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A NOTE ON THE TURÁN FUNCTION OF EVEN CYCLES

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ABSTRACT. The *Turán function* $\mathrm{ex}(n, F)$ is the maximum number of edges in an F-free graph on n vertices. The question of estimating this function for $F = C_{2k}$, the cycle of length $2k$, is one of the central open questions in this area that goes back to the 1930s. We prove that

$$\mathrm{ex}(n, C_{2k}) \leq (k-1)\, n^{1+1/k} + 16(k-1)n,$$

improving the previously best known general upper bound of Verstraëte [*Combin. Probab. Computing* **9** (2000), 369–373] by a factor $8 + o(1)$ when $n \gg k$.

1. Introduction

The *Turán function* $\mathrm{ex}(n, F)$ of a forbidden graph F is the maximum number of edges in an F-free graph on n vertices. It is named so as to honor the fundamental paper of Turán [20] from 1941 that determined this function for cliques. If we forbid C_{2k}, the cycle of length $2k$, then the problem of determining its Turán function goes back even further, to a 1938 paper of Erdős [6] one of whose results is essentially that $\mathrm{ex}(n, C_4) = \Theta(n^{3/2})$.

Determining the Turán function for even cycles is considered to be one of the key problems in extremal combinatorics. However, despite the efforts of many leading researchers, it remains wide open. In the proceedings of the 1963 Smolenice Symposium on Graph Theory and Its Applications, Erdős [7, Page 33] stated without proof that $\mathrm{ex}(n, C_{2k}) \leq \gamma_k n^{1+1/k}$ for some constant γ_k depending only on k. The first published proof of this appears in a paper of Bondy and Simonovits [3], whose Lemma 2 implies that $\mathrm{ex}(n, C_{2k}) \leq 20kn^{1+1/k}$ for all $n \gg k$. More recently, Verstraëte [21], as a by-product of his theorem on cycle lengths in graphs, showed that $\mathrm{ex}(n, C_{2k}) \leq 8(k-1)\, n^{1+1/k}$.

The case that is best understood so far is $k = 2$. Here we know that $\mathrm{ex}(n, C_4) = (1/2 + o(1))\, n^{3/2}$: the lower bound was proved by Erdős and Rényi [8] (and independently re-discovered by Brown [4]) and the upper bound by Erdős, Rényi, and Sós [9]. Moreover, we know $\mathrm{ex}(n, C_4)$ exactly when $n = q^2 + q + 1$ for any prime power $q \geq 16$ (Füredi [11, 12]) and when $n \leq 31$ (McCuaig [17], Clapham, Flockhart, and Sheehan [5], Yuansheng and Rowlinson [23]).

For arbitrary fixed k, Erdős and Simonovits [10] conjectured that $\mathrm{ex}(n, C_{2k}) = (1/2 + o(1))\, n^{1+1/k}$. This was disproved by Lazebnik, Ustimenko, and Woldar [15] for $k = 5$ who showed that

$$\mathrm{ex}(n, C_{10}) \geq (4 \cdot 5^{-6/5} + o(1))\, n^{6/5} = (0.5798... + o(1))\, n^{6/5},$$

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although they were not aware of this conjecture at the time, see the discussion in [16, Pages 504–505]. Füredi, Naor, and Verstraëte [13] showed that

$$0.5338\, n^{4/3} \le \mathrm{ex}(n, C_6) \le 0.6272\, n^{4/3},$$

thus disproving the case $k = 3$ of the above conjecture.

For $k \ge 3$, a lower bound of the form $\mathrm{ex}(n, C_{2k}) = \Omega(n^{1+1/k})$ is known only for $k = 3$ and $k = 5$ (first proved by Benson [1]). Some other constructions achieving it were found by Wenger [22], Lazebnik and Ustimenko [14], Mellinger [18], and Mellinger and Mubayi [19]. Unfortunately, for no other k is the rate of growth of $\mathrm{ex}(n, C_{2k})$ known, although there are various lower bounds. We refer the reader to the paper [16] that presents new constructions as well as gives numerous references.

In this note we prove the following result that improves the best known general upper bound of Verstraëte [21] by a factor $8 + o(1)$ when $n \gg k$.

Theorem 1.1. *For all $k \ge 2$ and $n \ge 1$, we have*

$$\mathrm{ex}(n, C_{2k}) \le (k-1)\, n^{1+1/k} + 16(k-1)n.$$

Essentially all the main ideas that we use to prove this result can be found in the papers [3, 21]. However, given the importance of this problem and the absence of any improvements in upper bounds on $\mathrm{ex}(n, C_{2k})$ for general k in the last decade, this result may help to draw more interest to this area and to introduce some beautiful ideas from [3, 21] to a larger audience, especially that our proof seems to be simpler and more transparent than those in [3, 21].

Although the bound of Theorem 1.1 can be somewhat improved (especially for small k), the author could not show that

$$\liminf_{k\to\infty}\ \liminf_{n\to\infty}\ \frac{\mathrm{ex}(n, C_{2k})}{kn^{1+1/k}}$$

is strictly below 1. It would be very interesting to decide if the limit inferior is 0 or not. Hopefully, the quest in this direction will lead to new ideas and insights.

