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Notes on 1089 and a Variation of the Kaprekar Operator

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Abstract

We study a variation of the Kaprekar operator $F(x)$ for all non-negative integers x and show that the range of F consists of 0, 99, 1089, and the integers of the form $1099\ldots98900\ldots0$, where $99\ldots9$ and $00\ldots0$ may be long, short, or disappear.

1 Introduction and Statement of the Main Result

Throughout this article, if $y \in \mathbb{R}$, then $\lfloor y \rfloor$ is the largest integer less than or equal to y and $\lceil y \rceil$ is the smallest integer larger than or equal to y. Unless stated otherwise, all other variables are nonnegative integers. For any $x \in \mathbb{N} \cup \{0\}$, we write the decimal expansion of x as

$$x = (a_k a_{k-1} \ldots a_1 a_0)_{10} = \sum_{0 \le j \le k} a_{k-j} 10^{k-j},$$

where $0 \le a_i \le 9$ for all $i = 0, 1, 2, \ldots, k$.

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The Kaprekar operator K is defined by the following operation: take any positive integer x having four decimal digits which are not all equal and the leading digit is not zero, say $x = (a_3a_2a_1a_0)_{10}$, $a_3 \neq 0$, and $a_i \neq a_j$ for some i, j, then rearrange a_3, a_2, a_1, a_0 as c_3, c_2, c_1, c_0 so that $c_3 \geq c_2 \geq c_1 \geq c_0$. Then

$$K(x) = (c_3c_2c_1c_0)_{10} - (c_0c_1c_2c_3)_{10}. \tag{1.1}$$

Observe that the second number on the right-hand side of (1.1) is obtained by reversing the decimal digits of the first. It is well known that no matter what x we start with, after repeating this process at most 7 steps, we always obtain the number 6174. For example, suppose $x = 1000$. Then

$$\begin{aligned}
&K(x) = 1000 - 1 = 999,\\
&K^2(x) = K(K(x)) = K(999) = K(0999) = 9990 - 0999 = 8991,\\
&K^3(x) = K(8991) = 9981 - 1899 = 8082,\\
&K^4(x) = 8820 - 0288 = 8532,\\
&K^5(x) = 8532 - 2358 = 6174,
\end{aligned}$$

and $K^m(x) = 6174$ for all $m \geq 6$. Here, it is important to keep in mind that the Kaprekar operator operates on the positive integers having four digits not all equal. So the decimal representation of $K(x)$ with nonzero leading digit may have only 3 digits but, to calculate $K(K(x))$, we must first write $K(x)$ as 4 digits number by adding 0 as the leading digit, as shown above in $K(999) = K(0999)$. We can generalize K to operate on any nonnegative integers as follows:

Definition 1.1 (Kaprekar operator on nonnegative integers). *Let* $g : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ *be given by* $g(0) = .0$ *If* $x = (a_ka_{k-1} \dots a_0)_{10}$, $a_k \neq 0$, *and* c_k, c_{k-1}, $\dots$, c_0 *is the permutation of* a_k, a_{k-1}, $\dots$, a_0 *such that* $c_k \geq c_{k-1} \geq \dots \geq c_0$, *then*

$$g(x) = (c_kc_{k-1} \dots c_1c_0)_{10} - (c_0c_1 \dots c_{k-1}c_k)_{10}.$$

In addition, for the purpose of this article, if x *is as above, then we always write the decimal representation of* $g(x)$ *as* $k+1$ *digits number, say* $g(x) = (b_kb_{k-1} \dots b_0)_{10}$.

Another trick is as follows: take any positive integer having three digits, say $x = (a_2a_1a_0)_{10}$, where $a_2 \neq 0$, $0 \leq a_j \leq 9$ for all j, and $a_i \neq a_j$ for some i, j. Then calculate $g(x)$, say $g(x) = b = (b_2b_1b_0)_{10}$. Then compute $f(b) = b + \text{reverse}(b) = (b_2b_1b_0)_{10} + (b_0b_1b_2)_{10}$. No matter what x we start with, we always obtain $f(b) = 1089$. We generalize this to the following operator:

Definition 1.2. *Let f be the reverse and add an operator. Let $F : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ be defined by $F = f \circ g$. In addition, to calculate $F(x) = f(g(x))$, we always keep the same convention in Definition 1.1, where the number of decimal digits of x and $g(x)$ are equal.*

