Permutations

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Permutations of finite sets play a central role in algebraic and enumerative combinatorics. In addition to having many interesting enumerative properties *per se*, permutations also arise in almost every area of mathematics and indeed in all the sciences. Here we will discuss three different topics involving permutations, focusing on combinatorics but also giving some hints about connections with other areas.

1 Increasing and decreasing subsequences

1.1 The Erdős-Szekeres theorem

Let S_n denote the symmetric group of all permutations of $[n] := \{1, 2, \ldots, n\}$. We write permutations $w \in S_n$ as *words*, i.e., $w = a_1 a_2 \cdots a_n$, where $w(i) = a_i$. An *increasing subsequence* of w of *length* k is a subsequence $a_{i_1}, \ldots, a_{i_k}$ (so $i_1 < \cdots < i_k$) satisfying $a_{i_1} < \cdots < a_{i_k}$, and similarly for *decreasing subsequence.* For instance, if $w = 5642713$, then 567 is an increasing subsequence and 643 is a decreasing subsequence. Let $\mathrm{is}(w)$ (respectively, $\mathrm{ds}(w)$) denote the length of the longest increasing (respectively, decreasing) subsequence of w. If $w = 5642713$ as above, then $\mathrm{is}(w) = 3$ (corresponding to 567) and $\mathrm{ds}(w) = 4$ (corresponding to 5421 or 6421).

The subject of increasing and decreasing subsequences began in

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1935, and there has been much recent activity. There have been major breakthroughs in understanding the distribution of $\mathrm{is}(w)$, $\mathrm{ds}(w)$, and related statistics on permutations, and many unexpected and deep connections have been obtained with such areas as representation theory and random matrix theory. A more extensive survey of this topic appears in [51].

The first result on increasing and decreasing subsequences is a famous theorem of Erdős and Szekeres [16].

Theorem 1.1. *Let $p, q \geq 1$. If $w \in S_{pq+1}$, then either $\mathrm{is}(w) > p$ or $\mathrm{ds}(w) > q$.*

Seidenberg [42] gave an exceptionally elegant proof of Theorem 1.1 based on the pigeonhole principle which has been reproduced many times. Theorem 1.1 is best possible in that there exists $w \in S_{pq}$ with $\mathrm{is}(w) = p$ and $\mathrm{ds}(w) = q$. Schensted [41] found a quantitative strengthening of this result based on his rediscovery of an algorithm (now called the RSK algorithm) of Robinson [38] which has subsequently become a central topic in algebraic combinatorics.

In order to explain Schensted's work, define a *partition* λ of an integer $n \geq 0$ to be an integer sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and $\sum \lambda_i = n$. We then write $\lambda \vdash n$. The number of $\lambda_i > 0$ is the *length* of λ, denoted $\ell(\lambda)$. The *conjugate partition* $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ to λ has $\lambda_i - \lambda_{i+1}$ parts equal to i. Note that $\lambda'' = \lambda$, and that $\lambda'_1 = \ell(\lambda)$, $\lambda_1 = \ell(\lambda')$.

A *standard Young tableau* (SYT) of *shape* $\lambda \vdash n$ is a left-justified array of integers, with λ_i entries in the ith row, such that every integer $1, 2, \dots, n$ appears once, and every row and column is increasing. An example of an SYT of shape 4421 (short for $(4, 4, 2, 1, 0, \dots)$) is

$$\begin{array}{cccc} 1 & 3 & 5 & 6 \\ 2 & 7 & 9 & 11 \\ 4 & 10 & & \\ 8 & & & \end{array} .$$

Schensted defines a bijection $w \xrightarrow{\mathrm{rsk}} (P, Q)$ between permutations $w \in S_n$ and pairs (P, Q) of SYT of the same shape $\lambda = (\lambda_1, \lambda_2, \dots) \vdash$

n. His main theorem states that

$$\mathrm{is}(w) = \lambda_1, \quad \mathrm{ds}(w) = \lambda'_1 = \ell(\lambda). \tag{1}$$

This result was greatly extended by Greene [25], as follows. Suppose that the parts of λ are given by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$, and of the conjugate partition λ' by $\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_k > 0$.

Theorem 1.2. *Let $w \in S_n$, $1 \leq i \leq \ell$, and $1 \leq j \leq k$. Then the size (number of terms) of the longest union of i increasing subsequences of w is $\lambda_1 + \cdots + \lambda_i$, while the size of the longest union of j decreasing subsequences of w is $\lambda'_1 + \cdots + \lambda'_j$.*

For instance, let $w = 247951368$. The longest increasing subsequence is 24568, so $\lambda_1 = 5$. The largest union of two increasing subsequences is 24791368 (the union of 2479 and 1368), so $\lambda_1 + \lambda_2 = 8$. (Note that it is impossible to find a union of length 8 of two increasing subsequences that contains an increasing subsequence of length $\lambda_1 = 5$.) Finally w itself is the union of the three increasing subsequences 2479, 1368, and 5, so $\lambda_1 + \lambda_2 + \lambda_3 = 9$. Hence $(\lambda_1, \lambda_2, \lambda_3) = (5, 3, 1)$ (and $\lambda_i = 0$ for $i > 3$).

Schensted's theorem (1) leads to a formula for the number

$$f_{p,q}(n) = \#\{w \in S_n \,:\, \mathrm{is}(w) = p,\ \mathrm{ds}(w) = q\}.$$

Namely, if f^λ is the number of SYT of shape λ (given explicitly by the famous *hook length formula* of Frame-Robinson-Thrall [50, Cor. 7.21.6]), then

$$f_{p,q}(n) = \sum_{\substack{\lambda \vdash n \\ \lambda_1 = p,\ \lambda'_1 = q}} (f^\lambda)^2.$$

This formula has a number of applications. For instance, if $n = pq$ then there is exactly one $\lambda \vdash n$ satisfying $\lambda_1 = p$ and $\lambda'_1 = q$, namely, $\lambda = \langle p^q \rangle$, the partition with q parts equal to p. Hence

$$f_{p,q}(pq) = \left(f^{\langle p^q \rangle}\right)^2.$$

Applying the hook length formula yields the surprising formula

$$f_{p,q}(pq) = \left(\frac{(pq)!}{1^1 2^2 \cdots p^p (p+1)^p \cdots q^p (q+1)^{p-1} \cdots (p+q-1)^1} \right)^2.$$

1.2 Statistical properties of is(w)

Let us now turn to statistical properties of is(w). The most basic property is the expectation $E(n)$:

$$E(n) = \frac{1}{n!} \sum_{w \in S_n} \mathrm{is}(w).$$

It is not hard to deduce from the Erdős-Szekeres theorem that $E(n) \geq \sqrt{n}$. Hammersley made a number of contributions regarding $E(n)$, in particular showing the asymptotic formula $E(n) \sim c\sqrt{n}$ for some $\frac{\pi}{2} \leq c \leq e$ and giving a heuristic argument that $c = 2$. It follows from Schensted's theorem that

$$E(n) = \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left(f^\lambda\right)^2. \tag{2}$$

Using this formula both Vershik and Kerov [58] and Logan and Shepp [31] showed that $c \geq 2$, and Vershik and Kerov added a clever argument based on the RSK algorithm that $c \leq 2$. Hence $c = 2$ as suggested by Hammersley. The work of Vershik-Kerov and Logan-Shepp also determined the limiting shape Ψ (scaled so that the Young diagram has area 1) of the partition $\lambda \vdash n$ that maximizes f^λ. Moreover, "most" permutations $w \in S_n$ have limiting shape approaching (in a suitable sense) Ψ as $n \to \infty$. If we rotate the the Young diagram of λ 90° counterclockwise and normalize it to have area 1, then the limiting shape is bounded by the x and y-axes, together with the curve

$$\begin{aligned} x &= y + 2\cos\theta \\ y &= \frac{2}{\pi}(\sin\theta - \theta\cos\theta), \end{aligned}$$

for $0 \leq \theta \leq \pi$. See Figure 1.

