol, and H is any solution of the congruence

$$hH \equiv -1 \pmod{k}.$$

2.1.2. *Farey fractions.* The sequence $\mathcal{F}_N$ of *proper Farey fractions of order N* is the set of all $\frac{h}{k}$ with $(h, k) = 1$ and $0 \leqq \frac{h}{k} < 1$, arranged in increasing order.

Thus, we have

$$\mathcal{F}_1 = \left\{\frac{0}{1}\right\}, \quad \mathcal{F}_2 = \left\{\frac{0}{1}, \frac{1}{2}\right\}, \quad \mathcal{F}_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right\}, \quad \mathcal{F}_4 = \left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\right\},$$

etc.

For a given N, let h_p, h_s, k_p, and k_s be such that $\frac{h_p}{k_p}$ is the immediate predecessor of $\frac{h}{k}$ and $\frac{h_s}{k_s}$ is the immediate successor of $\frac{h}{k}$ in $\mathcal{F}_N$. It will be convenient to view each F_N cyclically, i.e. to view $\frac{0}{1}$ as the immediate successor of $\frac{N-1}{N}$.

2.1.3. *Ford circles and the Rademacher path.* Let h and k be integers with $(h, k) = 1$ and $0 \leqq h < k$. The *Ford circle* [7] $C(h, k)$ is the circle in $\mathbb{C}$ of radius $\frac{1}{2k^2}$ centered at the point

$$\frac{h}{k} + \frac{1}{2k^2}i.$$

The *upper arc* $\gamma(h, k)$ *of the Ford circle* $C(h, k)$ is the arc of the circle

$$\left|\tau - \left(\frac{h}{k} + \frac{1}{2k^2}i\right)\right| = \frac{1}{2k}$$

from the initial point

$$(2.4) \qquad \alpha_I(h, k) := \frac{h}{k} - \frac{k_p}{k(k^2 + k_p^2)} + \frac{1}{k^2 + k_p^2}i$$

to the terminal point

$$(2.5) \qquad \alpha_T(h, k) := \frac{h}{k} + \frac{k_s}{k(k^2 + k_s^2)} + \frac{1}{k^2 + k_s^2}i,$$

traversed clockwise.

Note that we have $\alpha_I(0, 1) = \alpha_T(N - 1, N)$.

Every Ford circle is in the upper half plane. For $\frac{h_1}{k_1}, \frac{h_2}{k_2} \in \mathcal{F}_N$, $C(h_1, k_1)$ and $C(h_2, k_2)$ are either tangent or do not intersect.

The *Rademacher path* $P(N)$ of order N is the path in the upper half of the τ-plane from i to $i + 1$ consisting of

$$(2.6) \qquad \bigcup_{\frac{h}{k} \in \mathcal{F}_N} \gamma(h, k)$$

traversed left to right and clockwise. In particular, we consider the left half of the Ford circle $C(0, 1)$ and the corresponding upper arc $\gamma(0, 1)$ to be translated to the right by 1 unit. This is legal given the periodicity of the function which is to be integrated over $P(N)$.

2.2. Euler and Cauchy get us off the ground. Recall Euler's generating function for $p(n)$,

$$(2.7) \qquad f(q) := \sum_{n=0}^{\infty} p(n)q^n = \prod_{m=1}^{\infty} \frac{1}{1 - q^m}.$$

Let us now fix n. The function $f(q)/q^{n+1}$ has a pole of order $n + 1$ at $q = 0$, and an essential singularity at every point of the unit circle $|q| = 1$. The Laurent series of $f(q)/q^{n+1}$ about $q = 0$ is therefore

$$\sum_{j=0}^{\infty} p(j)q^{j-n-1} = \sum_{j=-n-1}^{\infty} p(j + n + 1)q^j,$$

for $0 < |q| < 1$, and so the residue of $f(q)/q^{n+1}$ at $q = 0$ is $p(n)$.

Thus, Cauchy's residue theorem implies that

$$(2.8) \qquad p(n) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(q)}{q^{n+1}} \, dq,$$

where $\mathcal{C}$ is any positively oriented, simple closed contour enclosing the origin and inside the unit circle.

2.3. The choice of $\mathcal{C}$. Since

$$\frac{f(q)}{q^{n+1}} = \frac{1}{q^{n+1}} \prod_{k=1}^{\infty} \frac{1}{1 - q^k} = \frac{1}{q^{n+1}} \prod_{k=1}^{\infty} \prod_{j=0}^{k-1} \frac{1}{1 - e^{2\pi i j/k} q}$$

we see that although every point of along $|q| = 1$ is an essential singularity of $f(q)/q^{n+1}$, in some sense $q = 1$ is the "heaviest" singularity, $q = -1$ is "half as heavy," $q = e^{2\pi i/3}$ and $e^{4\pi i/3}$ are each "one third as heavy," etc.

The integral (2.8) is evaluated by approximating the integrand for each h, k by an elementary function which is very nearly equal to $f(q)/q^{n+1}$ near the singularity $e^{2\pi i h/k}$. The contour $\mathcal{C}$ is chosen in such a way that the error introduced by this approximation is carefully kept under control.

We introduce the change of variable $q = \exp(2\pi i \tau)$ so that the unit disk $|q| \leqq 1$ in the q-plane maps to the infinitely tall, unit wide strip in the τ plane where $0 \leqq \Re\tau \leqq 1$ and $\Im\tau \geqq 0$. The contour $\mathcal{C}$ is then taken to be the preimage of the

Rademacher path $P(N)$ (see (2.6)) under the map $q \mapsto \exp(2\pi i \tau)$. Better yet, let us replace q with $\exp(2\pi i \tau)$ in (2.8) to express the integration in the τ-plane:

$$p(n) = \int_{P(N)} f(e^{2\pi i \tau}) e^{-2\pi i n \tau} d\tau$$

$$= \sum_{\frac{h}{k} \in \mathcal{F}_N} \int_{\gamma(h,k)} f(e^{2\pi i \tau}) e^{-2\pi i n \tau} d\tau$$

$$= \sum_{k=1}^{N} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \int_{\gamma(h,k)} f(e^{2\pi i \tau}) e^{-2\pi i n \tau} d\tau$$

2.4. Another change of variable. Next, we change variables again, taking

$$(2.9) \qquad \tau = \frac{iz + h}{k}$$

so that $z = -ik\left(\tau - \frac{h}{k}\right)$, for each $\tau \in C(h,k)$. Thus $C(h,k)$ (in the τ-plane) maps to the clockwise-oriented circle $K_k^{(-)}$ (in the z-plane) centered at $1/2k$ with radius $1/2k$.

