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ORDERS OF ZEROS OF POLYNOMIALS IN SOLUTIONS TO THE FUCHSIAN DIFFERENTIAL EQUATION

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We estimate the orders of zeros of polynomials $f(x) = P(u_1(x), u_2(x), \ldots, u_n(x))$ in the fundamental system of solutions to a linear Fuchsian differential equation. We introduce the notions of A- and (∞, A)-algebraic independence and prove that the system of functions $x^t, u_1(x), u_2(x), \ldots, u_n(x)$ is (∞, A)-algebraically independent. Bibliography: 6 titles.

1 Introduction

The study of the algebraic independence of a family of functions and estimation of the orders of zeros of polynomials in such families play an important role in number theory. Estimates for the orders of zeros of polynomials in solutions to linear differential equations are intensively studied in the theory of transcendental numbers (see [1, 2]). Such problems, but over the field of positive characteristics arise when applying the Stepanov method (see [3]). In this paper, we obtain an estimate for the order of zero of a polynomial in the fundamental system of solutions to a linear Fuchsian equation on the Riemann sphere. This estimate is used to prove the (∞, A)-algebraic independence of the system of functions $x^t, \ln(x - a_1), \ldots, \ln(x - a_n)$ over the field $\mathbb{C}$ with sufficiently large natural number t. Such estimates in the case of polynomial in components of solutions to a regular system were obtained in [4, 5]. In this paper, instead of components of one solution to a Fuchsian system, we deal with the fundamental system of solutions to a scalar Fuchsian equation.

We recall some definitions and results (see [6] for details).

1.1. Equations and their singularities. We consider the scalar differential equation

$$u^{(n)} + q_1(x)u^{(n-1)} + \ldots + q_n(x)u = 0, \quad x, u(x) \in \mathbb{C}, \tag{1.1}$$

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on the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. By singular points $a_1, \ldots, a_m$ in $\mathbb{C}$ of Equation (1.1) we mean singular points of its coefficients $q_1(x), \ldots, q_n(x)$ (the points where at least one of the coefficients has singularity). We say that $x = \infty$ is a singular point of an equation if the point $\xi = 0$ is a singular point of the transformed equation under the change of variables $x = 1/\xi$.

Solutions to Equation (1.1) form the n-dimensional vector space $\mathscr{L}$. The basis for this space is called the *fundamental system of solutions* to Equation (1.1). Let the basis consist of functions $u_1(x), \ldots, u_n(x)$. The solutions to Equation (1.1) can be analytically continued along any path γ in the punctured Riemann sphere $\overline{\mathbb{C}} \setminus \{a_1, \ldots, a_m\}$ along which the coefficients $q_1(x), \ldots, q_n(x)$ are holomorphic.

By the *representation of monodromy* or *monodromy* of Equation (1.1) we mean the representation

$$\chi : \pi_1(\overline{\mathbb{C}} \setminus \{a_1, \ldots, a_m\}, x_0) \to \mathrm{GL}(n, \mathbb{C}) \tag{1.2}$$

given by a mapping $[\gamma] \to G_\gamma$ of homotopy classes of loops such that the basis $(u_1(x), \ldots, u_n(x))$ for the space of solutions to Equation (1.1) in a neighborhood of a nonsingular point $x = x_0$, under the analytic continuation along the loop γ going out from the point x_0 and lying in $\overline{\mathbb{C}} \setminus \{a_1, \ldots, a_m\}$, is transformed into another basis $(\widetilde{u}_1(x), \ldots, \widetilde{u}_n(x))$. Two bases are connected by the nonsingular transition matrix G_γ corresponding to the loop γ: $(u_1, \ldots, u_n) = (\widetilde{u}_1, \ldots, \widetilde{u}_n) G_\gamma$.

By the *monodromy matrix* of an equation at a singular point a_k (with respect to the basis $(u_1, \ldots, u_n)$) we mean the matrix G_k corresponding to a simple loop γ_k going around a_k. The homotopy classes of loops $[\gamma]$ are associated with the matrices G_γ, which generates the monodromy representation (1.2).