A major breakthrough in understanding the behavior of is(w) was achieved in 1999 by Baik, Deift, and Johansson [4]. They determined the entire limiting distribution of is(w) as $n \to \infty$. It turns out to be given by the suitably scaled Tracy-Widom distribution, which

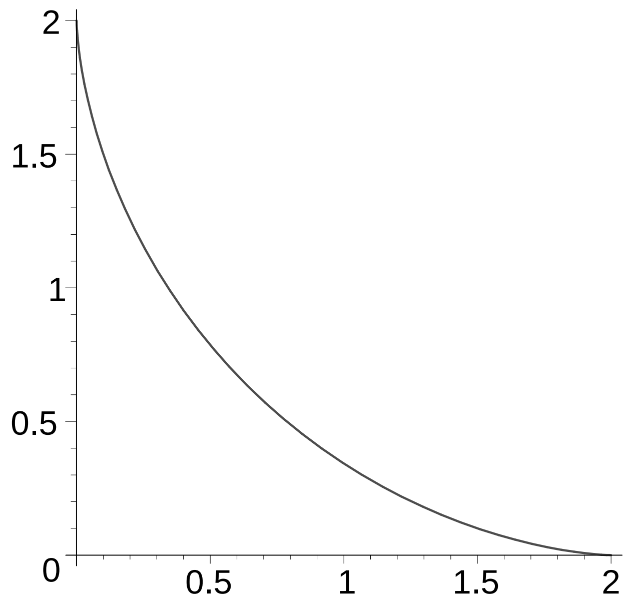

Figure 1: The curve $y = \Psi(x)$

had first appeared in connection with the distribution of the largest eigenvalue of a random hermitian matrix.

To describe these results, write $\mathrm{is}_n(w)$ for $\mathrm{is}(w)$ in order to indicate that $w \in S_n$. Let $\mathrm{Ai}(x)$ denote the *Airy function*, viz., the unique solution to the second-order differential equation

$$\mathrm{Ai}''(x) = x\,\mathrm{Ai}(x),$$

subject to the condition

$$\mathrm{Ai}(x) \sim \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}} \text{ as } x \to \infty.$$

Let $u(x)$ denote the unique solution to the nonlinear third order equation

$$u''(x) = 2u(x)^3 + xu(x), \tag{3}$$

subject to the condition

$$u(x) \sim -\mathrm{Ai}(x), \text{ as } x \to \infty.$$

Equation (3) is known as the *Painlevé II equation*, after Paul Painlevé (1863–1933).

Now define the *Tracy-Widom distribution* to be the probability distribution on $\mathbb{R}$ given by

$$F(t) = \exp\left(-\int_t^\infty (x-t)u(x)^2\, dx\right). \tag{4}$$

It is easily seen that $F(t)$ is indeed a probability distribution, i.e., $F'(t) \geq 0$, $\lim_{t\to\infty} F(t) = 1$, and $\lim_{t\to-\infty} F(t) = 0$. Let χ be a random variable with distribution F, and let χ_n be the random variable on S_n defined by

$$\chi_n(w) = \frac{\mathrm{is}_n(w) - 2\sqrt{n}}{n^{1/6}}.$$

We can now state the remarkable results of Baik, Deift, and Johansson.

Theorem 1.3. *As $n \to \infty$, we have*

$$\chi_n \to \chi \quad \textit{in distribution},$$

i.e., for all $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathrm{Prob}(\chi_n \leq t) = F(t). \tag{5}$$

As an example of the use of Theorem 1.3, we state the following improvement to the formula $E(n) \sim 2\sqrt{n}$.

Corollary 1.4.

$$\begin{aligned} \lim_{n\to\infty} \frac{E(\mathrm{is}_n) - 2\sqrt{n}}{n^{1/6}} &= \int t\, dF(t) \\ &= -1.7711\cdots. \end{aligned} \tag{6}$$

The starting point for the proof of Theorem 1.3 is a formula of Gessel [23] that determines the numbers

$$u_k(n) = \#\{w \in S_n \,:\, \mathrm{is}_n(w) \leq k\}.$$

For instance, it is known that $u_2(n) = C_n := \frac{1}{n+1}\binom{2n}{n}$ (a Catalan number), but no simple formula exists for general $u_k(n)$. Define

$$\begin{aligned} U_k(x) &= \sum_{n\geq 0} u_k(n)\frac{x^{2n}}{n!^2},\ k\geq 1 \\ I_i(2x) &= \sum_{n\geq 0} \frac{x^{2n+i}}{n!\,(n+i)!},\ i\geq 0. \end{aligned}$$

The function I_i is the *hyperbolic Bessel function* of the first kind of order i.

Theorem 1.5. *We have*

$$U_k(x) = \det\left(I_{|i-j|}(2x)\right)_{i,j=1}^{k}. \qquad (7)$$

Gessel's theorem (Theorem 1.5) reduces the theorem of Baik, Deift, and Johansson to "just" analysis, viz., the Riemann-Hilbert problem in the theory of integrable systems, followed by the method of steepest descent to analyze the asymptotic behavior of integrable systems. For further information see the survey [13] of Deift.

1.3 Matchings

There are many extensions and generalizations of the theory of increasing and decreasing subsequences. See for instance equation (15) and the last paragraph of Section 2.3 below. We will conclude this section with a different generalization: crossings and nestings of matchings. A (complete) *matching* on the set $[2n]$ may be defined as a partition $M = \{B_1, \ldots, B_n\}$ of $[2n]$ into n two-element blocks B_i. Thus $B_1 \cup B_2 \cup \cdots \cup B_n = [2n]$, $B_i \cap B_j = \emptyset$ if $i \neq j$, and $\#B_i = 2$. (These conditions are not all independent.) It is not immediately apparent what should be the analogue of increasing and decreasing subsequences; in particular, what plays the role of the symmetry $a_1 \cdots a_n \mapsto a_n \cdots a_1$ that interchanges increasing and decreasing subsequences of permutations?

Write $\mathfrak{M}_n$ for the set of matchings on $[2n]$. We represent a matching $M \in \mathfrak{M}_n$ by a diagram of $2n$ vertices $1, 2, \ldots, 2n$ on a horizontal line

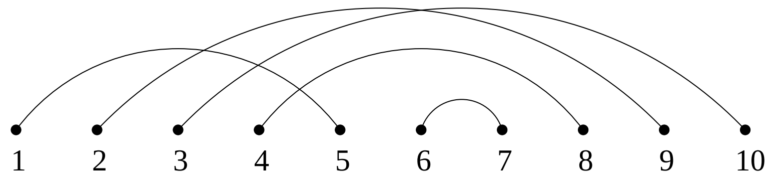

Figure 2: A matching on [10]

in the plane, with an arc between vertices i and j and lying above the vertices if $\{i, j\}$ is a block of M. Sometimes we identify the block $\{i, j\}$ with the arc connecting i and j. Figure 2 shows the diagram corresponding to the matching

$$M = \{\{1, 5\}, \{2, 9\}, \{3, 10\}, \{4, 8\}, \{6, 7\}\}.$$

Let $M \in \mathfrak{M}_n$. A *crossing* of M consists of two arcs $\{i, j\}$ and $\{k, l\}$ with $i < k < j < l$. Similarly a *nesting* of M consists of two arcs $\{i, j\}$ and $\{k, l\}$ with $i < k < l < j$. The maximum number of mutually crossing arcs of M is called the *crossing number* of M, denoted $\mathrm{cr}(M)$. Similarly the *nesting number* $\mathrm{ne}(M)$ is the maximum number of mutually nesting arcs. For the matching M of Figure 2, we have $\mathrm{cr}(M) = 3$ (corresponding to the arcs $\{1, 5\}$, $\{2, 9\}$, and $\{3, 10\}$), while also $\mathrm{ne}(M) = 3$ (corresponding to $\{2, 9\}$, $\{4, 8\}$, and $\{6, 7\}$).

Define

$$f_n(i, j) = \#\{M \in \mathfrak{M}_n \,:\, \mathrm{cr}(M) = i,\ \mathrm{ne}(M) = j\}.$$

It is well-known that

$$\sum_j f_n(0, j) = \sum_i f_n(i, 0) = C_n. \tag{8}$$

In other words, the number of matchings $M \in \mathfrak{M}_n$ with no crossings (or with no nestings) is the Catalan number C_n. Equation (8) was given the following generalization by Chen et al. [10].

Theorem 1.6. *For all i, j, n we have $f_n(i, j) = f_n(j, i)$.*

The key to proving Theorem 1.6 is to find a "matching analogue" of the RSK algorithm $w \overset{\text{rsk}}{\to} (P, Q)$ which Schensted connected with increasing and decreasing subsequences (equation (1)). This algorithm associates a matching M on $[2n]$ with a pair $\Phi(M) = (R, S)$ of *oscillating tableaux* of length n and the same shape. A standard Young tableau of size n may be regarded as a sequence of partitions of length n, starting with the empty partition and adding one square at a time to the diagram of the partition. An oscillating tableau is defined similarly, except that we can either add or delete a square at each step. An example of an oscillating tableau of length 6 and shape 31 (the partition $(3, 1)$) is $(\emptyset, 1, 11, 21, 2, 21, 31)$. The pair (R, S) can also be regarded as a single oscillating tableau of length $2n$ and shape $\emptyset$, by merging R with the reverse of S (identifying the last partition in R with the last in S). Oscillating tableaux were first defined (though not with that name) by Berele [7] in connection with the representation theory of the symplectic group. The matching analogue of equation (1) is the following.

