APPENDICES

A Proof of Lemma 1

Proof. If $\mu(A) > 0$ then $K(x, A) = \int_A k(x'|x)\mu(dx') > 0$ for all $x \in \mathsf{X}$; i.e., it is possible to get from any point $x \in \mathsf{X}$ to any non measure zero set A in one step. This implies that X is aperiodic and μ-irreducible. Since X is μ-irreducible, it is also ψ-irreducible, where ψ is the maximal irreducibility measure (Meyn and Tweedie, 1993, Section 4.2). The measure ψ dominates all other irreducibility measures, so ψ dominates μ (denoted by $\psi \succ \mu$). It is also true that $\mu \succ \psi$, so these two measures are actually equivalent. Indeed, if $\mu(A) = 0$, then it follows that $K^l(x, A) = 0$ for all $x \in \mathsf{X}$ and all $l \in \mathbb{N}$, which implies that $\psi(A) = 0$.

Since X is ψ-irreducible and admits an invariant probability distribution, it is positive, hence recurrent (Meyn and Tweedie, 1993, Chap. 10). In order to establish Harris recurrence, we must introduce the notion of harmonic functions. A function $h : \mathsf{X} \to \mathbb{R}$ is called harmonic for K if $h = Kh$, where $Kh(x) := \int_{\mathsf{X}} h(x')K(x, dx')$ for all $x \in \mathsf{X}$. One method of establishing Harris recurrence is to show that every bounded harmonic function is constant (Nummelin, 1984, Theorem 3.8). Suppose h is a bounded, harmonic function. Since X is ψ-irreducible and recurrent, h is constant ψ-a.e. (Nummelin, 1984, Proposition 3.1.3). Thus, there exists a set N with $\psi(N) = 0$ such that $h(x) = c$ for all $x \in \overline{N}$, where $\overline{N}$ denotes the complement set of N. Since $\psi \succ \mu \succ K(x, \cdot)$ for all $x \in \mathsf{X}$, $K(x, N) = 0$. Now, for any $x \in \mathsf{X}$, we have

$$h(x) = \int_{\mathsf{X}} h(x')K(x, dx') = \int_{\overline{N}} h(x')K(x, dx') + \int_N h(x')K(x, dx') = c + 0 = c,$$

which implies that $h \equiv c$. It follows that X is Harris recurrent. After all, X is positive and Harris recurrent, hence positive Harris recurrent by definition (Meyn and Tweedie, 1993, Chap. 10). □

B Technical Details

B.1 Upper bounds for conditional expectations

We begin by establishing some inequalities involving t, which will help us evaluate conditional expectations. First, note that

$$t = \sum_{i=1}^{q} \frac{m_i}{m_i\sigma_\theta^2 + \sigma_e^2} \geq \sum_{i=1}^{q} \frac{m_i}{(1+m_i)\max\{\sigma_e^2, \sigma_\theta^2\}} = \left(\sum_{i=1}^{q} \frac{m_i}{m_i+1}\right) \frac{1}{\max\left\{\sigma_\theta^2, \sigma_e^2\right\}} . \tag{1}$$

Lemma 2. *Let* $m^* = \max\{m_1, \ldots, m_q\}$*. Then for each* $i = 1, ..., q$*, we have*

$$\frac{m^*\sigma_\theta^2 + \sigma_e^2}{M} \geq \frac{1}{t} \geq \frac{m_i\sigma_\theta^2 + \sigma_e^2}{M(1+m_i)} .$$

Proof. The first inequality holds because

$$\frac{1}{t} = \left(\sum_{i=1}^{q} \frac{m_i}{m_i\sigma_\theta^2 + \sigma_e^2}\right)^{-1} \leq \left(\sum_{i=1}^{q} \frac{m_i}{m^*\sigma_\theta^2 + \sigma_e^2}\right)^{-1} = \frac{m^*\sigma_\theta^2 + \sigma_e^2}{M} .$$

For the second inequality, first note that

$$t = \sum_{i=1}^{q} \frac{m_i}{m_i\sigma_\theta^2 + \sigma_e^2} \leq \sum_{i=1}^{q} \frac{m_i}{m_i\sigma_\theta^2} = \frac{q}{\sigma_\theta^2} \quad \text{and} \quad t = \sum_{i=1}^{q} \frac{m_i}{m_i\sigma_\theta^2 + \sigma_e^2} \leq \sum_{i=1}^{q} \frac{m_i}{\sigma_e^2} = \frac{M}{\sigma_e^2} .$$