For example, suppose $x = 100$. Then $g(x) = 99 = 099$ and so $F(x) = f(099) = 990 + 099 = 1089$. By using a computer or a straightforward calculation, it is not difficult to notice the following pattern:

$$\begin{aligned}
&\text{if } 10 \le x < 10^2, \text{ then } F(x) = 0 \text{ or } 99;\\
&\text{if } 10^2 \le x < 10^3, \text{ then } F(x) = 0 \text{ or } 1089;\\
&\text{if } 10^3 \le x < 10^4, \text{ then } F(x) = 0,\ 10890, \text{ or } 10989;\\
&\text{if } 10^4 \le x < 10^5, \text{ then } F(x) = 0,\ 109890, \text{ or } 109989.
\end{aligned}$$

In general, we have the following result.

Theorem 1.3. *Let $F = f \circ g$, $k \ge 2$, and $10^k \le x < 10^{k+1}$. Let $x = (a_k a_{k-1} \dots a_0)_{10}$, $a_k \ne 0$, and $0 \le a_i \le 9$ for all $i = 0, 1, \dots, k$. If $k = 2$, then $F(x) = 0$ or 1089. Suppose that $k \ge 3$ and $c_k, c_{k-1}, \dots, c_0$ is the permutation of $a_k, a_{k-1}, \dots, a_0$ such that $c_k \ge c_{k-1} \ge \cdots \ge c_0$. Let $m = z(x)$ be the largest element of the set $\{j \in \{0, 1, \dots, k\} \mid c_{k-j} > c_j\}$. If $a_i = a_j$ for all i, j, then $F(x) = 0$. If $a_i \ne a_j$ for some i, j, then*

$$F(x) = 10\underbrace{99\dots9}_{y(x)}89\underbrace{00\dots0}_{z(x)},$$

where $y(x) = k - 2 - z(x)$.

Although the result is easy to observe for $k = 2, 3, 4$, it is more difficult when k is large. As far as we know, there is no proof for a general k. We hope that this article will help explain something related to 6174, 1089, and other similar magic numbers. Finally, it is an interesting open problem to determine whether or not a given number in the range of F is a Lychrel number. We leave this problem for the interested reader. For more information on 6174 and the Kaprekar operator, see for instance in [5], [6], and [7]. For related articles on 1089 and 2178, see for example [1], [2], [3], [4], [8], [9], and [10].

2 Proof of the Main Result

Proof. We first consider the case $k = 2$. Since $10^2 \le x < 10^3$, it can be written in the decimal representation as $x = (a_2a_1a_0)_{10}$, where $a_2 \neq 0$ and $0 \le a_i \le 9$ for $i = 0, 1, 2$. If $a_2 = a_1 = a_0$, then $F(x) = 0$. So suppose that a_2, a_1, a_0 are not all the same and let c_2, c_1, c_0 be the permutation of a_2, a_1, a_0 such that $c_2 \ge c_1 \ge c_0$. Then $c_2 > c_0$ and

$$\begin{aligned}
g(x) &= (c_2c_1c_0)_{10} - (c_0c_1c_2)_{10} \\
&= (10^2c_2 + 10c_1 + c_0) - (10^2c_0 + 10c_1 + c_2) \\
&= 10^2(c_2 - c_0 - 1) + 10(9) + 10 - (c_2 - c_0) \\
&= (d_2d_1d_0)_{10},
\end{aligned}$$

where $d_2 = c_2 - c_0 - 1$, $d_1 = 9$, and $d_0 = 10 - (c_2 - c_0)$. Then it is easy to see that

$$F(x) = (d_2d_1d_0)_{10} + (d_0d_1d_2)_{10} = 1089.$$

Next, let $k \ge 3$, $10^k \le x < 10^k$, and write $x = (a_ka_{k-1}\dots a_0)_{10}$, where $a_k \neq 0$ and $0 \le a_i \le 9$ for all $i = 0, 1, \dots, k$. If $a_i = a_j$ for all i, j, then $F(x) = 0$ and we are done. So suppose that $a_i \neq a_j$ for some i, j. Let c_k, c_{k-1}, $\dots$, c_0 be the permutation of a_k, a_{k-1}, $\dots$, a_0 such that $c_k \ge c_{k-1} \ge \dots \ge c_0$. Then

$$\begin{aligned}
g(x) &= (c_kc_{k-1}\dots c_0) - (c_0c_1\dots c_k)_{10} \\
&= \sum_{j=0}^{k} c_{k-j}10^{k-j} - \sum_{j=0}^{k} c_j10^{k-j} \\
&= \sum_{j=0}^{k} (c_{k-j} - c_j)10^{k-j}. \qquad (2.2)
\end{aligned}$$

