ON SOME RESULTS ON PRACTICAL NUMBERS

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Abstract

A positive integer n is said to be a practical number if every integer in $[1, n]$ can be represented as a sum of distinct divisors of n. In this article, we consider practical numbers of a given polynomial form. We give a necessary and sufficient condition for a quadratic polynomial to contain infinitely many practical numbers, using which we solve the first part of a conjecture of Wu. In the final section, we prove that every number of the form $8k + 1$ can be expressed as a sum of a practical number and a square. We also prove that for every $j \in \{0, \ldots, 7\} \setminus \{1\}$ there exist infinitely many natural numbers k such that $8k + j$ cannot be written as a sum of a square and a practical number.

1. Introduction

A positive integer n is said to be a practical number if every integer in $[1, n]$ can be expressed as a sum of distinct divisors of n. Practical numbers were introduced by Srinivasan in [6]. In [7], Stewart showed that $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where $p_1 < p_2 < \cdots < p_k$ are primes and $\alpha_i \geq 1$ are integers is practical if either $n = 1$ or $p_1 = 2$ and for all $2 \leq i \leq k$ we have $p_i \leq \sigma(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{i-1}^{\alpha_{i-1}}) + 1$ (here σ denotes the sum of divisors function). This implies that if m is a practical number then for all natural numbers $n \leq \sigma(m) + 1$, mn is a practical number. We will be using this property several times in this article.

There are many properties of practical numbers that are similar to the properties of prime numbers. All practical numbers except for 1 are even while all primes

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except for 2 are odd. Let $\pi(x)$ and $P(x)$ denote the number of primes less than or equal to x and the number of practical numbers less than or equal to x, respectively. From the Prime Number Theorem, we have $\pi(x) \sim \frac{x}{\log x}$. In [8], Weingartner showed that there exists a positive constant c such that $P(x) \sim \frac{cx}{\log x}$.

In [2], Margenstern made a Goldbach-type conjecture that every even positive integer can be expressed as a sum of two practical numbers. In the same paper, he conjectured that there are infinitely many positive integers m such that $m-2, m$, and $m+2$ are all practical numbers. Melfi proved both conjectures in [3].

In Section 2, we consider quadratic representations of practical numbers. Using our results, we prove the first part of [9, Conjecture 1.1] and generalize [9, Theorem 1.2].

In Section 3, we prove that every number of the form $8k+1$ can be represented as a sum of a square and a practical number. We also show that for all $j \in \{0, \dots, 7\} \setminus \{1\}$, there exist infinitely many natural numbers k such that $8k+j$ cannot be written as a sum of a square and a practical number.

2. Practical Numbers of the Form $an^2 + bn + c$

Let $q(n) = an^2 + bn + c$ be any quadratic polynomial having positive integer coefficients. For every prime p, define

$$m_q(p) := \sup\{k \in \mathbb{N} \cup \{0\} : q(n) \equiv 0 \bmod p^k \text{ has a solution}\}.$$

We will first prove that for every quadratic polynomial q there exists a prime p such that $m_q(p) = \infty$. In order to prove this, we will use the following version of Hensel's lemma.

Hensel's Lemma. *Suppose that $f(x)$ is a polynomial with integral coefficients. If $f(a) \equiv 0 \pmod{p^j}$ and $f'(a) \not\equiv 0 \pmod{p}$, then there is a unique $t \pmod{p}$ such that $f(a+tp^j) \equiv 0 \pmod{p^{j+1}}$.*

Proof. See [4, Theorem 2.23] for proof. □

Lemma 2.1. *Let $q(n) = an^2 + bn + c$ be any quadratic polynomial having positive integer coefficients. There exists a prime p such that $m_q(p) = \infty$.*

Proof. If $b^2 - 4ac = 0$ then $q(n) = \frac{(2an+b)^2}{4a}$. Let p be any prime number such that $p \nmid 2a$. For any natural numbers k and n such that $2an \equiv -b \mod p^k$, we have $q(n) \equiv 0 \mod p^{2k}$. Hence for any prime p not dividing $2a$, we have $m_q(p) = \sup\{k \in \mathbb{N} \cup \{0\} : q(n) \equiv 0 \mod p^k \text{ has a solution}\} = \infty$.

Let us assume $b^2 - 4ac \neq 0$. We can apply Schur's theorem [5], which states that there are infinitely many prime divisors of q. Consequently, there exists a prime

$p \nmid (b^2-4ac)$ and a natural number s such that $q(s) \equiv 0 \mod p$. We claim that $q'(s) \not\equiv 0 \mod p$ because if $q'(s) \equiv 0 \mod p$, it would imply that $(q'(s))^2-4aq(s) = (2as+b)^2-4a(as^2+bs+c) = b^2-4ac \equiv 0 \mod p$, which contradicts $p \nmid (b^2-4ac)$. Let

$$u = \sup\{k \in \mathbb{N} \cup \{0\} : q(r) \equiv 0 \bmod p^k \text{ and } r \equiv s \bmod p \text{ has a solution}\}.$$