So we now have

$$(2.10) \qquad p(n) = \sum_{k=1}^{N} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \int_{z_I(h,k)}^{z_T(h,k)} f(e^{2\pi i h/k - 2\pi z/k}) e^{-2\pi i n(iz+h)/k} \frac{i}{k} \, dz$$

$$(2.11) \qquad = \sum_{k=1}^{N} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{i}{k} e^{-2\pi i n h/k} \int_{z_I(h,k)}^{z_T(h,k)} e^{2n\pi z/k} f(e^{2\pi i h/k - 2\pi z/k}) \, dz,$$

where $z_I(h,k)$ (resp. $z_T(h,k)$) is the image of $\alpha_I(h,k)$ (see (2.4)) (resp. $\alpha_T(h,k)$ [see (2.5)]) under the transformation (2.9).

So the transformation (2.9) maps the upper arc $\gamma(h,k)$ of $C(h,k)$ in the τ-plane to the arc on $K_k^{(-)}$ which initiates at

$$(2.12) \qquad z_I(h,k) = \frac{k}{k^2 + k_p^2} + \frac{k_p}{k^2 + k_p^2} i$$

and terminates at

$$(2.13) \qquad z_T(h,k) = \frac{k}{k^2 + k_s^2} - \frac{k_s}{k^2 + k_s^2} i.$$

2.5. Exploiting a modular transformation. It is incredibly fortunate that

$$f(q) = f(e^{2\pi i \tau}) = \frac{e^{\pi i \tau/12}}{\eta(\tau)},$$

so that we may take advantage of the modular functional equation (2.2) satisfied by $\eta(\tau)$ in our effort to evaluate (2.11). Equation (2.2) rewritten in terms of $f(q)$ is

$$(2.14) \qquad f(e^{2\pi i(iz+h)/k}) = \omega(h,k) e^{\pi(z^{-1}-z)/12k} \sqrt{z} f(e^{2\pi i(iz^{-1}+H)/k}),$$

where $(h,k) = 1$ and H is any solution to the congruence $hH \equiv -1 \pmod{k}$, and $\sqrt{z}$ indicates the principle branch.

Note that when $|z|$ is close to 0, the left hand side of (2.14) is close to $f(e^{2\pi i h/k})$, i.e. for $|z|$ small, (2.14) gives a good approximation for f evaluated at the "heavy" singularity $e^{2\pi i h/k}$. Next, observe that the final factor on the right hand side of (2.14),

$$f(e^{2\pi i(iz^{-1}+H)/k}) = f\left(\exp\left(\frac{2\pi iH}{k} - \frac{2\pi}{zk}\right)\right),$$

is close to $f(0) = 1$ when $|z|$ is small, so that

$$f(e^{2\pi i(iz^{-1}+H)/k}) - 1$$

is close to 0 when $|z|$ is small.

Applying this information to (2.11), we find that

$$
\begin{aligned}
p(n) = &\sum_{k=1}^{N} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{i}{k} e^{-2\pi i nh/k} \omega(h,k) \\
&\times \int_{z_I(h,k)}^{z_T(h,k)} e^{2n\pi z/k} e^{\pi(z^{-1}-z)/12k} \sqrt{z} f(e^{2\pi i(iz^{-1}+H)/k}) \, dz \\
= &\sum_{k=1}^{N} \frac{i}{k} \sum_{\substack{0 \le h < k \\ (h,k)=1}} e^{-2\pi i nh/k} \omega(h,k) \\
&\times \int_{z_I(h,k)}^{z_T(h,k)} e^{\pi(24nz+z^{-1}-z)/12k} \sqrt{z} \left\{1 + \left[f(e^{2\pi i(iz^{-1}+H)/k}) - 1\right]\right\} \, dz \\
= &\sum_{k=1}^{N} \frac{i}{k} \sum_{\substack{0 \le h < k \\ (h,k)=1}} e^{-2\pi i nh/k} \omega(h,k) \left(I_{h,k} + I_{h,k}^*\right),
\end{aligned}
$$

(2.15)

where

(2.16)
$$I_{h,k} := \int_{z_I(h,k)}^{z_T(h,k)} e^{\pi(24nz+z^{-1}-z)/12k} \sqrt{z} \, dz$$

and

(2.17)
$$I_{h,k}^* := \int_{z_I(h,k)}^{z_T(h,k)} e^{\pi(24nz+z^{-1}-z)/12k} \sqrt{z} \left[f(e^{2\pi i(iz^{-1}+H)/k}) - 1\right] \, dz.$$

2.6. Estimating $I_{h,k}^*$. The next goal is to show that $I_{h,k}^*$ is small when N is large. Note that we can change the path of integration of (2.17) from an arc of the circle that is the image of the Ford circle under the transformation (2.9) connecting $z_I(h,k)$ and $z_T(h,k)$ to the line segment connecting $z_I(h,k)$ and $z_T(h,k)$ without altering the value of the integral. On the segment connecting $z_I(h,k)$ and $z_T(h,k)$, we have

(2.18) $\quad |z| \le \max\{|z_I(h,k)|, |z_T(h,k)|\} = \max\left\{\sqrt{\frac{1}{k^2+k_p^2}}, \sqrt{\frac{1}{k^2+k_s^2}}\right\} \le \frac{\sqrt{2}}{N}$

Obviously, the length of the segment connecting $z_I(h,k)$ and $z_T(h,k)$ can be easily calculated for any particular h and k. However, we wish to have an upper bound for the length that holds for a given N.

The length of the segment is $\leqq |z_I(h,k)| + |z_T(h,k)| \leqq \frac{2\sqrt{2}}{N}$.