Theorem 1.7. *Let M be a matching on $[2n]$ and let $\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n-1}, \lambda^{2n} = \emptyset)$ be the corresponding oscillating tableau of length $2n$ and shape $\emptyset$. Then $\mathrm{ne}(M)$ is equal to the the largest part of any of the λ^i's, while $\mathrm{cr}(M)$ is equal to the most number of parts of any of the λ^i's.*

Theorem 1.6 follows readily from Theorem 1.7, via the bijections $M \mapsto \Phi(M) \mapsto \Phi(M)' \mapsto \Phi^{-1}(\Phi(M)')$, where $\Phi(M)'$ is obtained from $\Phi(M)$ by conjugating all the partitions that appear in it.

We can now ask for a matching analogue of the formula $E(n) \sim 2\sqrt{n}$ for the expectation of $\mathrm{is}(w)$, or more generally, of the limiting distribution given by Theorem 1.3. This question can be reduced to finding the limiting distribution of $\mathrm{ds}(w)$ where w is a fixed-point free involution in S_{2n}. This problem had earlier been solved by Baik and Rains [5][6], yielding the following result.

Theorem 1.8. *We have for random (uniform) $M \in \mathfrak{M}_n$ and all*

$t \in \mathbb{R}$ that

$$\lim_{n\to\infty} \mathrm{Prob}\left(\frac{\mathrm{ne}_n(M)-\sqrt{2n}}{(2n)^{1/6}} \leq \frac{t}{2}\right) = F(t)^{1/2}\exp\left(\frac{1}{2}\int_t^\infty u(s)ds\right),$$

where $F(t)$ is the Tracy-Widom distribution and $u(s)$ the Painlevé II function. The same result holds with ne_n replaced with cr_n. In particular, the expectation $M(n)$ of $\mathrm{ne}_n(M)$ or $\mathrm{cr}_n(M)$ satisfies $M(n) \sim \sqrt{2n}$.

2 Alternating permutations

2.1 The basic generating function

A permutation $w = a_1a_2\cdots a_n \in S_n$ is called *alternating* if $a_1 > a_2 < a_3 > a_4 < \cdots$. In other words, $a_i < a_{i+1}$ for i even, and $a_i > a_{i+1}$ for i odd. Similarly w is *reverse alternating* if $a_1 < a_2 > a_3 < a_4 > \cdots$. Let E_n denote the number of alternating permutations in S_n. (Set $E_0 = 1$.) For instance, $E_4 = 5$, corresponding to the permutations 2143, 3142, 3241, 4132, and 4231. The number E_n is called an *Euler number* because Euler considered the numbers E_{2n+1}, though not with the combinatorial definition just given. For a more extensive survey of alternating permutations and Euler numbers, see [53].

The involution

$$a_1a_2\cdots a_n \mapsto n+1-a_1, n+1-a_2, \cdots, n+1-a_n \tag{9}$$

on S_n shows that E_n is also the number of *reverse* alternating permutations in S_n.

The fundamental enumerative property of alternating permutations is due to Desiré André [1] in 1879. We have

$$\begin{aligned} \sum_{n\geq 0} E_n\frac{x^n}{n!} &= \sec x + \tan x \qquad (10)\\ &= 1 + x + \frac{x^2}{2!} + 2\frac{x^3}{3!} + 5\frac{x^4}{4!} + 16\frac{x^5}{5!} + 61\frac{x^6}{6!}\\ &\quad +272\frac{x^7}{7!} + 1385\frac{x^8}{8!} + 7935\frac{x^9}{9!} + 50521\frac{x^{10}}{10!} + \cdots. \end{aligned}$$

Note that $\sec x$ is an even function (i.e, $\sec(-x) = \sec x$), while $\tan x$ is odd ($\tan(-x) = -\tan x$). It follows from equation (10) that

$$\sum_{n\geq 0} E_{2n}\frac{x^{2n}}{(2n)!} = \sec x \tag{11}$$

$$\sum_{n\geq 0} E_{2n+1}\frac{x^{2n+1}}{(2n+1)!} = \tan x. \tag{12}$$

For this reason E_{2n} is sometimes called a *secant number* and E_{2n+1} a *tangent number*. Note that equations (11) and (12) can be used to *define* $\sec x$ and $\tan x$, and hence also $\cos x = 1/\sec x$ and $\sin x = \tan x/\sec x$. The subject of "combinatorial trigonometry" seeks to develop as much of trigonometry as possible based on these combinatorial definitions. As a first step, the reader is invited to show that $\sec^2 x = 1 + \tan^2 x$ [50, Exer. 5.7].

2.2 Other occurrences of Euler numbers

There are numerous situations not directly related to alternating permutations in which the Euler numbers appear. A selection of these occurrences of Euler numbers is given here.

- *Complete increasing binary trees.* A *(plane) binary tree* on the vertex set $[n]$ is defined recursively by having a root vertex v and a left and right subtree of v which are themselves binary trees or are empty. A binary tree is *complete* if every vertex either has two children or is an endpoint. A binary tree on the vertex set $[n]$ is *increasing* if every path from the root is increasing. Figure 3 shows the two complete binary trees with five vertices. Each one has eight increasing labelings, so there are 16 complete increasing binary trees on [5].

  Theorem 2.1. *The number of complete increasing binary trees on* $[2m+1]$ *is the Euler number* E_{2m+1}. *(There is a similar but more complicated statement for the vertex set* $[2m]$ *which we do not give here.)*

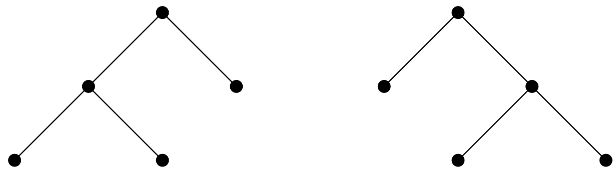

Figure 3: The two complete binary trees with five vertices

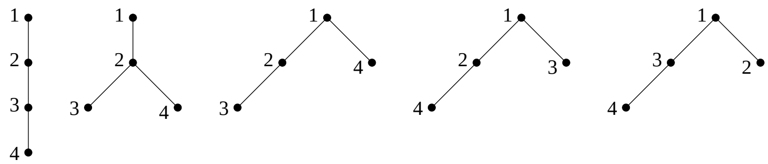

Figure 4: The five increasing (1,2)-trees with four vertices

- *Flip equivalence.* The Euler numbers are related to increasing binary trees in another way. The *flip* of a binary tree at a vertex v is the binary tree obtained by interchanging the left and right subtrees of v. Define two increasing binary trees T and T' on the vertex set $[n]$ to be *equivalent* if T' can be obtained from T by a sequence of flips. Clearly this definition of equivalence is an equivalence relation; the equivalence classes are in an obvious bijection with increasing *(1,2)-trees* on the vertex set $[n]$, that is, increasing (rooted) trees so that every non-endpoint vertex has one or two children. (These are not plane trees, i.e., the order in which we write the children of a vertex is irrelevant.) Figure 4 shows the five increasing (1,2)-trees on four vertices, so $f(4) = 5$.

  Theorem 2.2. *We have $f(n) = E_n$ (an Euler number).*

- *Simsun permutations.*

  Define a *simsun permutation* to be a permutation $w = a_1 \cdots a_n \in S_n$ such that for all $1 \leq k \leq n$, the subword of w consisting of $1, 2, \ldots, k$ (in the order they appear in w) does not have three consecutive decreasing elements. For instance, $w = 425163$ is not simsun since if we remove 5 and 6 from w we obtain

4213, which contains the three consecutive decreasing elements 421. Simsun permutations were named after Rodica Simion and Sheila Sundaram and were first described in print by Sundaram [56, §3]. They are a variant of a class of permutations due to Foata and Schützenberger [17] known as *André permutations.* For further information on simsun permutations see Chow [12]. We have chosen here to deal only with simsun permutations because their definition is a little simpler than that of André permutations. Simion and Sundaram prove in their paper the following basic result on simsum permutations.

Theorem 2.3. *The number of simsun permutations in S_n is the Euler number E_{n+1}.*

Another proof of Theorem 2.3 was given by Maria Monks (private communication, 2008). She gives a bijection between simsun permutations in S_n and the increasing $(1,2)$-trees on the vertex set $[n]$ discussed above. Simsun permutations have an interesting connection with the *cd*-index of S_n, discussed below.