A singular point $x = a$ of Equation (1.1) is said to be *regular* if there exists $N \in \mathbb{Z}$ such that for any solution $u(x)$ and any sector S with vertex $x = a$

$$\frac{|u(x)|}{|x-a|^N} \to 0, \quad x \to a, \quad x \in S,$$

and *Fuchsian* if the coefficient $q_j(x)$ has a pole of order at most j at the point $x = a$, i.e.,

$$q_j(x) = \frac{r_j(x)}{(x-a)^j}, \quad j \in \{1, \ldots, n\}, \tag{1.3}$$

where the functions $r_j(x)$ are holomorphic at the point $x = a$.

We say that an equation is *Fuchsian* if all its singular points are Fuchsian.

Theorem 1.1 (Fuchs)**.** *A singular point of the scalar differential equation* (1.1) *is a regular singular point if and only if it is Fuchsian.*

1.2. Exponents of solutions and the Fuchsian relation. On the space of solutions $\mathscr{L}$, we can define a function $\varphi_a : \mathscr{L} \to \mathbb{Z} \cup \{\infty\}$.

By a *valuation* $\varphi_a(u)$ of a function $u(x)$ at a point $x = a$ we mean the integer

$$\varphi_a(u(x)) := \sup\Big\{K \in \mathbb{Z} \mid \forall \lambda < K : \frac{u(x)}{|x-a|^\lambda} \to 0, \quad x \to a, \quad x \in S\Big\}, \quad \varphi(0) := \infty,$$

where S is a sector with vertex $x = a$.

We consider a valuation φ_a on the space $\mathscr{L}$ of solutions to Equation (1.1) at a regular singular point $x = a$. The valuation φ_a possesses the following properties:

(1) $\varphi_a(u+v) \geqslant \min(\varphi_a(u), \varphi_a(v))$ and $\varphi_a(u) \neq \varphi_a(v)$ implies $\varphi_a(u+v) = \min(\varphi_a(u), \varphi_a(v))$,

(2) $\varphi_a(c \cdot u) = \varphi_a(u)$ for any $c \in \mathbb{C}\backslash\{0\}$,

(3) $\varphi_a(u)$ is preserved under the analytic continuation around the point $z = a$.

A valuation at a regular singular point $z = a$ determines a filtration on the space $\mathscr{L}$. We denote by $\infty > \psi^1 > \ldots > \psi^q$ all various finite values $\varphi_a(u)$, $u \in \mathscr{L}$. There are subspaces $\mathscr{L}^s = \{u \in \mathscr{L} \mid \varphi_a(u) \geqslant \psi^s\}$, $s = 1, \ldots, q$, determining a *filtration* of the space $\mathscr{L}$ of solutions

$$0 \subset \mathscr{L}^1 \subset \ldots \subset \mathscr{L}^q = \mathscr{L}, \tag{1.4}$$

which, by property (3) of a valuation, is invariant under the analytic continuation around the point $z = a$.

The basis $\overline{u}(x) = (u_1(x), \ldots, u_n(x))$ for the space $\mathscr{L}$ of solutions to Equation (1.1) is said to be *associated* with the filtration (1.4) at a regular singular point $z = a$ if the valuation takes all its values there (counted according to their multiplicities).

For any associated basis $\overline{u}(x)$ at a singular point $x = a_k$ the *Levelt decomposition*

$$\overline{u}(x) = (h_k^1(x), \ldots, h_k^n(x))(x - a_k)^{\Lambda_k}(x - a_k)^{E_k}$$

holds, where $h_k^s(x)$ are holomorphic at $x = a_k$, $h_k^s(a_k) \neq 0$ for $s = 1, \ldots, n$,

$$\Lambda_k = \operatorname{diag}(\varphi_{a_k}(u_1), \ldots, \varphi_{a_k}(u_n))$$

is a diagonal integer matrix, $E_k = \frac{1}{2\pi\sqrt{-1}} \ln G_k$ and for the eigenvalues ρ_k^j of the matrix E_k

$$0 \leqslant \operatorname{Re} \rho_k^j < 1, \quad j = 1, \ldots, n, \quad k = 1, \ldots, m.$$

If $\varphi_{a_k}(u_1) \geqslant \ldots \geqslant \varphi_{a_k}(u_n)$ and the matrix E_k is upper triangular, then the basis $\overline{u}(x)$ is called the *Levelt* basis.