Bearing in mind that on the segment, $\Re z < \frac{1}{k}$ and $\Re\left(\frac{1}{z}\right) = k$, it can be shown that the integrand in (2.17) is less than $c|z|^{1/2}$, where

$$c = e^{2n\pi/k^2} \sum_{m=1}^{\infty} p(24m-1)t^{24m-1},$$

by mimicking the argument in [**3**, p. 107].

Since z is on the segment connecting $z_I(h,k)$ to $z_T(h,k)$, $|z|$ is bounded above by $\sqrt{2}/N$, so the integrand is bounded above by $c\sqrt{\sqrt{2}/N} = c2^{1/4}N^{-1/2}$. Thus

$$|I_{h,k}^*| < \frac{c2^{1/4}}{N^{1/2}}\frac{2\sqrt{2}}{N} = CN^{-3/2},$$

where $C = 2^{7/4}c$.

Finally, it can be shown (see, e.g., [**3**, p. 108]) that $|I_{h,k}^*| = O(N^{-1/2})$.

2.7. Estimations associated with $I_{h,k}$. The work of the preceding section allows us to rewrite (2.15) as

$$(2.19) \qquad p(n) = \sum_{k=1}^{N} \frac{i}{k} \sum_{\substack{0 \leqq h < k \\ (h,k)=1}} e^{-2\pi i n h/k}\omega(h,k)I_{h,k} + O(N^{-1/2}),$$

where, as before,

$$(2.20) \qquad I_{h,k} := \int_{z_I(h,k)}^{z_T(h,k)} e^{\pi(24nz+z^{-1}-z)/12k}\sqrt{z}\ dz.$$

We proceed by re-expressing $I_{h,k}$ as

$$(2.21) \qquad I_{h,k} = \int_{K_k^{(-)}} - \int_{0}^{z_I(h,k)} - \int_{z_T(h,k)}^{0},$$

where the integrands of all three integrals are the same as that of the right hand side of (2.20).

The length of the arc connecting 0 and $z_I(h,k)$ is less than

$$\frac{\pi}{2}|z_I(h,k)| < \frac{\pi}{2}\frac{\sqrt{2}}{N}.$$

On the arc, $|z| < \sqrt{2}/N$.

We had previously seen that the absolute value of the integrand is $< c|z|^{1/2}$, so

$$\int_{0}^{z_I(h,k)} e^{\pi(24nz+z^{-1}-z)/12k}\sqrt{z}\ dz < c\sqrt{\frac{\sqrt{2}}{N}}\frac{\pi}{\sqrt{2N}} = CN^{-3/2}.$$

An analogous estimate applies to $\int_{z_T(h,)}^{0}$.

2.8. The formula for $p(n)$. We may now write

(2.22)
$$p(n) = \sum_{k=1}^{N} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{i}{k} \, e^{-2\pi i n h/k} \int_{K_k^{(-)}} e^{2n\pi z/k} \omega(h,k) e^{\pi(z^{-1}-z)/12k} \sqrt{z} \, dz + O(N^{-1/2}).$$

Let $N \to \infty$ to obtain

(2.23)
$$p(n) = i \sum_{k=1}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{e^{-2\pi i n h/k}}{k} \omega(h,k) \int_{K_k^{(-)}} \sqrt{z} \exp\left\{ \frac{\pi}{12zk} + \frac{2\pi z}{k}\left(n - \frac{1}{24}\right) \right\} \, dz.$$

Next, apply the transformation $z = 1/w$ so that $dz = -1/w^2 \, dw$:

(2.24)
$$p(n) = \frac{1}{i} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{\substack{0 \le h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \omega_{h,k} \int_{1-\infty i}^{1+\infty i} w^{-5/2} \exp\left\{ \frac{\pi w}{12k} + \frac{2\pi}{wk}\left(n - \frac{1}{24}\right) \right\} \, dw.$$

The integral in (2.24) can be evaluated in terms of Bessel functions. To make this evaluation easier to see, we set $w = \frac{12k}{\pi} t$, so that $dw = \frac{12k}{\pi} dt$, to obtain

(2.25)
$$p(n) = 2\pi \left(\frac{\pi}{12}\right)^{3/2} \sum_{k=1}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} k^{-5/2} \, e^{-2\pi i n h/k} \omega(h,k)$$
$$\times \frac{1}{2\pi i} \int_{\pi/12k-\infty i}^{\pi/12k+\infty i} t^{-5/2} \exp\left\{ t + \frac{\pi^2}{6k^2 t}\left(n - \frac{1}{24}\right) \right\} \, dt.$$

Now recall the Bessel function of the first kind of purely imaginary argument is given by [**38**, p. 181, Eq. (1)]

$$I_\nu(z) = \frac{(z/2)^\nu}{2\pi i} \int_{-\infty}^{(0+)} t^{-\nu-1} \exp\left(t + \frac{z^2}{4t}\right) \, dt.$$

Taking into account the remark preceding Eq. (8) on p. 177 of [**38**], we may, since $\pi/12k > 0$, alter the path of integration to obtain

(2.26)
$$I_\nu(z) = \frac{(z/2)^\nu}{2\pi i} \int_{\pi/12k-i\infty}^{\pi/12k+i\infty} t^{-\nu-1} \exp\left(t + \frac{z^2}{4t}\right) \, dt.$$

Setting $\nu = 3/2$ and $z = \frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}$ in (2.26) and applying the result to (2.25), we find

(2.27)
$$p(n) = \frac{2\pi}{(24)^{3/2}} \left(n - \frac{1}{24}\right)^{-3/4} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{\substack{0 \le h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \omega_{h,k} I_{3/2}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right)$$

Bessel functions of half-odd order can be written in terms of elementary functions. In particular,

$$I_{3/2}(z) = \sqrt{\frac{2z}{\pi}} \frac{d}{dz}\left(\frac{\sinh z}{z}\right),$$

so the final form of the formula for $p(n)$ is

THEOREM 2.1 (Rademacher).