Let $A = \{j \in \{0, 1, \dots, k\} \mid c_{k-j} > c_j\}$. Since $c_k > c_0$, we see that $0 \in A$, and so $A \neq \varnothing$. Let m be the largest element of A. If $m \ge \lceil \frac{k}{2} \rceil$, then $k - m \le k - \lceil \frac{k}{2} \rceil = \lfloor \frac{k}{2} \rfloor \le m$, which implies $c_{k-m} \le c_m$ which contradicts the fact that $m \in A$. Therefore, $0 \le m < \lceil \frac{k}{2} \rceil$. Since m is the largest element of

A and $c_k \geq c_{k-1} \geq \cdots \geq c_0$, we assert that the following relations hold:

$$c_{k-j} > c_j \quad \text{for} \quad 0 \leq j \leq m, \tag{2.3}$$

$$c_{k-j} \leq c_j \quad \text{for} \quad j > m, \tag{2.4}$$

$$c_{k-j} = c_j \quad \text{for} \quad m < j \leq \left\lfloor \frac{k}{2} \right\rfloor, \tag{2.5}$$

$$c_{k-j} = c_j \quad \text{for} \quad \left\lceil \frac{k}{2} \right\rceil \leq j < k - m, \tag{2.6}$$

$$c_{k-j} < c_j \quad \text{for} \quad k - m \leq j \leq k. \tag{2.7}$$

For (2.3), we know that $c_{k-m} > c_m$ and if $0 \leq j < m$, then $c_{k-j} \geq c_{k-m} > c_m \geq c_j$. So (2.3) is verified. By the choice of m, (2.4) follows immediately. If $j \leq \lfloor \frac{k}{2} \rfloor$, then $k - j \geq k - \lfloor \frac{k}{2} \rfloor = \lceil \frac{k}{2} \rceil \geq j$, and so $c_{k-j} \geq c_j$. This and (2.4) imply (2.5). Replacing j by $k - j$ in (2.5), we obtain (2.6). Changing j to $k - j$ in (2.3), we obtain (2.7).

Next, we divide the sum in (2.2) into 3 parts: $0 \leq j \leq m$, $m < j < k - m$, and $k - m \leq j \leq k$. By (2.5) and (2.6), the second part is zero. Therefore, (2.2) becomes

$$g(x) = \sum_{0 \leq j \leq m} (c_{k-j} - c_j) 10^{k-j} + \sum_{k-m \leq j \leq k} (c_{k-j} - c_j) 10^{k-j}. \tag{2.8}$$

The terms $c_{k-j} - c_j$ in (2.8) are positive in the first sum and negative in the second. Then we write

$$\begin{aligned} 10^{k-m} &= \left(\sum_{m+1 \leq j \leq k-1} 9 \cdot 10^{k-j} \right) + 10 \\ &= \left(\sum_{m+1 \leq j \leq k-m-1} 9 \cdot 10^{k-j} \right) + \left(\sum_{k-m \leq j \leq k-1} 9 \cdot 10^{k-j} \right) + 10. \end{aligned}$$

Let $d_{k-m} = c_{k-m} - c_m - 1$ and $d_0 = 10 + c_0 - c_k$. Then

$$\begin{aligned} (c_{k-m} - c_m) 10^{k-m} &+ \sum_{k-m \leq j \leq k} (c_{k-j} - c_j) 10^{k-j} \\ &= d_{k-m} 10^{k-m} + 10^{k-m} + \sum_{k-m \leq j \leq k} (c_{k-j} - c_j) 10^{k-j} \\ &= d_{k-m} 10^{k-m} + \left(\sum_{m+1 \leq j \leq k-m-1} 9 \cdot 10^{k-j} \right) \\ &\quad + \sum_{k-m \leq j \leq k-1} (9 + c_{k-j} - c_j)\, 10^{k-j} + d_0, \end{aligned} \tag{2.9}$$

where d_{k-m}, d_0, and the coefficients of 10^{k-j} in the above equation are nonnegative and are less than 10. Therefore, (2.8) and (2.9) imply that we can write $g(x)$ in the decimal expansion as:

$$g(x) = (d_k d_{k-1} \dots d_0)_{10} = \sum_{0 \le j \le k} d_{k-j} 10^{k-j},$$

where $0 \le d_i \le 9$ for all $i = 0, 1, 2, \dots, k$, and d_{k-j} satisfies the following relations:

$$d_{k-j} = c_{k-j} - c_j \quad \text{for} \quad 0 \le j < m, \tag{2.10}$$

$$d_{k-m} = c_{k-m} - c_m - 1, \tag{2.11}$$

$$d_{k-j} = 9 \quad \text{for} \quad m+1 \le j \le k-m-1, \tag{2.12}$$

$$d_{k-j} = 9 + c_{k-j} - c_j \quad \text{for} \quad k-m \le j \le k-1, \tag{2.13}$$

$$d_0 = 10 + c_0 - c_k. \tag{2.14}$$

Since the decimal expansion of $g(x)$ has $k+1$ digits, that of $f(g(x))$ has at most $k+2$ digits. Then

$$F(x) = f(g(x)) = (d_k d_{k-1} \dots d_0)_{10} + (d_0 d_1 \dots d_k)_{10} = (e_{k+1} e_k \dots e_0)_{10},$$

where $0 \le e_i \le 9$ for all $i = 0, 1, \dots, k+1$. From elementary arithmetic, recall the fact that $e_0 = d_0 + d_k - 10\varepsilon_0$, where $\varepsilon_0 = 0$ if $d_0 + d_k < 10$, and $\varepsilon_0 = 1$ if $d_0 + d_k \ge 10$. In addition, $e_j = d_j + d_{k-j} + \varepsilon_{j-1} - 10\varepsilon_j$ for $1 \le j \le k$, where $\varepsilon_{j-1} = 0$ if there is no carry in the addition in the $(j-1)$th position and $\varepsilon_{j-1} = 1$ otherwise; while $\varepsilon_j = 0$ if $d_j + d_{k-j} + \varepsilon_{j-1} < 10$, and $\varepsilon_j = 1$ if $d_j + d_{k-j} + \varepsilon_{j-1} \ge 10$. Moreover, $e_{k+1} = 0$ if there is no carry in the addition in the kth position and $e_{k+1} = 1$ otherwise. We now calculate $e_0, e_1, \dots, e_k$, e_{k+1} by using this fact and the relations in (2.10) to (2.14). We obtain

$$e_0 = d_0 + d_k - 10\varepsilon_0 = (10 + c_0 - c_k) + (c_k - c_0) - 10\varepsilon_0 = 10 - 10\varepsilon_0,$$

which implies $\varepsilon_0 = 1$ and $e_0 = 0$. Then

$$e_1 = d_1 + d_{k-1} + 1 - 10\varepsilon_1 = (9 + c_1 - c_{k-1}) + (c_{k-1} - c_1) + 1 - 10\varepsilon_1 = 10 - 10\varepsilon_1,$$

which implies $\varepsilon_1 = 1$ and $e_1 = 0$. In general, we replace j by $k-j$ in (2.13) to get $d_j = 9 + c_j - c_{k-j}$ for $1 \le j \le m$; and if $\varepsilon_{j-1} = 1$ and $2 \le j \le m-1$, then

$$e_j = d_j + d_{k-j} + 1 - 10\varepsilon_j = (9 + c_j - c_{k-j}) + (c_{k-j} - c_j) + 1 - 10\varepsilon_j = 10 - 10\varepsilon_j,$$

which implies $\varepsilon_j = 1$ and $e_j = 0$. Applying this observation for $j = 2, 3, \ldots, m-1$, respectively, we obtain

$$\varepsilon_2 = 1, e_2 = 0, \varepsilon_3 = 1, e_3 = 0, \ldots, \varepsilon_{m-1} = 1, e_{m-1} = 0.$$

Then

$$\begin{aligned} e_m &= d_m + d_{k-m} + 1 - 10\varepsilon_m \\ &= (9 + c_m - c_{k-m}) + (c_{k-m} - c_m - 1) + 1 - 10\varepsilon_m = 9 - 10\varepsilon_m, \end{aligned}$$

which implies $\varepsilon_m = 0$ and $e_m = 9$. Then $e_{m+1} = d_{m+1} + d_{k-m-1} - 10\varepsilon_{m+1} = 9 + 9 - 10\varepsilon_{m+1}$, which implies $\varepsilon_{m+1} = 1$ and $e_{m+1} = 8$. In general, we replace j by $k - j$ in (2.12) to obtain $d_j = 9$ for $m + 1 \leq j \leq k - m - 1$; and if $\varepsilon_{j-1} = 1$ and $m + 2 \leq j \leq k - m - 1$, then

$$e_j = d_j + d_{k-j} + \varepsilon_{j-1} - 10\varepsilon_j = 9 + 9 + 1 - 10\varepsilon_j = 19 - 10\varepsilon_j,$$

which implies $\varepsilon_j = 1$ and $e_j = 9$. Applying this observation for $j = m + 2, m + 3, \ldots, k - m - 1$, respectively, we obtain

$$\varepsilon_{m+2} = 1, e_{m+2} = 9, \varepsilon_{m+3} = 1, e_{m+3} = 9, \ldots, \varepsilon_{k-m-1} = 1, e_{k-m-1} = 9.$$