- *Orbits of chains of partitions.* A *partition* π of the set $[n]$ is a collection $\{B_1, \dots, B_k\}$ of nonempty subsets of $[n]$ (called the *blocks* of π) such that $\bigcup B_i = [n]$ and $B_i \cap B_j = \emptyset$ if $i \neq j$. Let Π_n denote the set of all partitions of $[n]$. If $\pi, \sigma \in \Pi_n$, then we say that π is a *refinement* of σ, denoted $\pi \leq \sigma$, if every block of π is contained in a block of σ. The relation $\leq$ is a partial order, so Π_n becomes a partially ordered set (poset). Note that the symmetric group S_n acts on Π_n in an obvious way, viz., if $B = \{a_1, \dots, a_j\}$ is a block of π and $w \in S_n$, then $w \cdot B := \{w(a_1), \dots, w(a_j)\}$ is a block of $w \cdot \pi$.

  Let $\mathcal{M}(\Pi_n)$ denote the set of all maximal chains of Π_n, i.e., all chains
  $$\pi_0 < \pi_1 < \cdots < \pi_{n-1},$$
  so that for all $0 \leq i \leq n-2$, π_{i+1} is obtained from π_i by merging two blocks of π_i. Thus π_i has exactly $n-i$ blocks. In particular, π_0 is the partition into n singleton blocks, and

π_{n-1} is the partition into one block $[n]$. The action of S_n on Π_n induces an action on $\mathcal{M}(\Pi_n)$. For instance, when $n = 5$ a class of orbit representatives is given by the five chains below. We write e.g. $12-34-5$ for the partition $\{\{1,2\},\{3,4\},\{5\}\}$, and we omit the first and last element of each chain.

$$\begin{array}{ccccc}
12-3-4-5 & < & 123-4-5 & < & 1234-5 \\
12-3-4-5 & < & 123-4-5 & < & 123-45 \\
12-3-4-5 & < & 12-34-5 & < & 125-34 \\
12-3-4-5 & < & 12-34-5 & < & 12-345 \\
12-3-4-5 & < & 12-34-5 & < & 1234-5
\end{array}$$

Theorem 2.4. *The number of orbits of the action of S_n on $\mathcal{M}(\Pi_n)$ is the Euler number E_{n-1}.*

Theorem 2.4 was first proved by Stanley [44, Thm. 7.7] by showing that the number of orbits satisfies the same recurrence as E_{n-1}. By elementary representation theory, the number of orbits of S_n acting on $\mathcal{M}(\Pi_n)$ is the multiplicity of the trivial representation in this action. This observation suggests the problem of decomposing S_n-actions on various sets of chains in Π_n into irreducible representations. The first results in this direction appear in [44, §7]. Many further results were obtained by Sundaram [56]. Theorem 2.4 can also be proved by giving a simple bijection between the orbits and increasing $(1,2)$-trees on $[n-1]$, as defined earlier in this section

- *Polytope volumes.* Euler numbers occur as (normalized) volumes of certain convex polytopes. The first polyope, which we call the *zigzag order polytope* $\mathcal{P}_n$, consists of all points $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ satisfying

$$x_i \geq 0, \quad 1 \leq i \leq n$$
$$x_1 \geq x_2 \leq x_3 \geq \cdots x_n.$$

To compute its volume, for each alternating permutation $w = a_1 a_2 \cdots a_n \in S_n$, let $w^{-1} = b_1 b_2 \cdots b_n$. Let

$$\mathcal{P}_w = \{(x_1, \ldots, x_n) \in \mathcal{P}_n \,:\, x_{b_1} \leq x_{b_2} \leq \cdots \leq x_{b_n}\}.$$

It is easy to see that each $\mathcal{P}_w$ is a simplex with volume $1/n!$. One can check using the theory of P-partitions [48, §4.5] that the $\mathcal{P}_w$'s have disjoint interiors and union $\mathcal{P}_n$. Hence $\mathrm{vol}(\mathcal{P}_n) = E_n/n!$.

The second polytope is called the *zigzag chain polytope* $\mathcal{C}_n$. It consists of all points $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ satisfying

$$x_i \geq 0, \quad 1 \leq i \leq n$$

$$x_i + x_{i+1} \leq 1, \quad 1 \leq i \leq n-1.$$

The polytope $\mathcal{C}_n$ first arose in [14] and [43]. A "naive" method for computing the volume is the following. For $0 \leq t \leq 1$ let

$$f_n(t) = \int_{x_1=0}^{t} \int_{x_2=0}^{1-x_1} \int_{x_3=0}^{1-x_2} \cdots \int_{x_n=0}^{1-x_{n-1}} dx_1\, dx_2 \cdots dx_n. \tag{13}$$

Clearly $f(1) = \mathrm{vol}(\mathcal{C}_n)$. Differentiating equation (13) yields $f_n'(t) = f_{n-1}(1-t)$. There are various ways to solve this recurrence for $f_n(t)$ (with the initial conditions $f_0(t) = 1$ and $f_n(0) = 0$ for $n > 0$), yielding

$$\sum_{n\geq 0} f_n(t) x^n = (\sec x)(\cos(t-1)x + \sin tx).$$

Putting $t = 1$ gives

$$\sum_{n\geq 0} f_n(1) x^n = \sec x + \tan x,$$

so we conclude that $\mathrm{vol}(\mathcal{C}_n) = E_n/n!$.

A more sophisticated proof uses the earlier mentioned fact that $\mathrm{vol}(\mathcal{P}_n) = E_n/n!$. Given $(x_1, \ldots, x_n) \in \mathbb{R}_n$, define $\varphi(x_1, \ldots, x_n) = (y_1, \ldots, y_n) \in \mathbb{R}^n$ by

$$y_i = \begin{cases} 1 - x_i, & \text{if } i \text{ is odd} \\ x_i, & \text{if } i \text{ is even.} \end{cases}$$

It is easily checked that φ is an affine transformation taking $\mathcal{P}_n$ onto $\mathcal{C}_n$. Since the homogeneous part of φ has determinant ± 1, it follows that φ is a volume-preserving bijection from $\mathcal{P}_n$ onto $\mathcal{C}_n$, so $\mathrm{vol}(\mathcal{C}_n) = \mathrm{vol}(\mathcal{P}_n) = E_n/n!$. This argument appeared in Stanley [46, Thm. 2.3 and Exam. 4.3].

The polytope $\mathcal{C}_n$ has an interesting connection to tridiagonal matrices. An $n \times n$ matrix $M = (m_{ij})$ is *tridiagonal* if $m_{ij} = 0$ whenever $|i - j| \geq 2$. Let $\mathcal{T}_n$ be the set of all $n \times n$ tridiagonal doubly stochastic matrices M, i.e., $n \times n$ (real) tridiagonal matrices with nonnegative entries and with row and column sums equal to 1. Thus $\mathcal{T}_n$ is a convex polytope in a real vector space of dimension n^2 (or of dimension $3n - 2$ if we discard coordinates that are always 0). It is easy to see that if we choose the $n - 1$ entries $m_{12}, m_{23}, \ldots, m_{n-1,n}$ arbitrarily, then they determine a unique tridiagonal matrix M with row and column sums 1. Moreover, in order for M to be doubly stochastic it is necessary and sufficient that $m_{i,i+1} \geq 0$ and

$$m_{12} + m_{23} \leq 1, \quad m_{23} + m_{34} \leq 1, \ldots, \; m_{n-2,n-1} + m_{n-1,n} \leq 1.$$

It follows that $\mathcal{T}_n$ is linearly equivalent to $\mathcal{C}_{n-1}$ (in fact, $\mathcal{T}_n$ projects bijectively to $\mathcal{C}_{n-1}$). Moreover, the relative volume (volume normalized so that a fundmental parallelopiped in the intersection of the integer lattice with the affine span of $\mathcal{T}_n$ has volume 1) of $\mathcal{T}_n$ is $E_{n-1}/(n-1)!$.

- *Singularities.* V. I. Arnold [2] (see also [3] for a followup) has shown that the Euler number E_{n+1} is equal to the number of components of the space of real polynomials $f(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x$ whose critical points (zeros of $f'(x)$) are all real and whose $n-1$ critical values (the numbers $f(c)$ where c is a critical point) are all different. For instance, when $n = 3$ the polynomials $x^3 + ax^2 + bx$ form a real plane. The critical points are real if and only if $b \leq a^2/3$. Two critical values coincide in the region $b < a^2/3$ if and only if $b = a^2/4$ or $b = 0$. These two curves cut the region $b < a^2/3$ into $E_4 = 5$ components. Arnold interprets this result in terms of morsifications of the function x^{n+1}; see his paper for further details. Arnold goes on to deduce a number of interesting properties of Euler numbers.

He also extends the theory to morsifications of the functions $x^n + y^2$ and $xy + y^n$, thereby producing B_n and C_n analogues of Euler numbers (which correspond to the root system A_n).