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \omega(h,k) \frac{d}{dn} \left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right)}{\sqrt{n - \frac{1}{24}}} \right).$$

3. Restricted Partition Functions

3.1. Partition Identities. Euler [6] observed that the algebraic identity

$$(3.1) \qquad \prod_{j=1}^{\infty} (1 + q^j) = \prod_{j=1}^{\infty} \frac{1}{1 - q^{2j-1}}$$

implies the following theorem about integer partitions:

THEOREM 3.1 (Euler). *The number of partitions of n into distinct parts equals the number of partitions of n into odd parts.*

While such a result tells us that there are the same number of partitions of n into distinct parts as there are partitions of n using only odd parts, we do not know *how many* such partitions of n there are. The circle method has been applied by P. Hagis [13] and L. K. Hua [22] to address this question.

THEOREM 3.2 (Hagis). *Let $\delta(n)$ denote the number of partitions of n into distinct parts. Then*

$$(3.2) \qquad \delta(n) = \frac{\pi}{\sqrt{24n+1}} \sum_{\substack{k \geq 1 \\ 2 \nmid k}} \frac{1}{k} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{-2\pi n h/k} \frac{\omega(h,k)}{\omega(2h,k)} I_1\left(\frac{\pi\sqrt{24n+1}}{6\sqrt{2}k} \right).$$

J.W.L. Glaisher [8] generalized Euler's result to

THEOREM 3.3 (Glaisher). *The number of partitions of n where no part appears more than $j-1$ times equals the number of "j-regular partitions of n", i.e. partitions of n where no part is a multiple of j.*

Clearly, Euler's theorem is the $j = 2$ case of Glaisher's theorem. Glaisher's theorem follows immediately from the identity

$$(3.3) \qquad \prod_{k=1}^{\infty} (1 + q^k + q^{2k} + \cdots + q^{(j-1)k}) = \prod_{\substack{k \geq 1 \\ k \not\equiv 0 \,(\mathrm{mod}\ j)}} \frac{1}{1 - q^k}.$$

THEOREM 3.4 (Hagis [20]). *Let $\delta_j(n)$ denote the number of j-regular partitions of n. Then*

$$(3.4) \quad \delta_j(n) = \frac{2\pi}{j\sqrt{24n+j-1}} \sum_{\substack{0 < d < \sqrt{j} \\ d|j}} \sqrt{d(j-d^2)} \sum_{\substack{k \geq 1 \\ (k,j)=d}} \frac{1}{k}$$

$$\times \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{-2\pi n h/k} \frac{\omega(h,k)}{\omega(\frac{jh}{d}, \frac{k}{d})} I_1\left(\frac{\pi}{6k} \sqrt{\frac{(24n+j-1)(j-d^2)}{j}} \right).$$

Another well known partition identity of this type is

Theorem 3.5 (I. Schur [**34**]). *The number of partitions of n into distinct parts which differ by at least three and where no consecutive multiples of three appear equals the number of partitions of n into parts congruent to $\pm 1 \pmod 6$.*

Theorem 3.6 (I. Niven [**29**]). *Let $S(n)$ denote the number of partitions of n into parts congruent to $\pm 1 \pmod 6$. Then*

$$(3.5) \quad S(n) = \frac{\pi}{\sqrt{36n-3}} \sum_{d|6} \sqrt{(d-2)(d-3)} \sum_{\substack{k \geq 1 \\ (k,6)=d}} \frac{1}{k}$$

$$\times \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{-2\pi nh/k} \frac{\omega(h,k)\omega(\frac{6h}{d},\frac{k}{d})}{\omega(\frac{2h}{(k,2)},\frac{k}{(k,2)})\omega(\frac{3h}{(k,3)},\frac{k}{(k,3)})} I_1\left(\frac{\pi\sqrt{d(12n-1)}}{3\sqrt{6}k}\right).$$

Recently, the author found [**36**] For $r = 0, 1, 2, 3, 4$,

$$p_r(n) = \frac{2^{(r+1)/2}\sqrt{3}}{\pi} \sum_{\substack{k \geq 1 \\ (k,2^{\max(r,1)})=1}} \sqrt{k} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{-2\pi inh/k} \frac{\omega(h,k)\omega(2^r h,k)}{\omega(2h,k)}$$

$$\times \frac{d}{dn}\left\{ \frac{\sinh\left(\frac{\pi\sqrt{(24n-2^r+1)(1+2^{r-1})}}{2^{r/2}\cdot 6k}\right)}{\sqrt{24n-2^r+1}}\right\}$$

$$+ \frac{\sqrt{3}}{\pi} \sum_{j=1+\lfloor\frac{r}{2}\rfloor}^{r} 2^{(2-j+r)/2} \sum_{\substack{k \geq 1 \\ (k,2^r)=2^j}} \sqrt{k} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{-2\pi inh/k} \frac{\omega(h,k)\omega(2^{r-j}h,2^{-j}k)}{\omega(h,\frac{k}{2})}$$

$$\times \frac{d}{dn}\left\{ \frac{\sinh\left(\frac{\pi\sqrt{(24n-2^r+1)(-1+2^{2j-r})}}{6k}\right)}{\sqrt{24n-2^r+1}}\right\},$$

where

$$\sum_{n=0}^{\infty} p_r(n)q^n = \prod_{m=1}^{\infty} \frac{1+q^m}{1-q^{2^r m}}.$$

The $r = 1$ case corresponds to the Rademacher formula for $p(n)$, Theorem 2.1. The $r = 0$ case, due to Zuckerman [**39**, p. 321, Eq. (8.53)], simplifies to

$$(3.6) \qquad p_0(n) = \bar{p}(n) = \frac{1}{2\pi}\sum_{\substack{k \geq 1 \\ 2 \nmid k}} \sqrt{k} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\omega(h,k)^2}{\omega(2h,k)} e^{-2\pi inh/k} \frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi\sqrt{n}}{k}\right)}{\sqrt{n}}\right)$$

and the $r = 2$ case is

$$(3.7) \quad p_2(n) = pod(n) = \frac{2}{\pi\sqrt{6}} \sum_{d|4} \sqrt{(d-2)(5d-17)} \sum_{\substack{k \geq 1 \\ (k,4)=d}} \sqrt{k}$$