2. Auxiliary Results

We use the standard graph theory notation that can be found in e.g. Bondy and Murty's book [2]. Still, some terms are defined when they are used for the first time.

By a Θ-*graph* we mean a cycle of length at least $2k$ with a chord. We will need the following lemma that appears implicitly in the proof of Lemma 2 in [3] and is stated as a separate lemma in [21, Lemma 2].

Lemma 2.1. *Let F be a Θ-graph and $1 \le \ell \le v(F) - 1$. Let $V(F) = A \cup B$ be an arbitrary partition of its vertex set into two non-empty parts such that every path in F of length ℓ that begins in A necessarily ends in A. Then F is bipartite with parts A and B.*

Sketch of Proof. Let $n = v(F)$ and let its vertex set be $\mathbb{Z}_n$, the set of the residues modulo n, with i being adjacent to $i-1$ and $i+1$. We encode the partition $A \cup B$ by a 2-coloring χ of $\mathbb{Z}_n$. Let

$$P = \{m \in \mathbb{Z}_n : \forall i \in \mathbb{Z}_n\ \ \chi(i) = \chi(i+m)\}$$

consist of all periods of χ. Our assumption implies that $\ell \in P$. The smallest positive period $m \in P$ has to divide n and, in fact, $P = \{mi : i \in \mathbb{Z}_n\}$ is the set of multiples of m. Assume that $m > 2$ for otherwise we are easily done.

Let the chord connect 0 to r. Since $m > 2$, we cannot have that both r and $n - r$ are congruent to 1 modulo m, say $r \not\equiv 1 \pmod m$. Thus $r - 1 \notin P$. By the m-periodicity of χ, there is some $j \in \mathbb{Z}_n$ such that $\chi(j) \neq \chi(j + \ell + r - 1)$ and we can further assume that $-m < j \leq 0$. The ℓ-walk

$$j, j+1, \ldots, -1, 0, r, r+1, \ldots, j+\ell+r-1$$

in F connects A to B, which is a contradiction unless $\ell + r - 1 \geq n$. The remaining cases can be settled by similar arguments where the constructed l-path may change direction at a chord's endpoint. A complete proof can be found in [21, Lemma 2]. □

The following easy lemma will also be used.

Lemma 2.2. *Let $k \geq 3$. Any bipartite graph H of minimum degree at least k contains a Θ-subgraph.*

Proof. Take a longest path P in H. Let P visit vertices $x_1, \ldots, x_m$ in this order. The end-point x_1 has at least k neighbors in H. By the maximality of P, all of them lie on P. So pick any k neighbors $x_{i_1}, \ldots, x_{i_k}$ of x_1 where $i_1 < \cdots < i_k$. Every two neighbors of x_1 are at least 2 apart on P (because H is bipartite). Thus $i_k \geq 2k$ and the subpath of P between x_1 and x_{i_k} together with the edges $x_1 x_{i_2}$ and $x_1 x_{i_k}$ forms the required Θ-subgraph. □

3. Proof of Theorem 1.1

Suppose for the sake of contradiction that some C_{2k}-free graph G on n vertices violates Theorem 1.1. As is well known (for a proof see e.g. [3, Page 99] or [2, Theorem 2.5]), every graph G contains a subgraph of minimum degree at least half of the average degree of G:

$$\forall G \ \ \exists H \subseteq G \quad \delta(H) \geq \frac{d(G)}{2} = \frac{e(G)}{n}. \tag{1}$$

So take any $H \subseteq G$ of minimum degree at least δ, where we define

$$\delta = \frac{e(G)}{n} \geq (k-1)\, n^{1/k} + 16(k-1).$$

Fix an arbitrary vertex x in H. Let V_i consist of those vertices of H that are at distance i (with respect to the graph H) from x. Thus $V_0 = \{x\}$ and $V_1 = N(x)$ is the neighborhood of x in H. For $i \geq 0$, let $v_i = |V_i|$ and let

$$H_i = H[V_i, V_{i+1}]$$

be the bipartite subgraph of H induced by the disjoint sets V_i and V_{i+1}.

Claim 3.1. *For $1 \leq i \leq k-1$, neither of the graphs $H[V_i]$ and H_i contains a Θ-subgraph that is bipartite.*

Proof of Claim 3.1. Suppose on the contrary that a bipartite Θ-subgraph $F \subseteq H[V_i]$ exists. Let $Y \cup Z$ be the bipartition of F. Let $T \subseteq H$ be a breadth-first search tree in H with the root x (for definitions see e.g. [2, Section 6.1]). Let y be the vertex that is farthest from x such that every vertex of Y is a T-descendant of y. The paths inside T that connect y to Y branch at y. Pick one such branch, defined by some child z of y, and let A be the set of the T-descendants of z that lie in Y. Let $B = (Y \cup Z) \setminus A$. Since $Y \setminus A \neq \emptyset$, B is not an independent set of F.