The numbers $\beta_k^j = \varphi_k^j + \rho_k^j$, where $\varphi_k^j = \varphi_{a_k}(u_j)$, are called *exponents* of Equation (1.1) at the point $x = a_k$. The exponents are power asymptotics of solutions and also roots of the characteristic polynomial

$$\lambda(\lambda - 1)\ldots(\lambda - n + 1) + \ldots + \lambda(\lambda - 1)\ldots(\lambda - n + j + 1)r_j(a_k) + \ldots + r_n(a_k) = 0,$$

where $r_j(x)$ are taken from (1.3).

For the sum of exponents β_k^j of the Levelt bases of the Fuchsian differential equation (1.1) at singular points $a_1, \ldots, a_m$ *the Fuchsian relation* holds

$$\sum_{k=1}^{m} \sum_{j=1}^{n} \beta_k^j = \frac{(m-2)n(n-1)}{2}, \tag{1.5}$$

where n is the dimension of the space of solutions and m is the number of singular points of Equation (1.1).

As known, an element $u(x) \in \mathscr{L}$ of the space of solutions to Fuchsian differential equation at $x = a_k$ can be represented as

$$u(x) = \sum_{j,l \in \sigma} f_{jl}(x)(x - a_k)^{\rho_k^j} \ln^{b_l}(x - a_k),$$

where $f_{jl}(x)$ is the Laurent series with center $x = a_k$ and finite principal part, b_l are nonnegative integers, and σ is a finite set of indices. The *real exponent* $\widehat{\beta}_k$ of a solution $u(x)$ at the point $x = a_k$ is the minimum of $\operatorname{ord}_{a_k} f_{jl}(x) + \operatorname{Re}\rho_k^j$ over all $j, l \in \sigma$.

Lemma 1.1. *Let $\widetilde{\beta}_k^1, \dots, \widetilde{\beta}_k^n$ be real exponents of an arbitrary basis for the space of solutions to Equation* (1.1) *at the point $z = a_k$ (one for each k). Then*

$$\sum_{k=1}^{m}\sum_{j=1}^{n}\widetilde{\beta}_k^j \leqslant \frac{(m-2)n(n-1)}{2}.$$

Proof. This assertion is practically obvious since the set of real exponents $\beta_k^1, \dots, \beta_k^n$ of the associated basis at a given regular singular point $x = a_k$ is maximal among the sets of real exponents of other bases at this point. Thus, the sum of real exponents of associated bases over all singular points $a_1, \dots, a_m$ is exactly not less than the sum of real exponents over other bases. Since the real exponents are the real parts of exponents, we have

$$\sum_{k=1}^{m}\sum_{j=1}^{n}\widetilde{\beta}_k^j \leqslant \sum_{k=1}^{m}\sum_{j=1}^{n}\beta_k^j = \frac{(m-2)n(n-1)}{2},$$

where β_k^j are the exponents of the corresponding associated bases at the points $z = a_k$ ($\widetilde{\beta}_k^j = \operatorname{Re}\beta_k^j$). The lemma is proved. □

Definition 1.1. Functions $f_1(x), \dots, f_n(x)$ are called

- *algebraically dependent* over $\mathbb{C}$ if there exists a nonzero polynomial $P \in \mathbb{C}[x_1, \dots, x_n]$ such that $P(f_1(x), \dots, f_n(x)) \equiv 0$; otherwise, $f_1(x), \dots, f_n(x)$ are said to be *algebraically independent* over $\mathbb{C}$,
- A-*algebraically dependent* over $\mathbb{C}$ if there exists a nonzero polynomial $P \in \mathbb{C}[x_1, \dots, x_n]$ of degree at most A such that $P(f_1(x), \dots, f_n(x)) \equiv 0$; otherwise, $f_1(x), \dots, f_n(x)$ are said to be A-*algebraically independent* over $\mathbb{C}$,
- (∞, A)-*algebraically dependent* if there exists a nonzero polynomial $P \in \mathbb{C}[x_1, \dots, x_n]$ of the total degree at most A with respect to the variables $x_2, \dots, x_n$ such that $P(f_1(x), \dots, f_n(x)) \equiv 0$; otherwise, $f_1(x), \dots, f_n(x)$ are said to be (∞, A)-*algebraically independent* over $\mathbb{C}$.