$$\times \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\omega(h,k)\,\omega\left(\frac{4h}{d}, \frac{k}{d}\right)}{\omega\left(\frac{2h}{(k,2)}, \frac{k}{(k,2)}\right)} e^{-2\pi i n h/k} \frac{d}{dn} \left(\frac{\sinh\left(\frac{\pi\sqrt{d(8n-1)}}{4k}\right)}{\sqrt{8n-1}} \right),$$

where $pod(n)$ denotes the number of partitions of n where no odd part is repeated, and $\bar{p}(n)$ denotes the number of overpartitions of n. An *overpartition* of n is a finite weakly decreasing sequence of positive integers where the last occurrence of a given part may or may not be overlined. Thus the eight overpartitions of 3 are $(3), (\bar{3}), (2,1), (\bar{2},1), (2,\bar{1}), (\bar{2},\bar{1}), (1,1,1), (1,1,\bar{1})$. Overpartitions were introduced by S. Corteel and J. Lovejoy in [**5**] and have been studied extensively by them and others.

3.2. Distinct Parts. We have

$$\sum_{n=0}^{\infty} \delta(n) q^n = \prod_{\substack{j \geq 1 \\ 2 \nmid j}} \frac{1}{1-q^j} = \frac{f(q)}{f(q^2)} =: F(q),$$

where, as before $f(q) := \sum_{n \geq 0} p(n) q^n = \prod_{j \geq 1} (1-q^j)^{-1}$.

Proceeding as in the case of $p(n)$, we note

$$\delta(n) = \frac{1}{2\pi i} \int_C \frac{F(q)}{q^{n+1}}\,dq$$

$$= \frac{1}{2\pi i} \int_C \frac{f(q)}{f(q^2) q^{n+1}}\,dq$$

$$= \int_{P(N)} \frac{f(e^{2\pi i \tau})}{f(e^{4\pi i \tau})} e^{-2\pi i n \tau}\,d\tau$$

$$= \sum_{\frac{h}{k} \in \mathcal{F}_N} \int_{\gamma(h,k)} \frac{f(e^{2\pi i \tau})}{f(e^{4\pi i \tau})} e^{-2\pi i n \tau}\,d\tau$$

$$= \sum_{k=1}^{N} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \int_{\gamma(h,k)} \frac{f(e^{2\pi i \tau})}{f(e^{4\pi i \tau})} e^{-2\pi i n \tau}\,d\tau$$

$$= \sum_{k=1}^{N} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \int_{z_I(h,k)}^{z_T(h,k)} \frac{f(e^{2\pi i h/k - 2\pi z/k})}{f(e^{4\pi i h/k - 4\pi z/k})} e^{-2\pi i n (iz+h)/k} \frac{i}{k}\,dz$$

$$(3.8) \qquad = i \sum_{k=1}^{N} k^{-1} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \int_{z_I(h,k)}^{z_T(h,k)} e^{2n\pi z/k} \frac{f(e^{2\pi i h/k - 2\pi z/k})}{f(e^{4\pi i h/k - 4\pi z/k})}\,dz,$$

where, as before, $q = e^{2\pi i \tau}$, $\tau = (iz+h)/k$, and $P(N)$, $\gamma(h,k)$, $z_I(h,k)$, and $z_T(h,k)$ all have the same meaning as before.

At this point, we should like to transform

$$F(q) = f(q)/f(q^2) = f(e^{2\pi i h/k - 2\pi z/k})/f(e^{4\pi i h/k - 4\pi z/k}),$$

just as we had transformed $f(q) = f(e^{2\pi i h/k - 2\pi z/k})$ via (2.14) in the analogous analysis of $p(n)$.

It will be necessary to consider two cases. When k is even, $k/2$ is an integer, so we can obtain $f(q^2)$ from $f(q)$ by replacing k by $k/2$ in $f(e^{2\pi ih/k - 2\pi z/k})$. On the other hand, when k is odd, we instead replace h by $2h$ and z by $2z$ in $f(e^{2\pi ih/k - 2\pi z/k})$. Thus,

(3.9)

$$
F(e^{2\pi ih/k - 2\pi z/k}) = \begin{cases} \frac{\omega(h,k)}{\omega(h,k/2)} \exp\left(\frac{\pi z}{12k} - \frac{\pi}{12kz}\right) F\left(\exp\left(\frac{2\pi i(H_1 + iz^{-1})}{k}\right)\right), & \text{if } 2 \mid k, \\ \frac{\omega(h,k)}{\omega(2h,k)\sqrt{2}} \exp\left(\frac{\pi z}{12k} + \frac{\pi}{24kz}\right)/F\left(\exp\left(\frac{\pi i(H_2 + iz^{-1})}{k}\right)\right), & \text{if } 2 \nmid k, \end{cases}
$$

where H_j is a solution to the congruence $jhH_j \equiv -1 \pmod{k}$.

Thus,

$$
\delta(n) = i\sum_{\substack{k=1 \\ 2|k}}^{N} k^{-1} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{\omega(h,k)}{\omega(h,k/2)} e^{-2\pi inh/k} \int_{z_I(h,k)}^{z_T(h,k)} \exp\left[\frac{2\pi z}{k}\left(n + \frac{1}{24}\right) - \frac{\pi}{12kz}\right]
$$

$$
\times F\left(\exp\left(\frac{2\pi i(H_1 + iz^{-1})}{k}\right)\right) dz
$$

$$
+ \frac{i}{\sqrt{2}} \sum_{\substack{k=1 \\ 2\nmid k}}^{N} k^{-1} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{\omega(h,k)}{\omega(2h,k)} e^{-2\pi inh/k} \int_{z_I(h,k)}^{z_T(h,k)} \frac{\exp\left[\frac{2\pi z}{k}\left(n + \frac{1}{24}\right) + \frac{\pi}{24kz}\right]}{F\left(\exp\left(\frac{\pi i(H_2 + iz^{-1})}{k}\right)\right)} dz.
$$