2.3 Longest alternating subsequences

In Section 1.2 we discussed properties of the length $\mathrm{is}(w)$ of the longest increasing subsequence of a permutation $a_1 \cdots a_n \in S_n$. We can ask whether similar results hold for *alternating* subsequences of $w \in S_n$. In particular, for $w \in S_n$ define $\mathrm{as}(w)$ (or $\mathrm{as}_n(w)$ to make it explicit that $w \in S_n$) to be the length of the longest alternating subsequence of w. For instance, if $w = 56218347$ then $\mathrm{as}(w) = 5$, one alternating subsequence of longest length being 52834. Our source for most of the material in this section is the paper [52].

It turns out that the behavior of $\mathrm{as}(w)$ is much simpler than that of $\mathrm{is}(w)$. The primary reason for this is the following lemma, whose straightforward proof we omit.

Lemma 2.5. *Let $w \in S_n$. Then there is an alternating subsequence of w of maximum length that contains n.*

Lemma 2.5 allows us to obtain explicit formulas by induction. More specifically, define

$$\begin{aligned} a_k(n) &= \#\{w \in S_n : \mathrm{as}(w) = k\} \\ b_k(n) &= a_1(n) + a_2(n) + \cdots + a_k(n) \\ &= \#\{w \in S_n : \mathrm{as}(w) \le k\}. \end{aligned}$$

For instance, $b_1(n) = 1$, corresponding to the permutation $1, 2, \ldots, n$, while $b_2(n) = 2^{n-1}$, corresponding to the permutations $u_1, u_2 \ldots, u_i$, n, $v_1, v_2, \ldots, v_{n-i-1}$, where $u_1 < u_2 < \cdots < u_i$ and $v_1 > v_2 > \cdots > v_{n-i-1}$. Using Lemma 2.5, we can obtain the following recurrence for the numbers $a_k(n)$, together with the initial condition $a_0(0) = 1$:

$$a_k(n+1) = \sum_{j=0}^{n} \binom{n}{j} \sum_{\substack{2r+s=k-1 \\ r,s \ge 0}} (a_{2r}(j) + a_{2r+1}(j)) a_s(n-j). \qquad (14)$$

This recurrence can be used to obtain the following generating function for the numbers $a_k(n)$ and $b_k(n)$. No analogous formula is known for increasing subsequences.

Theorem 2.6. *Let*

$$\begin{aligned} A(x,t) &= \sum_{k,n\geq 0} a_k(n) t^k \frac{x^n}{n!} \\ B(x,t) &= \sum_{k,n\geq 0} b_k(n) t^k \frac{x^n}{n!}. \end{aligned}$$

Set $\rho = \sqrt{1-t^2}$. *Then*

$$\begin{aligned} B(x,t) &= \frac{2/\rho}{1-\dfrac{1-\rho}{t}e^{\rho x}} - \frac{1}{\rho} \\ A(x,t) &= (1-t)B(x,t). \end{aligned}$$

Many consequences can be derived from Theorem 2.6. In particular, there are explicit formulas for $a_k(n)$ and $b_k(n)$. For instance, for $k \leq 6$ we have

$$\begin{aligned} b_2(n) &= 2^{n-1} \\ b_3(n) &= \frac{1}{4}(3^n - 2n + 3) \\ b_4(n) &= \frac{1}{8}(4^n - 2(n-2)2^n) \\ b_5(n) &= \frac{1}{16}(5^n - (2n-5)3^n + 2(n^2 - 5n + 5)) \\ b_6(n) &= \frac{1}{32}(6^n - 2(n-3)4^n + (2n^2 - 12n + 15)2^n). \end{aligned}$$

We can also obtain explicit formulas for the moments of $\mathrm{as}(w)$. For instance, to obtain the mean (expectation)

$$D(n) = \frac{1}{n!} \sum_{w \in S_n} \mathrm{as}(w),$$

we compute

$$\begin{aligned}\sum_{n\geq 1} D(n)x^n &= \frac{\partial}{\partial t}A(x,1) \\ &= \frac{6x-3x^2+x^3}{6(1-x)^2} \\ &= x+\sum_{n\geq 2}\frac{4n+1}{6}x^n.\end{aligned}$$

Thus

$$D(n)=\frac{4n+1}{6},\quad n\geq 2,$$

a remarkably simple formula. Note that (not surprisingly) $D(n)$ is much larger than the expectation $E(n)$ of $\mathrm{is}(w)$, viz., $E(n)\sim 2\sqrt{n}$. Similarly the variance

$$V(n)=\frac{1}{n!}\sum_{w\in S_n}\left(\mathrm{as}(w)-D(n)\right)^2$$

is given by

$$V(n)=\frac{8}{45}n-\frac{13}{180},\quad n\geq 4.$$

Now that we have computed the mean and variance of $\mathrm{as}(w)$, we can ask whether there is an "alternating analogue" of the Baik-Deift-Johansson formula (5). In other words, can we determine the scaled limiting distribution

$$K(t)=\lim_{n\to\infty}\mathrm{Prob}\left(\frac{\mathrm{as}_n(w)-2n/3}{\sqrt{n}}\leq t\right),$$

for $t\in\mathbb{R}$? It turns out that the limiting distribution is Gaussian. It is a consequence of results of Pemantle and Wilson [35] and Wilf [60], and was proved directly by Widom [59]. More precisely, we have

$$K(t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{t\sqrt{45}/4}e^{-s^2}ds. \tag{15}$$

2.4 Umbral enumeration of subsets of alternating permutations

We now consider the enumeration of alternating permutations having additional properties, such as having alternating inverses or having no fixed points. The main tool is a certain character χ^{τ_n} of the symmetric group S_n, first considered by H. O. Foulkes [20][21], whose dimension in E_n. We will not define this character here but will simply state some results that follow from known connections between symmetric functions and permutation enumeration. We will use *umbral notation* [24][39] for Euler numbers. In other words, any polynomial in E is to be expanded in terms of powers of E, and then E^k is replaced by E_k. The replacement of E^k by E_k is always the *last* step in the evaluation of an umbral expression. For instance,

$$(E^2-1)^2 = E^4 - 2E^2 + 1 = E_4 - 2E_2 + 1 = 5 - 2\cdot 1 + 1 = 4.$$

Similarly,

$$\begin{aligned}
(1+t)^E &= 1 + Et + \binom{E}{2}t^2 + \binom{E}{3}t^3 + \cdots \\
&= 1 + Et + \frac{1}{2}(E^2 - E)t^2 + \frac{1}{6}(E^3 - 3E^2 + 2E)t^3 + \cdots \\
&= 1 + Et + \frac{1}{2}(E_2 - E_1)t^2 + \frac{1}{6}(E_3 - 3E_2 + 2E_1)t^3 + \cdots \\
&= 1 + 1\cdot t + \frac{1}{2}(1-1)t^2 + \frac{1}{6}(2 - 3\cdot 1 + 2\cdot 1)t^3 + \cdots \\
&= 1 + t + \frac{1}{6}t^3 + \cdots .
\end{aligned}$$

A typical result is the following. Let $g(n)$ (respectively, $g^*(n)$) denote the number of fixed-point-free alternating involutions (respectively, fixed-point-free reverse alternating involutions) in S_{2n}. Set

$$\begin{aligned} G(t) &= \sum_{n\geq 0} g(n)x^n \\ &= 1+t+t^2+2t^3+5t^4+17t^5+72t^6+367t^7+\cdots \\ G^*(t) &= \sum_{n\geq 0} g^*(n)x^n \\ &= 1+t^2+t^3+4t^4+13t^5+59t^6+308t^7+\cdots . \end{aligned}$$

Theorem 2.7. *We have the umbral generating functions*

$$\begin{aligned} G(t) &= \left(\frac{1+t}{1-t}\right)^{(E^2+1)/4} \\ G^*(t) &= \frac{G(t)}{1+t}. \end{aligned}$$

Another class of alternating permutations with a nice enumeration are cycles of length n. For this particular case the resulting formulas can be "deumbralized" to give more explicit formulas. Let $b(n)$ (respectively, $b^*(n)$) denote the number of alternating (respectively, reverse alternating) n-cycles in S_n.

Theorem 2.8. (a) *If n is odd then*

$$b(n) = b^*(n) = \frac{1}{n}\sum_{d|n}\mu(d)(-1)^{(d-1)/2}E_{n/d}.$$

(b) *If $n = 2^k m$ where $k \geq 1$, m is odd, and $m \geq 3$, then*

$$b(n) = b^*(n) = \frac{1}{n}\sum_{d|m}\mu(d)E_{n/d}.$$

(c) *If $n = 2^k$ and $k \geq 2$ then*

$$b(n) = b^*(n) = \frac{1}{n}(E_n - 1). \tag{16}$$

(d) *Finally, $b(2) = 1$, $b^*(2) = 0$.*

Note that curious fact that $b(n) = b^*(n)$ except for $n = 2$. Can Theorem 2.8 be proved combinatorially, especially in the cases when n in a prime power (when the sums have only two terms)?