If functions are algebraically independent over $\mathbb{C}$, then they are A-algebraically independent for any A.

2 The Main Results

Theorem 2.1. *Let $u_1(x), \dots, u_n(x)$ be the fundamental system of solutions to the Fuchsian equation* (1.1) *with singular points $a_1, \dots, a_m$ on the Riemann sphere, which is A-algebraically independent over $\mathbb{C}$, and let $P \in \mathbb{C}[x_1, \dots, x_n]$ be an arbitrary nonzero polynomial of degree at most A. Let the sum of real exponents of $u_1(x), \dots, u_n(x)$ at singular points $a_1, \dots, a_m$ be equal to S. Then, if $x = a$ is not singular for Equation* (1.1)*, then the function $f(x) = P(u_1(x), \dots, u_n(x))$ cannot have zero of order greater than T at the point $x = a$ if*

$$T = \frac{C_{A+n}^n}{2(n+1)}((m-2)(n+1)(C_{A+n}^n - 1) - 2AS + 2(n+1)).$$

Proposition 2.1. *The logarithms*

$$\ln(x-a_1), \ldots, \ln(x-a_n), \quad a_k \in \mathbb{C}, \quad a_k \neq a_l, \quad k \neq l \tag{2.1}$$

are algebraically independent over $\mathbb{C}$.

From Theorem 2.1 and Proposition 2.1 we obtain the following assertion.

Corollary 2.1. *The functions* $x^t, \ln(x-a_1), \ldots, \ln(x-a_n)$, *where*

$$t > T_0 = C_{A+n+1}^{n+1}\Big(\frac{1}{2}(n-1)(C_{A+n+1}^{n+1}-1)+1\Big)$$

are $(S=0)$ (∞, A)*-algebraically independent over* $\mathbb{C}$.

Proof. If the functions $x^t, \ln(x-a_1), \ldots, \ln(x-a_n)$ are (∞, A)-algebraically dependent over $\mathbb{C}$, then there exists a finite expression

$$\sum_{i_0, i_1, \ldots, i_n} C_{i_0, i_1, \ldots, i_n} x^{i_0 t} \ln^{i_1}(x-a_1) \ldots \ln^{i_n}(x-a_n) \equiv 0, \quad i_1 + \ldots + i_n, \leqslant A,$$

with nonzero set of complex coefficients $\{C_{i_0, i_1, \ldots, i_n}\}$. Let l be the minimal number such that not all coefficients $\{C_{l, i_1, \ldots, i_n}\}$ are zero. Then the above expression can be written as

$$x^{(l+1)t} \sum_{i_0 > l, i_1, \ldots, i_n} C_{i_0, i_1, \ldots, i_n} x^{(i_0 - l - 1)t} \ln^{i_1}(x-a_1) \ldots \ln^{i_n}(x-a_n)$$
$$+ x^{lt} \sum_{i_1, \ldots, i_n} C_{l, i_1, \ldots, i_n} \ln^{i_1}(x-a_1) \ldots \ln^{i_n}(x-a_n) \equiv 0.$$

We have

$$x^t \mid \sum_{i_1, \ldots, i_n} C_{l, i_1, \ldots, i_n} \ln^{i_1}(x-a_1) \ldots \ln^{i_n}(x-a_n),$$

which for $t > T_0$ contradicts Theorem 2.1 applied to the fundamental system of solutions $1, \ln(x-a_1), \ldots, \ln(x-a_n)$ to the Fuchsian equation of order $n+1$ with $m = n+1$ singular points such that $S \geqslant 0$. (The general method for constructing such an equation is described in the proof of Lemma 3.1.) □