Next, we expand the appearances of F as series:

$$
\delta(n) = i\sum_{\substack{k=1 \\ 2|k}}^{N} k^{-1} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{\omega(h,k)}{\omega(h,k/2)} e^{-2\pi inh/k} \int_{z_I(h,k)}^{z_T(h,k)} \exp\left[\frac{2\pi z}{k}\left(n + \frac{1}{24}\right) - \frac{\pi}{12kz}\right]
$$

$$
\times \sum_{m=0}^{\infty} \delta(m) \exp\left(\frac{2\pi i(H_1 + iz^{-1})m}{k}\right) dz
$$

$$
+ \frac{i}{\sqrt{2}} \sum_{\substack{k=1 \\ 2\nmid k}}^{N} k^{-1} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{\omega(h,k)}{\omega(2h,k)} e^{-2\pi inh/k} \int_{z_I(h,k)}^{z_T(h,k)} \exp\left[\frac{2\pi z}{k}\left(n + \frac{1}{24}\right) + \frac{\pi}{24kz}\right]
$$

$$
\times \sum_{m=0}^{\infty} \delta^*(m) \exp\left(\frac{\pi i(H_2 + iz^{-1})m}{k}\right) dz
$$

$$
(3.10) \quad = i\sum_{\substack{k=1 \\ 2|k}}^{N} k^{-1} \sum_{m=0}^{\infty} \delta(m) \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{\omega(h,k)}{\omega(h,k/2)} \exp\left[\frac{2\pi i}{k}(H_1 m - hn)\right]
$$

$$
\times \int_{z_I(h,k)}^{z_T(h,k)} \exp\left[\frac{2\pi z}{k}\left(n + \frac{1}{24}\right) - \frac{\pi}{kz}\left(2m + \frac{1}{12}\right)\right] dz
$$

$$
+ \frac{i}{\sqrt{2}} \sum_{\substack{k=1 \\ 2\nmid k}}^{N} k^{-1} \sum_{m=1}^{\infty} \delta^*(m) \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{\omega(h,k)}{\omega(2h,k)} \exp\left[\frac{\pi i}{k}(H_2 m - 2hn)\right]
$$

$$\times \int_{z_I(h,k)}^{z_T(h,k)} \exp\left[\frac{2\pi z}{k}\left(n+\frac{1}{24}\right)+\frac{\pi}{kz}\left(\frac{1}{24}-m\right)\right] dz$$

$$+\frac{i}{\sqrt{2}}\sum_{\substack{k=1\\2\nmid k}}^{N}k^{-1}\sum_{\substack{0\leq h<k\\(h,k)=1}}\frac{\omega(h,k)}{\omega(2h,k)}e^{-2\pi inh/k}\int_{z_I(h,k)}^{z_T(h,k)}\exp\left[\frac{2\pi z}{k}\left(n+\frac{1}{24}\right)+\frac{\pi}{24kz}\right]dz$$

where

$$\frac{1}{F(q)}=\sum_{n=0}^{\infty}\delta^*(n)q^n,$$

and we have used the fact that $\delta^*(0)=1$.

If the three sums in (3.10) are designated S_1, S_2, and S_3 respectively, it can be shown via Kloosterman sum estimation that $S_1, S_2 \to 0$ as $N \to \infty$ and only S_3 contributes to the final formula for $\delta(n)$.

(3.11)

$$\delta(n)=\frac{-i}{24\sqrt{2}}\sum_{\substack{k=1\\2\nmid k}}^{N}\frac{1}{k}\sum_{\substack{0\leq h<k\\(h,k)=1}}\frac{\omega(h,k)}{\omega(2h,k)}e^{-2\pi inh/k}\int_{z_I(h,k)}^{z_T(h,k)}\exp\left[\frac{2\pi z}{k}\left(n+\frac{1}{24}\right)+\frac{\pi}{24kz}\right]dz.$$

Change variables $t=\frac{\pi}{12kz}$ to obtain

$$\delta(n)=\frac{-i\pi}{24\sqrt{2}}\sum_{\substack{k=1\\2\nmid k}}^{\infty}\frac{1}{k^2}$$

$$\times\sum_{\substack{0\leq h<k\\(h,k)=1}}e^{-2\pi ihn/k}\frac{\omega(h,k)}{\omega(2h,k)}\int_{\pi/12-\infty i}^{\pi/12+\infty i}t^{-2}\exp\left(t+\frac{\pi^2(24n+1)}{288k^2t}\right)dt$$

(3.12) $$=\frac{\pi}{\sqrt{24n+1}}\sum_{\substack{k=1\\2\nmid k}}^{\infty}\frac{1}{k}\sum_{\substack{0\leq h<k\\(h,k)=1}}e^{-2\pi ihn/k}\frac{\omega(h,k)}{\omega(2h,k)}I_1\left(\frac{\pi\sqrt{24n+1}}{6k\sqrt{2}}\right).$$

3.3. Summary of calculations. We now summarize the required steps to find a Rademacher type formula for $a(n)$ where

$$\sum_{n=0}^{\infty}a(n)q^n=\prod_{j=1}^{J}\frac{f(q^{b_j})}{f(q^{c_j})}.$$

- Find $L:=\operatorname{lcm}(b_1,b_2,\ldots,b_J,c_1,c_2,\ldots,c_J)$.
- For each divisor d of L, there corresponds a case $\gcd(k,L)=d$.
  - To each case there corresponds to a summand of the form

$$\Omega_{h,k}C\Psi_k(z)F(z,h,k)$$

  which results from applying the modular transformation (2.2) to that case. $\Omega_{h,k}$ is a product of powers of the ω $24k$th root of 1, C is the constant that results, $\Psi_k(z)$ is the exponential expression, and $F(z,h,k)$ is the product of powers of f.

– Only those cases for which the coefficient of z^{-1} in $\log \Psi_1(z)$ is positive will contribute to the final formula; others can be shown to approach 0 via Kloosterman sum estimation.
– Map $z \mapsto \pi/(12kt)$.
– Evaluate integral in terms of the I_1 Bessel function.

4. Slater's list

In 1952, L. J. Slater published a list of 130 identities of Rogers-Ramanujan type [**35**]. Many of the infinite products can be realized as products of powers of η-functions, and have straightforward combinatorial interpretations as generating functions of restricted classes of partitions or overpartitons.

Let us recall some of the identities in Slater's list.