There are umbral formulas for enumerating alternating and reverse alternating permutations according to the length of their longest increasing subsequence. These results are alternating analogues of equation (7). For any integer i we use the notation $D^iF(x)$ for the nth formal derivative of the power series $F(x) = \sum_{n\geq 0} a_n x^n$, where in particular

$$D^{-1}F(x) = \sum_{n\geq 0} a_n \frac{x^{n+1}}{n+1}$$

and $D^{-i-1} = D^{-1}D^{-i}$ for all $i \geq 1$.

Theorem 2.9. *Let $\alpha_k(n)$ (respectively, $\alpha'_k(n)$) denote the number of alternating (respectively, reverse alternating) permutations $w \in S_n$ whose longest increasing subsequence has length at most n. Let*

$$\begin{aligned} \exp\left(E\tan^{-1}(x)\right) &= \sum_{n\geq 0} c_n(E)x^n \\ &= 1 + Ex + \frac{1}{2}E^2x^2 + \frac{1}{6}(E^3 - 2E)x^3 + \cdots . \end{aligned}$$

Define

$$\begin{aligned} A_1(x) &= \sum_{n\geq 0} c_n(E)\frac{x^n}{n!} \\ A_2(x) &= \sqrt{1+x^2}A_1(x) \\ A_3(x) &= \frac{A_1(x)}{\sqrt{1+x^2}}, \end{aligned}$$

and for $1 \leq r \leq 3$ and $k \geq 1$ define

$$B_{r,k}(x) = \det(D^{j-i}A_r(x))_{i,j=1}^k.$$

We then have:

- *If n is odd, then the coefficient of $x^n/n!$ in the umbral evaluation of $B_{1,k}(x)$ is $\alpha_k(n) = \alpha'_k(n)$.*

- *If n is even, then the coefficient of $x^n/n!$ in the umbral evaluation of $B_{2,k}(x)$ is $\alpha_k(n)$.*

- *If n is even, then the coefficient of $x^n/n!$ in the umbral evaluation of $B_{3,k}(x)$ is $\alpha'_k(n)$.*

Note that it follows from Theorem 2.9 that $\alpha_k(2m+1) = \alpha'_k(2m+1)$. This result is also an immediate consequence of the involution (9).

It is known (see [54]) that $\alpha_2(n) = C_{\lceil (n+1)/2 \rceil}$ for $n \geq 3$ and $\alpha'_2(n) = C_{\lceil n/2 \rceil}$ for $n \geq 1$, where $C_i = \frac{1}{i+1}\binom{2i}{i}$ (a Catalan number). Can these formulas be deduced directly from the case $k = 2$ of Theorem 2.9? Similarly, J. Lewis [29] has shown that

$$\alpha'_3(n) = \begin{cases} f^{(m,m,m)}, & n = 2m \\ f^{(m-1,m,m+1)}, & n = 2m+1, \end{cases}$$

where (as in Subsection 1.1) f^λ denotes the number of SYT of shape λ. No such results are known for $\alpha'_4(n)$ or $\alpha_3(n)$.

Is there an asymptotic formula for the expected length $L(n)$ of the longest increasing subsequence of an alternating permutation $w \in S_n$ as $n \to \infty$, analogous to the result (2) of Logan-Shepp and Vershik-Kerov for arbitrary permutations? It is easy to see that $L(n) \geq \sqrt{n}$. Is it true that $\lim_{n\to\infty} \frac{\log L(n)}{\log n} = \frac{1}{2}$? Is there in fact a limiting distribution for the (suitably scaled) length of the longest increasing subsequence of an alternating permutation $w \in S_n$ as $n \to \infty$, analogous to the Baik-Deift-Johansson theorem (5) for arbitrary permutations? Two closely related problems are the following. Let a^λ denote the number of SYT of shape $\lambda \vdash n$ and descent set $\{1,3,5,\dots\} \cap [n-1]$, as defined in [50, p. 361]. (These are just the SYT Q that arise from alternating permutations $w \in S_n$ by applying the RSK algorithm $w \mapsto (P,Q)$.) What is the (suitably scaled) limiting shape of λ as $n \to \infty$ that maximizes a^λ and similarly that maximizes $a^\lambda f^\lambda$? (For the shape that maximizes f^λ, see Figure 1. Could this shape also maximize a^λ?)

2.5 The cd-index of the symmetric group

Let $w = a_1 \cdots a_n \in S_n$. The *descent set* $D(w)$ of w is defined by

$$D(w) = \{i : a_i > a_{i+1}\} \subseteq [n-1].$$

A permutation w is thus alternating if $D(w) = \{1, 3, 5, \ldots\} \cap [n-1]$ and reverse alternating if $D(w) = \{2, 4, 6, \ldots\} \cap [n-1]$. For $S \subseteq [n-1]$ let

$$\beta_n(S) = \#\{w \in S_n : D(w) = S\}.$$

The numbers $\beta_n(S)$ are fundamental invariants of w that appear in a variety of combinatorial, algebraic, and geometric contexts. Here we explain how alternating permutations are related to the more general subject of permutations with a fixed descent set.

We first define for fixed n a noncommutative generating function for the numbers $\beta_n(S)$. Given a set $S \subseteq [n-1]$, define its *characteristic monomial* (or *variation*) to be the noncommutative monomial

$$u_S = e_1 e_2 \cdots e_{n-1}, \tag{17}$$

where

$$e_i = \begin{cases} a, & \text{if } i \notin S \\ b, & \text{if } i \in S. \end{cases}$$

For instance, $D(37485216) = \{2, 4, 5, 6\}$, so $u_{D(37485216)} = ababbba$. Define

$$\begin{aligned} \Psi_n = \Psi_n(a, b) &= \sum_{w \in S_n} u_{D(w)} \\ &= \sum_{S \subseteq [n-1]} \beta_n(S) u_S. \end{aligned} \tag{18}$$

Thus Ψ_n is a noncommutative generating function for the numbers $\beta_n(S)$. For instance,

$$\begin{aligned} \Psi_1 &= 1 \\ \Psi_2 &= a + b \\ \Psi_3 &= a^2 + 2ab + 2ba + b^2 \\ \Psi_4 &= a^3 + 3a^2b + 5aba + 3ba^2 + 3ab^2 + 5bab + 3b^2a + b^3. \end{aligned}$$

The polynomial Ψ_n is called the *ab-index* of the symmetric group S_n.

The main result of this section is the following.

Theorem 2.10. *There exists a polynomial $\Phi_n(c, d)$ in the noncommuting variables c and d such that*

$$\Psi_n(a, b) = \Phi_n(a + b, ab + ba).$$

The polynomial $\Phi_n(c, d)$ is called the *cd-index* of S_n. For instance, we have

$$\Psi_3(a, b) = a^2 + 2ab + 2ba + b^2 = (a + b)^2 + (ab + ba),$$

so $\Phi_3(c, d) = c^2+d$. Some values of $\Phi_n(c, d)$ for small n are as follows:

$$\begin{aligned}
\Phi_1 &= 1 \\
\Phi_2 &= c \\
\Phi_3 &= c^2 + d \\
\Phi_4 &= c^3 + 2cd + 2dc \\
\Phi_5 &= c^4 + 3c^2d + 5cdc + 3dc^2 + 4d^2 \\
\Phi_6 &= c^5 + 4c^3d + 9c^2dc + 9cdc^2 + 4dc^3 + 12cd^2 + 10dcd + 12d^2c.
\end{aligned}$$

If we define $\deg(c) = 1$ and $\deg(d) = 2$, then the number of *cd*-monomials of degree $n-1$ is the Fibonacci number F_n. It is not hard to see that all these monomials actually appear in $\Phi_n(c, d)$. Thus $\Phi_n(c, d)$ has F_n terms, compared with 2^{n-1} terms for $\Psi_n(a, b)$.

There are several known proofs of Theorem 2.10. Perhaps the most natural approach is to define an equivalence relation on S_n such that for each equivalence class C, we have that $\sum_{w\in C} u_{D(w)}$ is a monomial in $c = a+b$ and $d = ab+ba$. Such a proof was given by G. Hetyei and E. Reiner [26]. See [49, §1.6] for an exposition of this proof. The proof of Hetyei and Reiner leads to a somewhat complicated combinatorial interpretation of the coefficients of $\Phi_n(c, d)$ which makes it apparent that they are nonnegative. It is reasonable to ask whether there is a more "direct" description of the coefficients. Such a description was first given by D. Foata and M.-P. Schützenberger [17] in terms of the *André permutations* mentioned in Section 2.2. We state here the analogous result for simsun permutations (as defined in Section 2.2), due to R. Simion and S. Sundaram.