3 Proof of Theorem 2.1

Since the point $x = a$ is nonsingular for Equation (1.1), it is a holomorphy point of the function $f(x)$. It remains to estimate the possible order of zero of this function at the point $x = a$. Assume the contrary. We assume that $f(x)$ has zero of order $t > T$ at the point $x = a$. We denote by $\Phi_{\bar{i}}(x)$ the products

$$\Phi_{\bar{i}}(x) = u_1^{i_1}(x) \cdot u_2^{i_2}(x) \cdot \ldots \cdot u_n^{i_n}(x), \tag{3.1}$$

where $\bar{i} = (i_1, \ldots, i_n)$ is the multiindex with nonnegative integer components varying in $i_1 + \ldots + i_n \leqslant A$. This set of indices $\bar{i}$ is denoted by $\mathscr{A}$. It is easy to see that $D := \#\mathscr{A} = C_{A+n}^n$. The functions $u_1(x),\dots,u_n(x)$ form the fundamental system of solutions to Equation (1.1). Since the original system $u_1(x),\dots,u_n(x)$ is A-algebraically independent, the products

$$\Phi_{\bar i}(x), \quad \bar i = (i_1,\dots,i_n) \in \mathscr{A}, \tag{3.2}$$

are linearly independent over $\mathbb{C}$. It is easy to see that the space generated by the functions (3.2) is invariant under the operation of analytic continuation around singular points.

Lemma 3.1. *The space formed by the functions* (3.2) *is invariant under the operation of analytic continuation around singular points.*

Proof. Indeed, the monodromy matrix acting on the fundamental system of solutions of Equation (1.1) sends every solution to a linear combination of solutions. Under the action of the monodromy operator, the product (3.2) also goes to a linear combination of the products $\Phi_{\bar i}$ with indices $\bar i$ in $\mathscr{A}$. Consequently, the analytically extended system yields the same space. □

From the linear independence of the products (3.1) and Lemma 3.1 it follows that, based on these products regarded as solutions, one can construct the scalar differential equation

$$W(u(x),\Phi_{\bar i}(x)\mid \bar i\in\mathscr{A}) = \begin{pmatrix} u(x) & \Phi_{0,0,\dots,0}(x) & \dots & \Phi_{i_1,\dots,i_n}(x) & \dots \\ u'(x) & \Phi'_{0,0,\dots,0}(x) & \dots & \Phi'_{i_1,\dots,i_n}(x) & \dots \\ \vdots & \vdots & \ddots & \vdots & \\ u^{(D)}(x) & \Phi^{(D)}_{0,0,\dots,0}(x) & \dots & \Phi^{(D)}_{i_1,\dots,i_n}(x) & \dots \end{pmatrix} \equiv 0, \tag{3.3}$$

where $W(u,\Phi_{\bar i}(x)\mid \bar i\in\mathscr{A})$ is the Wronskian constructed from the function $u(x)$, $\Phi_{\bar i}(x)$, $\bar i\in\mathscr{A}$.

Singular points of Equation (3.3) are the singular points $a_1,\dots,a_m$ of the differential equation (1.1) and the false singular points $c_1,\dots,c_{m'}$ as well, where all solutions are holomorphic functions, but the coefficient at the higher order derivative in Equation (3.3), i.e., the Wronskian $W(\Phi_{\bar i}(x)\mid \bar i\in\mathscr{A})$ vanishes. If at least one of the solutions to Equation (3.3) has zero of order $t\geqslant D$ at the point $x=a$, $a\notin\{a_1,\dots,a_m\}$, then a coincides with some singular point c_k. Indeed, since D is the order of the equation, in the case $t\geqslant D$, the Wronskian $W(\Phi_{\bar i}(x)\mid \bar i\in\mathscr{A})$ which is a coefficient at the higher order derivative, vanishes. Consequently, $x=a$ is a false singular point. Let $a=c_1$, $\widetilde{\beta}^{\bar i}_k$ be the real exponents of the products $\Phi_{\bar i}(x)$ at points a_k, and let $\widehat{\beta}^s_k$ be the Levelt exponents of Equation (3.3) at false singular points. The real exponent $\widetilde{\beta}^{\bar i}_k$ of the product of solutions $u_1^{i_1}(x)\cdot\ldots\cdot u_n^{i_n}(x)$ at the point $z=a_k$ is the sum $\widetilde{\beta}^{\bar i}_k = i_1\widetilde{\beta}^1_k+\ldots+i_n\widetilde{\beta}^n_k$, where $\widetilde{\beta}^j_k$ are the real exponents of $u_j(x)$ at the point $z=a_k$.