Notice that $u \leq m_q(p)$. If u is finite, then there exists a natural number r such that $q(r) \equiv 0 \mod p^u$ and $r \equiv s \mod p$. Since $r \equiv s \mod p$, we have $q'(r) \equiv q'(s) \mod p$, which implies $q'(r) \not\equiv 0 \mod p$. From Hensel's lemma, there exists a unique $t \bmod p$ such that $q(r+tp^u) \equiv 0 \bmod p^{u+1}$. Since $r+tp^u \equiv r \equiv s \mod p$ and $q(r+tp^u) \equiv 0 \mod p^{u+1}$, it follows that $u+1 \in \{k \in \mathbb{N} \cup \{0\} : q(r) \equiv 0 \bmod p^k$ and $r \equiv s \bmod p$ has a solution$\}$. This is absurd as u is the supremum. Therefore, we can conclude that $u = \infty$. As $u \leq m_q(p)$, we have $m_q(p) = \infty$. □

Let us now work on giving a necessary and sufficient condition for a quadratic polynomial $q(n)$ to represent infinitely many practical numbers. We require a lemma in order to prove Theorem 2.3.

Lemma 2.2. *Let $q(n) = an^2+bn+c$ be any quadratic polynomial with integer coefficients and let $Q = \{p :\ p$ prime and there exists $n \in \mathbb{N}$ such that $p|q(n)\}$. The infinite product $\prod_{p\in Q}\left(1+\frac{1}{p}\right)$ diverges.*

Proof. Let f be any irreducible divisor of q and let $\rho_f(p)$ denote the number of solutions modulo p of the congruence $f(x) \equiv 0 \bmod p$. From the Prime Ideal Theorem, it follows that $\sum_p \frac{\rho_f(p)}{p}$ diverges (see [1, Corollary 3.2.2]). As $\rho_f(p) \leq 2$, it follows that $\sum_{p\in Q}\frac{1}{p}$ diverges. Hence $\prod_{p\in Q}(1+\frac{1}{p})$ diverges. □

Theorem 2.3. *Let q be a quadratic polynomial with positive integer coefficients. Let p_n denote the n-th prime number and let r be the least positive integer such that $m_q(p_r) = \infty$. There are infinitely many practical numbers of the form $q(n)$ if and only if $p_1^{m_q(p_1)}p_2^{m_q(p_2)}\cdots p_{r-1}^{m_q(p_{r-1})}p_r$ is a practical number.*

Proof. Suppose $p_1^{m_q(p_1)}p_2^{m_q(p_2)}\cdots p_{r-1}^{m_q(p_{r-1})}p_r$ is not a practical number. We have either $m_q(p_1) = m_q(2) = 0$ or there exists an i, with $2 \leq i \leq r$, such that $m_q(p_i) > 0$ and $p_i > \sigma\left(p_1^{m_q(p_1)}p_2^{m_q(p_2)}\cdots p_{i-1}^{m_q(p_{i-1})}\right)+1$. If $m_q(2) = 0$ then all $q(n)$ are odd and all $q(n) > 1$ are not practical numbers. Hence there are only finitely many practical numbers of the form $q(n)$. If there exists an i, with $2 \leq i \leq r$, such that $m_q(p_i) > 0$ and

$$p_i > \sigma(p_1^{m_q(p_1)}\cdots p_{i-1}^{m_q(p_{i-1})})+1,$$

then we claim that for all natural numbers n such that

$$q(n) > p_1^{m_q(p_1)}\cdots p_{r-1}^{m_q(p_{r-1})}p_r,$$

$q(n)$ is not a practical number. We claim that $q(n) > p_1^{m_q(p_1)} \cdots p_{r-1}^{m_q(p_{r-1})} p_r$ should have at least one prime factor $p \geq p_i$. Otherwise, if all the prime factors of $q(n)$ are less than p_i then $q(n)$ can be expressed as $q(n) = p_1^{e_1} \cdots p_{i-1}^{e_{i-1}}$ for some non-negative integers $e_1, \ldots, e_{i-1}$. For $1 \leq j \leq i-1$, as $p_j^{e_j} | q(n)$ we have $e_j \in \{k \in \mathbb{N} \cup \{0\} : q(n) \equiv 0 \bmod p_j^k$ has a solution$\}$ and hence $e_j \leq \sup\{k \in \mathbb{N} \cup \{0\} : q(n) \equiv 0 \bmod p_j^k$ has a solution$\} = m_q(p_j)$. Consequently, $e_j \leq m_q(p_j)$ for $1 \leq j \leq i-1$, and we have $q(n) = p_1^{e_1} \cdots p_{i-1}^{e_{i-1}} \leq p_1^{m_q(p_1)} \cdots p_{i-1}^{m_q(p_{i-1})}$. This contradicts $q(n) > p_1^{m_q(p_1)} \cdots p_{i-1}^{m_q(p_{i-1})}$. Hence there exists at least one prime factor of $q(n)$ which is greater than or equal to p_i. Let p be the smallest prime factor of $q(n)$ satisfying $p \geq p_i$. Now, $q(n) = p_1^{a_1} p_2^{a_2} \cdots p_{i-1}^{a_{i-1}} p^k Q$ where Q is either 1 or the least prime factor of Q is greater than p_i. As

$$p \geq p_i > \sigma(p_1^{m_q(p_1)} p_2^{m_q(p_2)} \cdots p_{i-1}^{m_q(p_{i-1})}) + 1 \geq \sigma(p_1^{a_1} \cdots p_{i-1}^{a_{i-1}}) + 1,$$

from [7, Theorem 1] we can conclude that $q(n)$ is not a practical number. Hence there are only finitely many $q(n)$ such that $q(n)$ is a practical number. This proves one part of the theorem.