Theorem 2.11. *Let μ be a monomial of degree $n-1$ in the noncommuting variables c, d, where $\deg(c) = 1$ and $\deg(d) = 2$. Replace each c in μ with 0, each d with 10, and remove the final 0. We get the characteristic vector of a set $S_\mu \subseteq [n-2]$. Then the coefficient of μ in $\Phi_n(c,d)$ is equal to the number of simsun permutations in S_{n-1} with descent set S_μ.*

For example, if $\mu = cd^2c^2d$ then we get the characteristic vector 01010001 of the set $S_\mu = \{2,4,8\}$. Hence the coefficient of cd^2c^2d in $\Phi_{10}(c,d)$ is equal to the number of simsun permutations in S_9 with descent set $\{2,4,8\}$.

Note that every cd-monomial, when expanded in terms of ab-monomials, is a sum of distinct monomials including $bababa\cdots$ and $ababab\cdots$. These monomials correspond to descent sets of alternating and reverse alternating permutations, respectively. Hence $\Phi_n(1,1) = E_n$. This fact also follows immediately from Theorems 2.3 and 2.11. Since the coefficients of $\Phi_n(c,d)$ are nonnegative and the expansion of every cd-monomial contains $bababa\cdots$ and $ababab\cdots$, it follows that

$$\beta_n(S) \le \beta_n(1,3,5,\dots) = \beta_n(2,4,6,\cdots) = E_n. \tag{19}$$

Moreover, with just a little more work it is easy to see that equality holds in equation (19) if and only if $S = \{1,3,5,\dots\}$ or $S = \{2,4,6,\dots\}$. This result is originally due to Niven [34] and de Bruijn [9]; the proof based on the cd-index appears in [47, Thm. 2.3(b)]. All three references actually prove a more general result about when $\beta_n(S) \le \beta_n(T)$.

3 Reduced decompositions

Let $w \in S_n$, and let

$$p = \ell(w) = \#\{(i,j) \,:\, i<j,\ w(i) > w(j)\},$$

the number of *inversions* of w, or the *length* of w when we regard S_n as a Coxeter group. Write $s_i = (i, i+1) \in S_n$, $1 \le i - 1$.

It is easy to see that w cannot be written as a product of fewer than p adjacent transpositions s_i. A *reduced decomposition* of w is a sequence $(r_1, r_2, \dots, r_p)$ of integers $1 \le r_i \le n-1$ such that $w = s_{r_1} s_{r_2} \cdots s_{r_p}$. Write $R(w)$ for the set of reduced decompositions of w and $r(w) = \#R(w)$, the number of reduced decompositions of w. A theorem of Tits [57][8, Thm. 3.3.1(ii)] asserts that any two reduced decompositions can be obtained from each other by applying the relations $s_i s_j = s_j s_i$ if $|i-j| \ge 2$, and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$. For instance, one of the reduced decompositions of $w = 24351$ is (using abbreviated notation where we omit commas and parentheses) 12324. It follows that $R(w) = \{12324, 12342, 13234, 31234\}$ and $r(w) = 4$.

Given a permutation $w \in S_n$, define a power series G_w in the infinitely many variables $x_1, x_2, \dots$ by

$$G_w = \sum_{(r_1,\dots,r_p)\in R(w)} \sum_{\substack{1\le i_1 \le \cdots \le i_p \\ i_j < i_{j+1} \text{ if } r_j < r_{j+1}}} x_{i_1} \cdots x_{i_p}. \tag{20}$$

Thus G_w is homogeneous of degree $p = \ell(w)$. For instance, let $w = 321$, so $R(w) = \{121, 212\}$. Then

$$G_{321} = \sum_{1\le i<j\le k} x_i x_j x_k + \sum_{1\le i\le j<k} x_i x_j x_k.$$

Note that G_{321} is a *symmetric function* of the x_i's. The homogeneous symmetric functions of degree p have a well-known basis consisting of the *Schur functions* s_λ for $\lambda \vdash p$. In particular, the expansion of G_{321} in terms of Schur functions is simply $G_{321} = s_{21}$ (where 21 is abbreviated notation for the partition $(2,1,0,0,\dots)$). The following key lemma was proved by Stanley [45]. Several other proofs have subsequently been given. Perhaps the most straightforward is a proof of Fomin and Stanley [19] based on the nilCoxeter algebra of S_n.

Lemma 3.1. *For all permutations $w \in S_n$, the power series G_w is a symmetric function.*

By Lemma 3.1 we can define coefficients $\alpha_{w\lambda}$ for $w \in S_n$ and $\lambda \vdash p = \ell(w)$ by

$$G_w = \sum_{\lambda \vdash p} \alpha_{w\lambda} s_\lambda. \tag{21}$$

Because G_w is an integer linear combination of monomials it follows that $\alpha_{w\lambda} \in \mathbb{Z}$, but at this point it is unclear whether $\alpha_{w\lambda} \geq 0$. Note that it follows immediately from the definition of G_w that $r(w)$ is the coefficient of $x_1 x_2 \cdots x_p$ in G_w. On the other hand, the coefficient of $x_1 x_2 \cdots x_p$ in s_λ (where $\lambda \vdash p$) is just f^λ, the number of SYT of shape λ. Hence we obtain the following "formula" for $r(w)$.

Corollary 3.2. *Let $w \in S_n$, $\ell(w) = p$. Then*

$$r(w) = \sum_{\lambda \vdash p} \alpha_{w\lambda} f^\lambda.$$

Of course the usefulness of Corollary 3.2 depends on what we can say about the numbers $\alpha_{w\lambda}$. (Recall from Section 1 that f^λ is given explicitly by the "hook length formula," so we may regard f^λ as known.) The simplest situation is when $G_w = s_\lambda$ for some $\lambda \vdash n$. We can say precisely when this happens. Define a permutation $a_1 a_2 \cdots a_n \in S_n$ to be 2143-*avoiding* or *vexillary* if there do not exist $1 \leq h < i < j < k \leq n$ for which $a_i < a_h < a_k < a_j$. (Numerous equivalent defintions exist.)

Suppose that $w = a_1 \cdots a_n \in S_n$. For $1 \leq i \leq n-1$ define

$$c_i = \#\{j \,:\, i < j \leq n,\ a_j < a_i\},$$

and define $\lambda(w)$ to the partition whose parts are the c_i's (sorted into decreasing order).

Theorem 3.3. *We have $G_w = s_\lambda$ for some λ if and only if w is vexillary. In this case $\lambda = \lambda(w)$, so $r(w) = f^{\lambda(w)}$.*

For instance, let $w = 5361472 \in S_7$. Then w is vexillary, and $(c_1, \ldots, c_6) = (4, 2, 3, 0, 1, 1, 0)$. Hence $\lambda(w) = 43211$, $G_w = s_{43211}$, and $r(w) = f^{43211} = 2310$.

An especially interesting special case of Theorem 3.3 occurs for the permutation $w_0 = n, n-1, \ldots, 1 \in S_n$. Note that w_0 is the unique longest permutation in S_n, and $\ell(w_0) = \binom{n}{2}$. Clearly w_0 is vexillary, and $\lambda(w_0) = (n-1, n-2, \ldots, 1)$. We obtain from the hook length formula the following corollary.

Corollary 3.4. *For $w_0 \in S_n$ we have*

$$r(w_0) = f^{(n-1,n-2,\ldots,1)} = \frac{\binom{n}{2}!}{1^{n-1}3^{n-2}5^{n-3}\cdots(2n-1)}.$$

Corollary 3.4 suggests to an enumerative combinatorialist the problem of finding a "simple" bijective proof that $r(w_0) = f^{(n-1,n-2,\ldots,1)}$. A remarkable such proof was given by Edelman and Greene [15]. It is also a consequence of the theory developed by Edelman and Greene that the coefficients $\alpha_{w\lambda}$ of equation (21) are nonnegative. An elegant combinatorial interpretation of $\alpha_{w\lambda}$ was later given by Fomin and Greene [18], which we now describe.

A *semistandard Young tableau* (SSYT) T of shape λ is a Young diagram of shape λ with a positive integer inserted into each square so that the rows are weakly increasing and columns strictly increasing. The *row reading word* of T is the sequence obtained by reading the entries of T beginning with the first row from right-to-left, then the second row right-to-left, etc.