The sum $\widetilde{S}$ of real exponents of $\Phi_{\bar i}(x)$, $\bar i\in\mathscr{A}$, at points $a_1,\dots,a_m$ can be expressed as

$$\widetilde{S} = \sum_{k=1}^{m}\sum_{\bar i\in\mathscr{A}} \widetilde{\beta}^{\bar i}_k = \frac{1}{n+1}DAS.$$

We calculated the number of exponents $\widetilde{\beta}^j_k$ in the expression for $\widetilde{S}$ and used the fact that they enter the sum in a symmetric way. By Lemma 1.1,

$$\sum_{\bar i\in\mathscr{A}}\sum_{k=1}^{m}\widetilde{\beta}^{\bar i}_k + \sum_{s=1}^{D}\sum_{k=1}^{m'}\widehat{\beta}^s_k \leqslant \frac{(M-2)D(D-1)}{2},$$

where $M = m + m'$, m is the number of singular points of Equation (1.1) and m' is the number of false singular points of Equation (3.3).

Let us estimate the sum of real exponents

$$\sum_{s=1}^{D}\sum_{k=1}^{m'} \widehat{\beta}_k^s \leqslant \frac{(M-2)D(D-1)}{2} - \frac{1}{n+1}DAS.$$

The sum of Levelt exponents at each false singular point $x = c_k$ is not less than

$$\sum_{s=1}^{D} \widehat{\beta}_k^s \geqslant \frac{D(D-1)}{2} + 1.$$

Then we obtain the following estimate for the maximal exponent $\widehat{\beta}_1^1$ at the point $x = a = c_1$:

$$\widehat{\beta}_1^1 \leqslant D + \frac{(M-2)D(D-1)}{2} - \frac{1}{n+1}DAS - m'\frac{D(D-1)}{2} - m' = D + \frac{(m-2)D(D-1)}{2}$$
$$- \frac{1}{n+1}DAS - m' \leqslant \frac{C_{A+n}^n}{2(n+1)}((m-2)(n+1)(C_{A+n}^n - 1) - 2AS + 2(n+1)) = T.$$

For $t > T$ we arrive at a contradiction.

Corollary 3.1. *Let $u_1(x)$, $u_2(x)$, $\ldots u_n(x)$ be the fundamental system of solutions to Equation* (1.1) *such that $x = 0$ is a nonsingular point, and let the assumptions of Theorem* 2.1 *be satisfied. Then the functions $\Phi_{\bar{i}}(x) = x^{i_0 t}u_1^{i_1}(x)\ldots u_n^{i_n}(x)$, $t > T$, $i_1 + \ldots + i_n \leqslant A$, $\bar{i} = (i_0, i_1, \ldots, i_n)$, where $i_0 \geqslant 0$ runs over a finite set of integers, are linearly independent over $\mathbb{C}$.*

Proof. Assume the contrary. Then we can construct the nontrivial linear combination

$$\sum_{\bar{i}\in\mathscr{A}} C_{\bar{i}} \cdot x^{i_0 t} \cdot u_1^{i_1}(x) \cdot u_2^{i_2}(x) \cdot \ldots \cdot u_n^{i_n}(x) \equiv 0$$

which is written in the form

$$x^t \sum_{i_0 \geqslant 1} C_{\bar{i}} \cdot x^{(i_0-1)t} \cdot u_1^{i_1}(x) \cdot u_2^{i_2}(x) \cdot \ldots \cdot u_n^{i_n}(x) + \sum_{\bar{i}, i_0 = 0} C_{\bar{i}} \cdot u_1^{i_1}(x) \cdot u_2^{i_2}(x) \cdot \ldots \cdot u_n^{i_n}(x) \equiv 0.$$

Then $\sum\limits_{i_1+\ldots+i_n\leqslant A} C_{\bar{i}} \cdot u_1^{i_1}(x) \cdot u_2^{i_2}(x) \cdot \ldots \cdot u_n^{i_n}(x)$ is divided by x^t. However, by Theorem 2.1, for $t > T$ this sum cannot be divided by x^t, and we arrive at a contradiction. □