(S. 5)
$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(2n+1)}}{(q^2;q^2)_n(-q;q^2)_{n+1}} = \prod_{m=1}^{\infty} (1+q^{2m})(1-q^{2m-1})$$

(S. 8)
$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q)_n}{(q)_n} = \prod_{m=1}^{\infty} \frac{1-q^{4m}}{1-q^m}$$

(S. 9 = S. 84)
$$\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q)_{2n+1}} = \prod_{m=1}^{\infty} (1+q^m)$$

(S. 10)
$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-1)_{2n}}{(q^2;q^2)_n(q^2;q^4)_n} = \prod_{m=1}^{\infty} \frac{1+q^{2m-1}}{1-q^{2m-1}}$$

(S. 11 = S.51 = S.64)
$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q;q^2)_n}{(q)_{2n+1}} = \prod_{m=1}^{\infty} \frac{1-q^{4m}}{1-q^m}$$

(S. 24)
$$\sum_{n=0}^{\infty} \frac{q^n(-1)_{2n}}{(q^2;q^2)_n} = \prod_{m=1}^{\infty} \frac{(1-q^{6m-3})^2(1-q^{6m})(1+q^m)}{1-q^m}$$

(S. 26)
$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q)_n}{(q;q^2)_{n+1}(q)_n} = \prod_{m=1}^{\infty} \frac{(1-q^{6m-3})^2(1-q^{6m})(1+q^m)}{1-q^m}$$

(S. 27)
$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}(-q;q^2)_n}{(q)_{2n+1}(-q^2;q^2)_n} = \prod_{m=1}^{\infty} \frac{(1+q^{6m-5})(1+q^{6m-1})}{(1-q^{6m-4})(1-q^{6m-2})}$$

(S. 52 = S. 85)
$$\sum_{n=0}^{\infty} \frac{q^{n(2n-1)}}{(q)_{2n}} = \prod_{m=1}^{\infty} (1+q^m)$$

(S. 76)
$$\sum_{n=0}^{\infty} \frac{q^{n(n+3)/2}(-q)_{n+1}(q^3;q^3)_n}{(q)_n(q)_{2n+2}} = \prod_{m=1}^{\infty} \frac{(1-q^{18m})(1-q^{18m-3})(1-q^{18m-15})}{(1-q^{2m-1})(1-q^m)}$$

(S. 77)
$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q)_n(q^3;q^3)_n}{(q)_n(q)_{2n+1}} = \prod_{m=1}^{\infty} \frac{(1-q^{6m})(1+q^m)}{1-q^m}$$

(S. 78)
$$1+\sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}(-1)_{n+1}(q^3;q^3)_{n-1}}{(q)_{n-1}(q)_{2n}} = \prod_{m=1}^{\infty} \frac{(1-q^{18m})(1-q^{18m-9})^2(1+q^m)}{1-q^m}$$

(S. 92)
$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(q^3;q^3)_n}{(q)_{2n+1}(q)_n} = \prod_{m=1}^{\infty} \frac{1-q^{9m}}{1-q^m}$$

(S. 107)
$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(q^3;q^6)_n(-q^2;q^2)_n}{(q^2;q^2)_{2n+1}(q;q^2)_n} = \prod_{m=1}^{\infty} \frac{(1-q^{6m})(1+q^{12m-3})(1+q^{12m-9})}{(1-q^{4m-2})(1-q^{2m})}$$

(S. 110 corrected)
$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}(q^3;q^6)_n(-q;q^2)_{n+1}}{(q^2;q^2)_{2n+1}(q;q^2)_n} = \prod_{m=1}^{\infty} \frac{(1-q^{12m})(1-q^{4m-2})}{1-q^m}$$

(S. 115)
$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}(q^6;q^6)_n(-q;q^2)_{n+1}}{(q^2;q^2)_n(q^2;q^2)_{2n+2}} = \prod_{m=1}^{\infty} \frac{(1-q^{36m})(1-q^{36m-27})(1-q^{36m-9})}{(1-q^{2m-1})(1-q^{4m})}$$

Denote the coefficient of q^n in the power series expansion of equation (S.j) above by $S_j(n)$. The following combinatorial interpretations are then immediate:

- $S_8(n) = S_{11}(n) = \delta_4(n) =$ the number of 4-regular partitions of n; see Theorem 3.4.
- $S_9(n) = S_{52}(n) = \delta(n) =$ the number of partitions into odd parts; see Theorem 3.2.
- $S_{10}(n) =$ the number of overpartitions of n with only odd parts.
- $S_{27}(n) =$ the number of overpartitions of n where overlined parts are odd nonmultiples of 3 and the nonoverlined parts are even nonmultiples of 6.
- $S_{76}(n) =$ the number of overpartitions of n where no nonoverlined part is congruent to 0, 3, or 15 (mod 18).
- $S_{77}(n) =$ the number of overpartitions of n where no nonoverlined part is a multiple of 6.
- $S_{92}(n) = \delta_9(n) =$ the number of 9-regular partitions of n; see Theorem 3.4.
- $S_{107}(n) =$ the number of overpartitions of n where overlined parts are even or ± 3 (mod 12) and nonoverlined parts are ± 2 (mod 6).
- $S_{110}(n) =$ the number of partitions of n into parts not congruent to $0, 2, 6, 10$ (mod 12).
- $S_{115}(n) =$ the number of partitions of n into parts not congruent to $0, \pm 9$ (mod 36) nor congruent to 2 (mod 4).

$S_5(n)$, $S_{24}(n)$, and $S_{78}(n)$ are not as easily interpreted in terms of partitions or overpartitions, because of the presence of a repeated factor in the numerator.