Theorem 3.5. *Let $w \in S_n$, $\ell(w) = p$, and $\lambda \vdash p$. The coefficient $\alpha_{w\lambda}$ is equal to the number of SSYT of shape λ whose row reading word is a reduced decomposition of w.*

As an example of Theorem 3.5, let $w = 4152736$. The SSYT whose row reading words are a reduced decomposition of w are

$$\begin{array}{lll} 1\,2\,3 & 1\,2\,3 & 1\,2\,3\,6 \\ 3\,4 & 3\,4\,6 & 3\,4 \\ 5\,6 & 5 & 5, \end{array}$$

with row reading words 3214365, 3216435, and 6321435. Hence

$$r(w) = f^{322} + f^{331} + f^{421} = 21 + 21 + 35 = 77.$$

The symmetric function G_w has a connection with representation theory. Let $\varphi\colon S_p \to \mathrm{GL}(m, \mathbb{C})$ be an ordinary (complex) representation of S_p of dimension m. The irreducible representations φ^λ of S_p are indexed by partitions $\lambda \vdash p$. Let $m_\varphi(\lambda)$ be the multiplicity

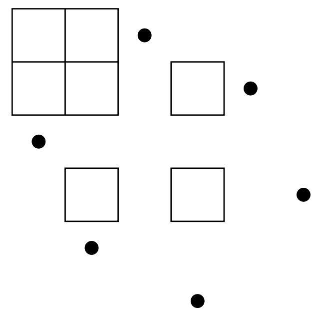

Figure 5: The diagram of the permutation $w = 351624$

of φ^λ in φ, and define the *(Frobenius) characteristic* of φ to be the symmetric function

$$\operatorname{ch}\varphi = \sum_{\lambda \vdash p} m_\varphi(\lambda) s_\lambda.$$

Now let μ be the (Young) diagram of any partition $\mu \vdash p$. There is a standard way, involving column antisymmetrizers of tabloids of shape μ, of constructing a module M_μ, called the *Specht module*, that affords the irreducible representation φ^μ. See e.g. Sagen [40, §2.3]. The Specht module construction can be carried out for *any* finite diagram D, i.e., any finite subset of an infinite grid of squares, giving a module M_D for a representation φ^D of S_p, where $p = \#D$.

For any permutation $w \in S_n$ we can define its diagram D_w as follows. Consider the squares of an infinite quadrant of squares as being coordinatized by $\mathbb{P} \times \mathbb{P}$, where $\mathbb{P} = \{1, 2, \dots\}$. If $1 \le i \le n$ and $w(i) = j$, then remove all squares (h, k) with $h = i$ and $k > j$, or with $k = j$ and $i > h$. What remains will be D_w, which is easily seen to have $p = \ell(w)$ squares. For instance, Figure 5 shows the diagram of $w = 351624$, with a dot in the center of the squares (i, j) for which $w(i) = j$ (using matrix coordinates, i.e., rows increase from top-to-bottom and columns from left-to-right). The following result is due to Kraśkiewicz and Pragacz [28] (first written up in 1986 but not published until 2004); see also Kraśkiewicz [27].

Theorem 3.6. *Let $w \in S_n$, with diagram D_w. Then $\operatorname{ch}\varphi^{D_w} = G_w$, so in particular $\dim \varphi^{D_w} = r(w)$.*

It is still not understood for arbitrary diagrams D how to decompose M_D into irreducible modules or even to give a "nice" description of $\dim M_D$, though there is a generalization of Theorem 3.6 due to Reiner and Shimozono [37]. Some further cases where $\dim M_D$ and $\operatorname{ch}\varphi^D$ can be computed are considered by Liu [30].

The symmetric functions G_w are also connected with flag varieties. The basic result is the following. Let Fl_n denote the set of all complete flags $0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$ of subspaces in $\mathbb{C}^n$ (so $\dim V_i = i$). The set Fl_n has the natural structure of a smooth projective variety; in fact, it is canonically isomorphic to the coset space $\mathrm{GL}(n, \mathbb{C})/B$, where B is the Borel subgroup of all invertible upper triangular matrices. For every $w \in S_n$ there is an affine subvariety Ω_w of Fl_n called a *Schubert variety* (see e.g. [22, Chap. 10]). The closure $\overline{\Omega}_w$ represents a class $[\overline{\Omega}_w]$ in the cohomology $H^{2\left(\binom{n}{2}-p\right)}(\mathrm{Fl}_n; \mathbb{C})$, and these classes form a basis for $H^*(\mathrm{Fl}_n; \mathbb{C})$.

We want a more explicit description of the cohomology ring $H^*(\mathrm{Fl}_n; \mathbb{C})$ with its distinguished basis $[\overline{\Omega}_w]$, $w \in S_n$. To this end define the *Schubert polynomial* $\mathfrak{S}_w = \mathfrak{S}_w(x_1, \ldots, x_{n-1})$ by

$$\mathfrak{S}_w = \sum_{(r_1,\ldots,r_p)\in R(w)} \sum_{\substack{1\leq i_1\leq\cdots\leq i_p \\ i_j<i_{j+1} \text{ if } r_j<r_{j+1} \\ i_j\leq j}} x_{i_1}\cdots x_{i_p}.$$

(There are several equivalent definitions; we choose the one most convenient here.) Note the similarity to the definition (20) of G_w. In fact, if we define for $w \in S_n$ and $N \geq 1$ the permutation $1_N \times w \in S_{n+N}$ by

$$(1_N \times w)(i) = w(i) + N \quad \text{if } 1 \leq i \leq n,$$

$$(1_N \times w)(n+1) < (1_N \times w)(n+2) < \cdots < (1_N \times w)(n+N),$$

then it is clear that

$$G_w = \lim_{N\to\infty} \mathfrak{S}_{1_N\times w}.$$

For this reason G_w is sometimes called a *stable Schubert polynomial*.

Now let $R_n = \mathbb{C}[x_1, x_2, \ldots, x_n]/I_n$, where the ideal I_n is generated by the elementary symmetric functions $e_1, e_2, \ldots, e_n$ in the x_i's. Then there is an algebra isomorphism

$$\varphi\colon R_n \to H^*(\mathrm{Fl}_n;\mathbb{C}),$$

such that for $w \in S_n$ we have $\varphi(\mathfrak{S}_{w_0 w}) = [\overline{\Omega}_w]$, where $w_0 = n, n-1, \dots, 1$. For a proof, see e.g. [32, (A.5)][33, §3.6.4]. This isomorphism shows the primary geometric significance of Schubert polynomials.

There is an interesting identity involving reduced decompositions that is related to Schubert polynomials.

Theorem 3.7. *Let $w \in S_n$ and $\ell(w) = p$. Then*

$$\sum_{(r_1,r_2,\dots,r_p)\in R(w)} r_1 r_2 \cdots r_p = p!\,\mathfrak{S}_w(1,1,\dots,1).$$

Theorem 3.7 was first proved by Macdonald [32, (6.11)]. Subsequently a simpler proof, based on the nilCoxeter algebra of S_n, was given by Fomin and Stanley [19]. This proof was extended to give a q-analogue of Theorem 3.7 originally conjectured by Macdonald [32, $(6.11_q?)$].

It is known [32, (4.7)][33, Prop. 2.6.7] that $\mathfrak{S}_w$ is a single monomial if and only if $w = a_1 \cdots a_n$ is 132-*avoiding* (also called *dominant*), i.e., there does not exist $i < j < k$ such that $a_i < a_k < a_j$. The number of 132-avoiding permutations in S_n is the Catalan number C_n. Note in particular that w_0 is 132-avoiding. It follows from Theorem 3.7 that if w is 132-avoiding then

$$\sum_{(r_1,r_2,\dots,r_p)\in R(w)} r_1 r_2 \cdots r_p = p!. \tag{22}$$

There is a curious analogue of equation (22), going back to Chevalley [11] and made more explicit by Stembridge [55] in the case $w = w_0$, connected with degrees of Schubert varieties, Bruhat order, etc. We simply state the result here; for further details see [36]. Let (i,j) denote the transposition interchanging i and j. Given $w \in S_n$ of length p, let

$$T(w) = \{((i_1,j_1),(i_2,j_2),\dots,(i_p,j_p)) \,:\, w = (i_1,j_1)(i_2,j_2)\cdots(i_p,j_p)$$

and $\ell((i_1,j_1)\cdots(i_k,j_k)) = k$ for all $1 \le k \le p\}$.

Then for $w = w_0$ (so $p = \binom{n}{2}$) we have

$$\sum_{((i_1,j_1),(i_2,j_2),\ldots,(i_p,j_p))\in T(w_0)} (j_1-i_1)(j_2-i_2)\cdots(j_p-i_p) = p!. \qquad (23)$$

For instance, if $w = 321$ then (abbreviating the transposition (i,j) as ij)

$$T(w) = \{(12,23,12),(23,12,23),(12,13,23),(23,13,12)\}.$$

Hence we get

$$1\cdot 1\cdot 1 + 1\cdot 1\cdot 1 + 1\cdot 2\cdot 1 + 1\cdot 2\cdot 1 = 3!.$$

Open problem. Is the similarity between equation (22) in the case $w = w_0$ and equation (23) just a coincidence? Can either equation be given a combinatorial proof?

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