If $\sigma_\theta^2 \leq \sigma_e^2$, then

$$\frac{1}{m_i\sigma_\theta^2 + \sigma_e^2}\frac{1}{t} \geq \frac{\sigma_e^2}{M(m_i\sigma_\theta^2 + \sigma_e^2)} \geq \frac{1}{M(1+m_i)} ,$$

else if $\sigma_\theta^2 > \sigma_e^2$, then

$$\frac{1}{m_i\sigma_\theta^2 + \sigma_e^2}\frac{1}{t} \geq \frac{\sigma_\theta^2}{q\left(m_i\sigma_\theta^2 + \sigma_e^2\right)} > \frac{1}{q\left(1+m_i\right)} \geq \frac{1}{M\left(1+m_i\right)} .$$

□

Note that $\mathrm{E}\big(\mu\big|\sigma_\theta^2, \sigma_e^2\big)$ is a convex combination of the $\overline{y}_i$. Hence, as a function of σ_θ^2 and σ_e^2, this conditional expectation is uniformly bounded by a constant. Along the same lines, for each fixed k, $\mathrm{E}\big(\theta_k\big|\sigma_\theta^2, \sigma_e^2\big)$ is a convex combination of $\mathrm{E}\big(\mu\big|\sigma_\theta^2, \sigma_e^2\big)$ and $\overline{y}_k$, so it too is uniformly bounded by a constant. Using these facts along with the forms of the conditional densities given in Subsection 3.1, we have

$$
\begin{aligned}
\mathrm{E}(\tilde{w}_1|\sigma^2) &= \mathrm{E}\bigg[\sum_i (\tilde{\theta}_i - \tilde{\mu})^2 \bigg| \sigma_\theta^2, \sigma_e^2\bigg] \\
&= \sum_i \Big[\mathrm{Var}\big[(\tilde{\theta}_i - \tilde{\mu})\big|\sigma_\theta^2, \sigma_e^2\big] + \Big(\mathrm{E}\big[(\tilde{\theta}_i - \tilde{\mu})\big|\sigma_\theta^2, \sigma_e^2\big]\Big)^2\Big] \\
&= \sum_i \frac{\sigma_\theta^2 \sigma_e^2}{\sigma_e^2 + m_i \sigma_\theta^2} + \sum_i \frac{\left(\sigma_e^2\right)^2}{(m_i\sigma_\theta^2 + \sigma_e^2)^2 t} - 2\sum_i \frac{\sigma_e^2}{(m_i\sigma_\theta^2 + \sigma_e^2)t} + \frac{q}{t} + const \\
&\le \sum_i \frac{\sigma_\theta^2 \sigma_e^2}{m_i\sigma_\theta^2 + \sigma_e^2} - \sum_i \frac{\sigma_e^2}{(m_i\sigma_\theta^2 + \sigma_e^2)t} + \frac{q}{t} + const \\
&\le \sum_i \frac{\sigma_e^2}{m_i} - \sum_i \frac{\sigma_e^2}{M(1+m_i)} + \frac{q}{t} + const
\end{aligned}
$$

where the final inequality uses Lemma 2. We now bound $\frac{q}{t}$ in two different ways. First, applying (1), we have

$$
\frac{q}{t} \le q\bigg(\sum_{i=1}^{q} \frac{m_i}{m_i+1}\bigg)^{-1} \max\left\{\sigma_\theta^2, \sigma_e^2\right\} \le q\bigg(\sum_{i=1}^{q} \frac{m_i}{m_i+1}\bigg)^{-1} \left(\sigma_\theta^2 + \sigma_e^2\right) .
$$