The following Rademacher type formulas were conjectured with the aid of *Mathematica*, and are believed to be new:

$$(4.1) \quad S_5(n) = \frac{2\pi}{\sqrt{24n+1}} \sum_{\substack{k \geq 1 \\ k \equiv 2 (\mathrm{mod}\ 4)}} \frac{1}{k}$$

$$\times \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \frac{\omega(h,\frac{k}{2})^2}{\omega(h,k)\omega(2h,\frac{k}{2})} I_1\left(\frac{\pi\sqrt{24n+1}}{3k\sqrt{2}}\right).$$

$$(4.2) \qquad S_{10}(n) = \frac{\pi}{4\sqrt{n}} \sum_{\substack{k \geq 1 \\ 2 \nmid k}} \frac{1}{k} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \frac{\omega(h,k)^2 \omega(4h,k)}{\omega(2h,k)^3} I_1\left(\frac{\pi\sqrt{n}}{k\sqrt{2}}\right).$$

$$(4.3) \qquad S_{24}(n) = \frac{\pi}{3\sqrt{2n}} \sum_{\substack{k \geq 1 \\ 2,3 \nmid k}} \frac{1}{k} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \frac{\omega(h,k)^2 \omega(6h,k)}{\omega(2h,k)\omega(3h,k)^2} I_1\left(\frac{\pi\sqrt{2n}}{k\sqrt{3}}\right)$$

$$(4.4) \quad S_{27}(n) = \frac{\pi}{9\sqrt{4n+1}} \sum_{d|4} (d-2)(2d-5) \sum_{\substack{k \geq 1 \\ (k,12)=d}} \frac{1}{k}$$
$$\times \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \frac{\omega(h,k)\omega(\frac{4h}{d},\frac{k}{d})\omega(\frac{6h}{\sqrt{d}},\frac{k}{\sqrt{d}})}{\omega(\frac{12h}{d},k)\omega(3h,\frac{k}{d})\omega(\frac{2h}{\sqrt{d}},\frac{k}{\sqrt{d}})} I_1\left(\frac{\pi\sqrt{d(4n+1)}}{2k\sqrt{3}}\right)$$

$$(4.5) \quad S_{76}(n) = \frac{\pi}{9\sqrt{2n+2}} \sum_{\substack{k \geq 1 \\ (k,18)=1}} \frac{1}{k}$$
$$\times \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \frac{\omega(h,k)^2 \omega(6h,k)\omega(9h,k)}{\omega(2h,k)\omega(3h,k)\omega(18h,k)^2} I_1\left(\frac{2\pi\sqrt{2n+2}}{3k}\right)$$

$$(4.6) \quad S_{77}(n) = \frac{\pi\sqrt{2}}{3\sqrt{12n+3}} \sum_{\substack{k \geq 1 \\ 2,3 \nmid k}} \frac{1}{k} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \frac{\omega(h,k)^2}{\omega(2h,k)\omega(6h,k)} I_1\left(\frac{\pi\sqrt{8n+2}}{3k}\right)$$

$$(4.7) \quad S_{78}(n) = \frac{\pi\sqrt{2}}{9\sqrt{n}} \sum_{\substack{k \geq 1 \\ (k,18)=1}} \frac{1}{k} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \frac{\omega(h,k)^2 \omega(18h,k)}{\omega(2h,k)\omega(9h,k)^2} I_1\left(\frac{2\pi\sqrt{2n}}{3k}\right)$$

$$(4.8) \quad S_{107}(n) = \frac{2\pi}{3\sqrt{24n+3}} \sum_{j=1}^{2} \sqrt{4j-3} \sum_{\substack{k \geq 1 \\ (k,12)=j}} \frac{1}{k}$$
$$\times \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \frac{\omega(\frac{2h}{j},\frac{k}{j})^2 \omega(3h,k)\omega(\frac{12h}{j},\frac{k}{j})}{\omega(\frac{4h}{j},\frac{k}{j})\omega(\frac{6h}{j},\frac{k}{j})^3} I_1\left(\frac{\pi\sqrt{(3j-1)(8n+1)}}{6k}\right)$$

$$(4.9) \quad S_{110}(n) = \frac{2\pi}{9\sqrt{16n+6}} \sum_{d|4} \sqrt{(d-2)(7d-13)} \sum_{\substack{k \geq 1 \\ (k,12)=d}} \frac{1}{k}$$

$$\times \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \frac{\omega(h,k)\omega(\frac{4h}{d},\frac{k}{d})}{\omega(\frac{2h}{\sqrt{d}},\frac{k}{\sqrt{d}})\omega(\frac{12h}{d},\frac{k}{d})} I_1\left(\frac{\pi\sqrt{1+d}\sqrt{8n+3}}{6k}\right)$$

$$(4.10) \quad S_{115}(n) = \frac{\pi}{27\sqrt{n+1}}\sum_{d|4}(d-2)(2d-5)\sum_{\substack{k \geq 1 \\ (k,36)=d}}\frac{1}{k}$$

$$\times \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \frac{\omega(h,k)\omega(\frac{4h}{d},\frac{k}{d})\omega(\frac{18h}{\sqrt{d}},\frac{k}{\sqrt{d}})}{\omega(9h,k)\omega(\frac{2h}{\sqrt{d}},\frac{k}{\sqrt{d}})\omega(\frac{36h}{d},\frac{k}{d})} I_1\left(\frac{2\sqrt{d}\pi\sqrt{n+1}}{3k}\right)$$

5. Numerical Test

Each of the formulas (4.2)–(4.10), along with Hagis's formula (3.2) and Niven's formula (3.5), was tested summing k from 1 to 10, and the value provided by the formula was compared with the actual value. In the chart below, the true value of the given function at $n = 100$ is provided along with the magnitude of the largest error in the formula (when truncated at $k = 10$) for $1 \leq n \leq 100$.

Eq. no.	function	value at $n = 100$	max error
(3.2)	$\delta(n)$	444 793	0.211
(3.5)	$S(n)$	20 901	0.318
(4.1)	$S_5(n)$	444 793	0.186
(4.2)	$S_{10}(n)$	29 025 326	0.210
(4.3)	$S_{24}(n)$	793 378 722	0.200
(4.4)	$S_{27}(n)$	369 566	0.188
(4.5)	$S_{76}(n)$	15 008 235 468	0.050
(4.6)	$S_{77}(n)$	23 399 621 246	0.133
(4.7)	$S_{78}(n)$	26 086 456 322	0.143
(4.8)	$S_{107}(n)$	4 690 080	0.166
(4.9)	$S_{110}(n)$	4 731 983	0.216
(4.10)	$S_{115}(n)$	4 105 275	0.162

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