Let ℓ be the distance between x and y. We have $\ell < i$ and $2k - 2i + 2\ell < 2k \leq v(F)$. By Lemma 2.1 we can find a path $P \subseteq F$ of length $2k - 2i + 2\ell$ that starts in some $a \in A$ and ends in $b \in B$. Since the length of P is even, we have $b \in Y$. Let P_a and P_b be the unique paths in T that connect y to respectively a and b. They intersect only in the vertex y since P_a starts with the edge yz while P_b uses some different child of y. Also, each of these paths has length $i - \ell$. But then the union of the paths P, P_a, and P_b forms a $2k$-cycle in H, a contradiction.

The same proof (where we let $Y = V(F) \cap V_i$) works for $H_i = H[V_i, V_{i+1}]$. $\square$

By (1), Lemma 2.2, and Claim 1 (and the simple fact that every graph has a bipartite subgraph with at least half the edges, see e.g. [2, Theorem 2.4]), we conclude that for $k \geq 3$ we have

$$d(H[V_i]) \leq 4k - 4 \text{ and } d(H_i) \leq 2k - 2, \quad \text{for } 1 \leq i \leq k - 1. \tag{2}$$

Also, if $k = 2$, then $H[V_1]$ has no path of length 2 and no vertex of V_2 can send more than one edge to V_1, so (2) still holds.

We are essentially done with the combinatorial part and what remains is some algebra. Here is a sketch for $n \gg k$. For $i \leq k - 1$, the inequalities in (2) imply that, on average, a vertex of V_i sends at least $\delta - O(k)$ edges to V_{i+1} while at most $2k - 2$ edges per vertex of V_{i+1} are sent back. Thus the first k ratios v_{i+1}/v_i are at least $(\delta - O(k))/(2k - 2)$ each. We conclude that $(\delta/(2k - 2))^k \leq n + o(n)$, giving $e(G) \leq (2k - 2 + o(1))\, n^{1+1/k}$. An extra factor of $2 + o(1)$ is saved by observing that, in view of $v_i \ll v_{i+1}$, a vertex of V_{i+1} sends only $k - 1 + o(1)$ edges back on average.

Let us provide all detailed calculations. Define $\varepsilon = 4(k - 1)^2/\delta$. Let us show inductively on $i = 0, 1, \ldots, k - 1$ that

$$e(H_i) \leq (k - 1 + \varepsilon)\, v_{i+1}, \tag{3}$$

which bounds the average degree of the vertices in V_{i+1} into V_i. Clearly, this is true for $i = 0$ since each vertex of V_1 sends only one edge to V_0. Suppose that we want to prove (3) for some $i > 0$. By (2) and the inductive assumption,

$$e(H_i) = \sum_{y \in V_i} d_{V_{i+1}}(y) \geq \big(\delta - (4k - 4) - (k - 1 + \varepsilon)\big)\, v_i = (\delta - 5k + 5 - \varepsilon)\, v_i. \tag{4}$$

Thus the average degree of the vertices of V_i with respect to H_i is at least $\delta - 5k + 5 - \varepsilon \geq 2k - 2$. Here we used the facts that

$$\delta \geq 16(k - 1) \quad \text{and} \quad \varepsilon \leq \frac{k - 1}{4}. \tag{5}$$

In particular, $V_{i+1} \neq \emptyset$. In order to satisfy the second inequality in (2), it must be the case that the average V_i-degree of a vertex in V_{i+1} is at most $2k - 2$, that is, $e(H_i) \leq (2k - 2)v_{i+1}$. Thus, by (4), we have

$$v_i \leq \frac{e(H_i)}{\delta - 5k + 5 - \varepsilon} \leq \frac{2k - 2}{\delta - 5k + 5 - \varepsilon}\, v_{i+1}.$$

By (2) we conclude that

$$\frac{2e(H_i)}{(1 + \frac{2k-2}{\delta - 5k + 5 - \varepsilon})\, v_{i+1}} \leq \frac{2e(H_i)}{v_{i+1} + v_i} = d(H_i) \leq 2k - 2.$$

which implies the desired bound (3). (Indeed, (5) implies that $\frac{2(k-1)^2}{\delta-5.25(k-1)} \le \frac{4(k-1)^2}{\delta}$ which, again by (5), gives $\frac{2(k-1)^2}{\delta-5k+5-\varepsilon} \le \varepsilon$ and the last inequality is exactly what we need for deducing (3).)

By (3) and (4), we conclude that for each $i = 0, \dots, k-1$ we have

$$\frac{v_{i+1}}{v_i} = \frac{v_{i+1}}{e(H_i)} \times \frac{e(H_i)}{v_i} \ge \frac{\delta - 5k + 5 - \varepsilon}{k - 1 + \varepsilon} \ge \frac{\delta}{k - 1 + 4\varepsilon},$$

where the last inequality follows again from (5). Since $\delta \ge (k-1)\, n^{1/k}$, we have

$$n \ge v(H) \ge v_k \ge \left(\frac{\delta}{k-1+4\varepsilon}\right)^k \ge \left(\frac{e(G)/n}{k-1+16(k-1)\, n^{-1/k}}\right)^k,$$

implying the desired upper bound on $e(G)$ (that is, a contradiction). This finishes the proof of Theorem 1.1.

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