For the other way, apply Lemma 2 and

$$
\frac{q}{t} \le \frac{q m^* \sigma_\theta^2}{M} + \frac{q\sigma_e^2}{M} .
$$

Hence,

$$
\begin{aligned}
\mathrm{E}(\tilde{w}_1|\sigma^2) \le{}& q\bigg(\sum_{i=1}^{q} \frac{m_i}{m_i+1}\bigg)^{-1} \sigma_\theta^2 \\
&+ \bigg[\sum_i \frac{1}{m_i} - \sum_i \frac{1}{M(1+m_i)} + q\bigg(\sum_{i=1}^{q} \frac{m_i}{m_i+1}\bigg)^{-1}\bigg]\sigma_e^2 + const ,
\end{aligned}
$$

and

$$
\mathrm{E}(\tilde{w}_1|\sigma^2) \le \frac{q m^*}{M}\sigma_\theta^2 + \bigg[\sum_i \frac{1}{m_i} - \sum_i \frac{1}{M(1+m_i)} + \frac{q}{M}\bigg]\sigma_e^2 + const .
$$

Now,

$$\begin{aligned}
\mathrm{E}(\tilde{w}_2|\sigma^2) &= \sum_i m_i \mathrm{E}\big[(\overline{y}_i - \tilde{\theta}_i)^2 \big| \sigma_\theta^2, \sigma_e^2\big] \\
&= \sum_i m_i \Big[\mathrm{Var}(\tilde{\theta}_i | \sigma_\theta^2, \sigma_e^2) + \Big(\mathrm{E}\big[(\tilde{\theta}_i - \overline{y}_i) \big| \sigma_\theta^2, \sigma_e^2\big]\Big)^2\Big] \\
&= \sum_i \frac{m_i \sigma_\theta^2 \sigma_e^2}{m_i \sigma_\theta^2 + \sigma_e^2} + \sum_i \frac{m_i \big(\sigma_e^2\big)^2}{(m_i \sigma_\theta^2 + \sigma_e^2)^2 t} + const \\
&\le \sum_i \sigma_e^2 + \sum_i \frac{m_i \sigma_e^2}{(m_i \sigma_\theta^2 + \sigma_e^2) t} \sigma_e^2 = (q+1)\sigma_e^2 + const \; .
\end{aligned}$$

B.2 Proof of Proposition 2

Proof. Recall equations (14) and (15) from the main body of the paper, which are

$$\mathrm{E}(\tilde{w}_1^s \mid \xi) \le \delta_3(s) w_1^s + \delta_2(s) w_2^s + const \; ,$$

and

$$\mathrm{E}(\tilde{w}_2^s \mid \xi) \le \delta_1(s) w_2^s + const \; .$$

If follows that

$$\begin{aligned}
\mathrm{E}(\epsilon \tilde{w}_1^s + \tilde{w}_2^s \mid \xi) &\le \epsilon \delta_3(s) w_1^s + (\delta_1(s) + \epsilon \delta_2(s)) w_2^s + const \\
&= \rho(\epsilon, s)(\epsilon w_1^s + w_2^s) + \epsilon(\delta_3(s) - \rho(\epsilon, s)) w_1^s + const
\end{aligned}$$

where $\rho(\epsilon, s) = \delta_1(s) + \epsilon \delta_2(s)$. Therefore, we will have a viable drift condition if

$$\rho(\epsilon, s) < 1 \;\; \text{and} \;\; \delta_3(s) - \rho(\epsilon, s) \le 0 \; . \tag{2}$$

Clearly, (2) requires that $\delta_1(s) < 1$ and $\delta_3(s) < 1$. We now show that these conditions are also sufficient for the existence of $\epsilon > 0$ such that (2) is satisfied.

There are two cases. In the first case, $\delta_1(s) \le \delta_3(s) < 1$. If we take $\epsilon = (\delta_3(s) - \delta_1(s))/\delta_2(s)$, then $\rho(\epsilon, s) = \delta_3(s) < 1$ and $\delta_3(s) - \rho(\epsilon, s) = 0$. In the second case, $\delta_3(s) < \delta_1(s) < 1$. Now take $\epsilon = \big(1 - \delta_1(s)\big) / \big(2\delta_2(s)\big)$. Then

$$\rho(\epsilon, s) = \delta_1(s) + \frac{1 - \delta_1(s)}{2} = \frac{1 + \delta_1(s)}{2} < 1 \; ,$$

and

$$\delta_3(s) - \rho(\epsilon, s) = \delta_3(s) - \frac{1 + \delta_1(s)}{2} < 0 \; .$$

Hence, if $\delta_1(s) < 1$ and $\delta_3(s) < 1$, then there is a viable drift condition. □

B.3 Finding an $s \in S$ such that $\delta_1(s) < 1$ and $\delta_3(s) < 1$

This section contains a proof of the following result.