Then

$$\begin{aligned} e_{k-m} &= d_{k-m} + d_m + 1 - 10\varepsilon_{k-m} \\ &= (c_{k-m} - c_m - 1) + (9 + c_m - c_{k-m}) + 1 - 10\varepsilon_{k-m} = 9 - 10\varepsilon_{k-m}, \end{aligned}$$

which implies $\varepsilon_{k-m} = 0$ and $e_{k-m} = 9$. Then

$$\begin{aligned} e_{k-m+1} &= d_{k-m+1} + d_{m-1} - 10\varepsilon_{k-m+1} \\ &= (c_{k-m+1} - c_{m-1}) + (9 + c_{m-1} - c_{k-m+1}) - 10\varepsilon_{k-m+1} \\ &= 9 - 10\varepsilon_{k-m+1}, \end{aligned}$$

which implies $\varepsilon_{k-m+1} = 0$ and $e_{k-m+1} = 9$. In general, we replace j by $k - j$ in (2.13) to obtain $d_j = 9 + c_j - c_{k-j}$ for $1 \leq j \leq m$; and if $\varepsilon_{k-j-1} = 0$ and $1 \leq j < m$, then

$$e_{k-j} = d_{k-j} + d_j - 10\varepsilon_{k-j} = (c_{k-j} - c_j) + (9 + c_j - c_{k-j}) - 10\varepsilon_{k-j} = 9 - 10\varepsilon_{k-j},$$

which implies $\varepsilon_{k-j} = 0$ and $e_{k-j} = 9$. Applying this observation for $j = m-2, m - 3, \ldots, 1$, respectively, we obtain

$$\varepsilon_{k-m+2} = 0, e_{k-m+2} = 9, \varepsilon_{k-m+3} = 0, e_{k-m+3} = 9, \ldots, \varepsilon_{k-1} = 0, e_{k-1} = 9.$$

Then

$$e_k = d_k + d_0 - 10\varepsilon_k = (c_k - c_0) + (10 + c_0 - c_k) - 10\varepsilon_k = 10 - 10\varepsilon_k,$$

which implies $\varepsilon_k = 1$ and $e_k = 0$. Then $e_{k+1} = 1$. To conclude, we obtain $e_j = 0$ for $0 \leq j < m$, $e_m = 9$, $e_{m+1} = 8$, $e_j = 9$ for $m + 2 \leq j \leq k - 1$, $e_k = 0$, and $e_{k+1} = 1$. This completes the proof. □

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References

[1] D. Acheson, 1089 *and all that*, available at https://plus.maths.org/content/1089-and-all

[2] D. Acheson, 1089 *and All That: A Journey into Mathematics*, Oxford University Press, 2002.

[3] E. Behrends, The mystery of the number 1089 – how Fibonacci numbers come into play, *Elemente der Mathematik*, **70,** no. ,) (2015), 144–152.

[4] A. Bogomolny, 1089 *and a Property of* 3*-digit Numbers*, available at https://www.cut-the-knot.org/Curriculum/Arithmetic/S1089.shtml

[5] K. E. Eldridge, S. Sagong, The determination of Kaprekar convergence and loop convergence of all three-digits numbers, *The American Mathematical Monthly*, **95,** no. 2, (1988), 105–112.

[6] D. R. Kaprekar, An interesting property of the number 6174, *Scripta Mathematica*, **21,** (1955), 304.

[7] T. Prince, Kaprekar constant revisited, *International Journal of Mathematical Archive*, **4,** no. 5, (2013), 52–58.

[8] N. J. A. Sloane, 2178 and all that, *The Fibonacci Quarterly*, **52,** no. 2, (2014), 99–120.

[9] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, available at https://oeis.org/A023108

[10] R. Webster, G. Williams, On the trail of reverse divisors: 1089 and all that follow, *Mathematical Spectrum*, **45,** no. 3, (2013), 96–102.