4 Proof of Proposition 2.1

The assertion that the logarithms (2.1) are algebraically independent over $\mathbb{C}$ is equivalent to the fact that the functions

$$f_a^{\bar{i}}(x) = f_{a_1,\ldots,a_n}^{i_1,\ldots,i_n}(x) = \ln^{i_1}(x - a_1) \cdot \ldots \cdot \ln^{i_n}(x - a_n), \quad i_1, \ldots, i_n \in \mathbb{Z}_+, \tag{4.1}$$

are linearly independent over $\mathbb{C}$. We consider the linear combination of the functions (4.1)

$$L(x) = \sum_{\bar{i}\in\mathscr{A}} C_{\bar{i}} \cdot f_a^{\bar{i}}(x), \quad a = (a_1, \ldots, a_n), \quad \bar{i} = (i_1, \ldots, i_n), \tag{4.2}$$

and prove this assertion by induction. We start with induction on n. For $n = 1$ from (4.1) we have

$$L(x) = \sum_{i=0}^{s} C_i \cdot \ln^i(x - a_1).$$

If $s = 0$, then $L(x) \equiv 0$ implies $C_0 = 0$. Let $s' \geqslant 0$. We assume that for $s = s'$ from $L(x) \equiv 0$ it follows that $C_0 = \ldots = C_{s'} = 0$. Let $s = s' + 1$. Denote by $\sigma_1 f$ the analytic continuation of $f(x)$ around the point a_1. Then $\sigma_1 L(x) - L(x) \equiv 0$. Since the degree of this expression is at most s', all its coefficients vanish by assumption. It is easy to see that from its coefficient it is possible to recover all the coefficients of $L(x)$, except for C_0, and all they turn out to be zero. Since $L(x) \equiv 0$, we have $C_0 = 0$.

Let us prove the induction step. Assume that for $n = m$ the functions (4.1) are algebraically independent. We show that the same is true for $n = m + 1$. We use induction on l, $i_n \leqslant l$, of the last factor of the linear combination (4.2). It is obvious that the case $l = 0$ corresponds to the induction hypothesis for $n = m$.

We assume that the functions (4.1) with $i_n \leqslant l$ are linearly independent over $\mathbb{C}$ and show that the functions (4.1) are also linearly independent for $i_n \leqslant l + 1$.

We denote by $\sigma_n f$ the analytic continuation of $f(x)$ around the point a_n. We have

$$\begin{aligned}
\sigma_n f_a^{\bar{i}}(x) &= \ln^{i_1}(x - a_1) \cdot \ldots \cdot \ln^{i_{n-1}}(x - a_{n-1}) \cdot (\ln(x - a_n) + 2\pi i)^{i_n} \\
&= \ln^{i_1}(x - a_1) \cdot \ldots \cdot \ln^{i_{n-1}}(x - a_{n-1}) \cdot \left(\binom{i_n}{0} \ln^{i_n}(x - a_n) \right. \\
&\left. + \binom{i_n}{1} \ln^{(i_n-1)}(x - a_n) \cdot (2\pi i)^1 + \ldots + \binom{i_n}{i_n} (2\pi i)^{i_n} \right).
\end{aligned}$$

We note that the expression $\sigma_n f_a^{\bar{i}} - f_a^{\bar{i}}(z)$ has the form (4.2) and contains the factors $\ln(x - a_n)$ of degree at most $i_n - 1$.

Now, we assume that there exists the linear combination (4.2) with nonzero coefficients that identically vanishes, $L(x) \equiv 0$. Let all the terms of $L(x)$ have degrees $i_n \leqslant l + 1$. We consider the expression $\mathscr{L}(x) = \sigma_n L(x) - L(x)$ which is also equal to zero, but the degree of $\ln(x - a_n)$ does not exceed l. By the induction hypothesis, all the coefficients of $\mathscr{L}(x)$ vanish. Then the coefficients of $L(x)$ also vanish since all the terms of $L(x)$ containing $\ln(x - a_n)$ are uniquely recovered from $\mathscr{L}(x)$. The remaining terms of $L(x)$ also vanish by the induction hypothesis since they form the sum equal to zero identically. The proposition is proved.

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