Proposition 4. *If $M + 2b \geq q + 3$ and $\Delta_1 < 2\exp\big(\Psi\big(\frac{q}{2} + a\big)\big)$, then there exists $s \in S$ such that $\delta_1(s) < 1$ and $\delta_3(s) < 1$.*

We will prove Proposition 4 by establishing that

- $\delta_1(s) < 1$ for any $s \in (0, 1)$ if $M + 2b \geq q + 3$, and
- $\delta_3(s_0) < 1$ for some small positive s_0 if $\Delta_1 < 2\exp\big(\Psi\big(\frac{q}{2} + a\big)\big)$.

It is well known that $\Psi'(x) > 0$ for all $x > 0$ and that $\Psi(x + 1) = \Psi(x) + \frac{1}{x}$. A couple of common values of the digamma function that we will encounter later are $\Psi(1) = -\gamma$ and $\Psi(\frac{1}{2}) = -\gamma - 2\log(2)$, where $\gamma := \lim_{p\to\infty}\big(1 + \frac{1}{2} + \cdots + \frac{1}{p} - \log(p)\big) \doteq 0.577$ is Euler's constant. Also, the recurrence formula yields: $\Psi(\frac{3}{2}) = -\gamma - 2\log(2) + 2$.

B.3.1 Bounding $\delta_1(s)$

Recall that $\delta_1(s)$ is actually a function of s, q and $\frac{M}{2} + b$. Indeed,

$$\delta_1\bigg(s, q, \frac{M}{2} + b\bigg) = \frac{\Gamma\big(\frac{M}{2} + b - s\big)}{\Gamma\big(\frac{M}{2} + b\big)}\Big(\frac{q+1}{2}\Big)^s .$$

Now, for any fixed $s > 0$, $\Gamma(x - s)\big/\Gamma(x)$ is decreasing in x for $x > s$, because

$$\frac{d}{dx}\Big[\log(\Gamma(x - s)) - \log(\Gamma(x))\Big] = \Psi(x - s) - \Psi(x) < 0 \quad \text{for all } x > s > 0 .$$

Therefore, with (s, q) fixed, $\delta_1\big(s, q, \frac{M}{2} + b\big)$ is decreasing in $\big(\frac{M}{2} + b\big)$ as long as $\frac{M}{2} + b > s$. Consequently, to show that $\delta_1\big(s, q, \frac{M}{2} + b\big) < 1$ for all $s \in (0, 1)$ if $M + 2b \geq q + 3$, we need only prove the following.

Lemma 3. $\delta_1\big(s, q, \frac{q+3}{2}\big) < 1$ *for all $s \in (0, 1)$ and $q \geq 2$.*

Proof. Fix $s \in (0, 1)$ and define

$$T(x) = \frac{\Gamma(x + s)}{\Gamma(x)}(x + s - 1)^{-s} \quad \text{for } x > 1 - s .$$

We claim that

1. $T(x)$ is strictly decreasing in x, and

2. $\lim_{x\to\infty} T(x) = 1$.

To prove claim 1, we will show that $Q(x) = \log(T(x))$ is decreasing in x. First, for $x > 0$

$$\Psi(x) = -\gamma + \sum_{p=1}^{\infty} \left(\frac{1}{p} - \frac{1}{x+p-1} \right)$$

(Abramowitz and Stegun, 1964, p.259). Note that $\left(\frac{1}{p} - \frac{1}{x+p-1}\right)$ is nonnegative for all p when $x \geq 1$ and negative for all p when $x < 1$. Hence, the above series is absolutely convergent for all $x > 0$. Clearly, $Q(x) = -s\log(x+s-1) + \log(\Gamma(x+s)) - \log(\Gamma(x))$ and its derivative can be expressed as follows

$$Q'(x) = -s\frac{1}{x+s-1} + \sum_{p=1}^{\infty} \left(\frac{1}{p} - \frac{1}{x+s+p-1} \right) - \sum_{p=1}^{\infty} \left(\frac{1}{p} - \frac{1}{x+p-1} \right) . \qquad (3)$$

The fraction in the first term of (3) can be written as the following absolutely convergent telescoping series

$$\frac{1}{x+s-1} = \sum_{p=1}^{\infty} \left(\frac{1}{x+s+p-2} - \frac{1}{x+s+p-1} \right) .$$

Therefore,

$$\begin{aligned} Q'(x) &= s\sum_{p=1}^{\infty} \left(\frac{1}{x+s+p-1} - \frac{1}{x+s+p-2} \right) + \sum_{p=1}^{\infty} \left(\frac{1}{x+p-1} - \frac{1}{x+s+p-1} \right) \\ &= \sum_{p=1}^{\infty} \left[-(1-s)\frac{1}{x+s+p-1} - s\frac{1}{x+s+p-2} + \frac{1}{x+p-1} \right] . \end{aligned}$$

The convexity of the function $h(z) = \frac{1}{z}$ on $\mathbb{R}_+$ combined with the fact that $(1-s)(x+s+p-1) + s(x+s+p-2) = x+p-1$ can be used to show that every term in the series above is negative. It follows that $Q(x)$ and $T(x)$ are both decreasing in x for $x > 1-s$.

We now prove claim 2. Fix $s \in (0,1)$ and define $S(x) = x^{-s}\Gamma(x+s)\big/\Gamma(x)$. As $x \to \infty$, $S(x) \to 1$ (Abramowitz and Stegun, 1964, p.257). As a consequence,

$$\lim_{x\to\infty} T(x) = \lim_{x\to\infty} S(x)\left(\frac{x}{x+s-1}\right)^s = 1 .$$

Finally, for fixed $s \in (0,1)$, note that $\frac{q+3}{2} - s > 1-s$ and

$$\delta_1\big(s, q, \frac{q+3}{2}\big) = \big(\frac{q+1}{2}\big)^s \frac{\Gamma\big(\frac{q+3}{2} - s\big)}{\Gamma\big(\frac{q+3}{2}\big)} = \Big(T\big(\frac{q+3}{2} - s\big)\Big)^{-1} .$$

It follows from claims 1 and 2 that $T\big(\frac{q+3}{2} - s\big) > 1$ and hence $\delta_1\big(s, q, \frac{q+3}{2}\big) < 1$. $\square$

B.3.2 Bounding $\delta_3(s)$

Recall that $\delta_3(s)$ is actually a function of s, m and a. If we define

$$A(s,q,a) = \frac{\Gamma(\frac{q}{2}+a-s)}{2^s\Gamma(\frac{q}{2}+a)} \ ,$$

then we have $\delta_3(s,m,a) = A(s,q,a)\Delta_1^s(m)$. Note that there exists an $s_0 \in S$ such that $\delta_3(s_0,m,a) < 1$ if and only if $\Delta_1(m) < A^*(q,a)$, where

$$A^*(q,a) := \sup_{s\in S} A^{-\frac{1}{s}}(s,q,a) = 2\sup_{s\in S}\left(\frac{\Gamma(\frac{q}{2}+a)}{\Gamma(\frac{q}{2}+a-s)}\right)^{\frac{1}{s}} .$$

We now establish a lower bound for $A^*(q,a)$. Define

$$g(s,q,a) = \log\left(\frac{1}{2}A^{-\frac{1}{s}}(s,q,a)\right) = \frac{1}{s}\left[\log\left(\Gamma\Big(\frac{q}{2}+a\Big)\right) - \log\left(\Gamma\Big(\frac{q}{2}+a-s\Big)\right)\right] .$$

Then

$$\lim_{s\to 0} g(s,q,a) = \frac{d\log(\Gamma(x))}{dx}\bigg|_{x=\frac{q}{2}+a} = \Psi\Big(\frac{q}{2}+a\Big) .$$

Hence,

$$A^*(q,a) \geq \lim_{s\to 0} 2\exp\big[g(s,q,a)\big] = 2\exp\left(\Psi\Big(\frac{q}{2}+a\Big)\right) . \tag{4}$$

We conclude that there exists $s \in S$ such that $\delta_3(s,m,a) < 1$ if $\Delta_1(m) < 2\exp\left(\Psi\Big(\frac{q}{2}+a\Big)\right)$.

Remark 1. It is easy to show that $\frac{\partial g(s,q,a)}{\partial s}\big|_{s=0} < 0$. In other words, for fixed q and a, $g(s,q,a)$ is decreasing in s in a neighborhood of $s=0$. Furthermore, numerical calculations suggest that $g(s,q,a)$ is decreasing on the entire set S for any fixed q and a. Hence, we believe the lower bound on $A^*(q,a)$ in (4) is sharp.

B.4 Specializing Proposition 3 to the case where $a = -\frac{1}{2}$

Here we study the condition $\Delta_1(m) < 2\exp\left(\Psi\Big(\frac{q}{2}+a\Big)\right)$ in the special case where $a = -\frac{1}{2}$. According to (4) from the main body of the paper, when $a = -\frac{1}{2}$, the posterior is improper if $q \leq 2$. Hence, we can restrict attention to the case $q \geq 3$. When $q \geq 4$, we have

$$2\exp\left(\Psi\Big(\frac{q}{2}+a\Big)\right) \geq 2\exp\left(\Psi\Big(\frac{3}{2}\Big)\right) = 2\exp(-\gamma - 2(\log 2 - 1)) \doteq 2.074 .$$

Now recall that

$$\Delta_1(m) = \min\left\{q\bigg(\sum_{i=1}^{q}\frac{m_i}{m_i+1}\bigg)^{-1}, \frac{qm^*}{M}\right\} .$$

Since $\frac{m_i}{1+m_i} \geq \frac{1}{2}$, we have

$$\Delta_1(m) \leq \frac{q}{\sum_i \frac{m_i}{m_i+1}} \leq \frac{q}{\sum_i \frac{1}{2}} = 2 < 2\exp\left(\Psi\left(\frac{q}{2}+a\right)\right) ,$$

so, when $a = -\frac{1}{2}$ and $q \geq 4$, the condition $\Delta_1(m) < 2\exp\left(\Psi\left(\frac{q}{2}+a\right)\right)$ is *always satisfied*.

Now, when $q = 3$, we have

$$2\exp\left(\Psi\left(\frac{q}{2}+a\right)\right) = 2\exp\left(\Psi(1)\right) = 2\exp(-\gamma) \doteq 1.123 .$$

For balanced data,

$$\Delta_1(m) \leq \frac{qm^*}{M} = 1 < 2\exp\left(\Psi(1)\right) .$$

Hence, when $a = -\frac{1}{2}$ and $q = 3$ and the data are balanced, the condition $\Delta_1(m) < 2\exp\left(\Psi\left(\frac{q}{2}+a\right)\right)$ is satisfied. Finally, if $q = 3$ and the data are unbalanced, then $\Delta_1(m) < 2\exp(-\gamma)$ if and only if

$$\sum_i \frac{m_i}{m_i+1} > \frac{3}{2\exp(-\gamma)} \doteq 2.67 \quad \text{or} \quad m^* < \frac{2\exp(-\gamma)}{3} M \doteq 0.374M . \qquad (5)$$

Table 5 displays all unbalanced data configurations with $q = 3$ and $m^* \leq 12$ that satisfy (5).

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m_1	m_2	m_3	m_1	m_2	m_3	m_1	m_2	m_3	m_1	m_2	m_3	m_1	m_2	m_3
3	4	4	7	8	10	7	10	11	6	9	12	8	11	12
4	5	5	7	9	10	7	11	11	6	10	12	8	12	12
5	6	6	7	10	10	8	8	11	6	11	12	9	9	12
5	7	7	8	8	10	8	9	11	6	12	12	9	10	12
6	6	7	8	9	10	8	10	11	7	7	12	9	11	12
6	7	7	8	10	10	8	11	11	7	8	12	9	12	12
6	8	8	9	9	10	9	9	11	7	9	12	10	10	12
7	7	8	9	10	10	9	10	11	7	10	12	10	11	12
7	8	8	6	9	11	9	11	11	7	11	12	10	12	12
7	9	9	6	10	11	10	10	11	7	12	12	11	11	12
8	8	9	6	11	11	10	11	11	8	8	12	11	12	12
8	9	9	7	8	11	5	11	12	8	9	12			
6	10	10	7	9	11	5	12	12	8	10	12			

Table 5: A complete list of all unbalanced configurations (m_1, m_2, m_3) with $m^* \leq 12$ that satisfy $\Delta_1(m) < 2\exp(-\gamma)$.