We claim that if $p_1^{m_q(p_1)} \cdots p_{r-1}^{m_q(p_{r-1})} p_r$ is a practical number then there are infinitely many practical numbers of the form $q(n)$. Assume, for the sake of contradiction, that there are only a finite number of practical numbers of the form $q(n)$. This assumption implies the existence of a real number A such that there are no practical numbers $q(n)$ for which $q(n) \geq A$. Let

$$Q = \{p : \ p \text{ prime and there exists } n \in \mathbb{N} \text{ such that } p | q(n)\}.$$

From the previous lemma, it follows that $\prod_{p \in Q}(1 + \frac{1}{p})$ diverges. Hence there exist primes $t_1, \ldots, t_s \in Q$ that are greater than p_r such that $\prod_{i=1}^{s}(1 + \frac{1}{t_i}) > a + b + c$. Let k be a natural number such that $p_r^k > A$ and $p_1^{m_q(p_1)} \cdots p_{r-1}^{m_q(p_{r-1})} p_r^k \geq t_1 \cdots t_s$. Since $m_q(p_r) = \infty$ there exists a solution $x_r \bmod p_r^k$ such that $q(x_r) \equiv 0 \bmod p_r^k$. For $1 \leq i \leq r-1$, there exists a solution $x_i \bmod p_i^{m_q(p_i)}$ such that $q(x_i) \equiv 0 \bmod p_i^{m_q(p_i)}$, and for $1 \leq j \leq s$ there exists $y_j \bmod t_j$ such that $q(y_j) \equiv 0 \bmod t_j$. Hence there exists an n, with $1 \leq n \leq D$, such that $q(n)$ is divisible by D, where $D = p_1^{m_q(p_1)} \cdots p_{r-1}^{m_q(p_{r-1})} p_r^k t_1 t_2 \cdots t_s$. Observe that as $p_1^{m_q(p_1)} \cdots p_{r-1}^{m_q(p_{r-1})} p_r^k$ is a practical number and $t_1 t_2 \cdots t_r \leq p_1^{m_q(p_1)} p_2^{m_q(p_2)} \cdots p_{r-1}^{m_q(p_{r-1})} p_r^k$, we can conclude that $D = p_1^{m_q(p_1)} p_2^{m_q(p_2)} \cdots p_{r-1}^{m_q(p_{r-1})} p_r^k t_1 t_2 \cdots t_s$ is also a practical number. As $q(n) = D(\frac{q(n)}{D})$ and

$$\sigma(D) = D\left(\frac{\sigma(D)}{D}\right) \geq n \prod_{j=1}^{s}\left(1 + \frac{1}{t_j}\right) > n(a+b+c) \geq \frac{an^2 + bn + c}{n} \geq \frac{q(n)}{n} \geq \frac{q(n)}{D},$$

we have that $q(n)$ is a practical number greater than A. This contradicts our assumption that there are no practical numbers of the form $q(n)$ which are greater than A. Hence there are infinitely many practical numbers of the form $q(n)$. This proves the other part of the theorem. $\square$

Now before proving a corollary of Theorem 2.3, we prove a lemma.

Lemma 2.4. *Let a, b, and c be positive integers such that $2 \nmid ab$ and $2|c$. For all positive integers k, there exists a natural number n such that $an^2 + bn + c \equiv 0 \mod 2^k$.*

Proof. We prove this lemma using induction on k. For $k = 1$, as $a.1^2 + b.1 + c \equiv 0 \mod 2$, the statement is true for $k = 1$. Suppose the statement is true for $k = l$, then there exists an n such that $an^2 + bn + c \equiv 0 \mod 2^l$. For any natural number x, let $m = 2^l x + n$. We have $a(2^l x + n)^2 + b(2^l x + n) + c \equiv an^2 + 2^l bx + bn + c \mod 2^{l+1}$. If $x \equiv \frac{an^2+bn+c}{2^l} \mod 2$ then $a(2^l x + n)^2 + b(2^l x + n) + c \equiv 0 \mod 2^{l+1}$. Hence the statement is true for $l + 1$ and the lemma follows from the principle of mathematical induction. $\square$

Corollary 2.5. *Let a, b, and c be positive integers such that $2 \nmid ab$ and $2|c$. There are infinitely many practical numbers of the form $an^2 + bn + c$.*

Proof. Let $q(n) = an^2 + bn + c$. From the previous lemma, we have $m_q(2) = \infty$. As 2 is a practical number, from Theorem 2.3 we can conclude that there are infinitely many practical numbers of the form $an^2 + bn + c$. $\square$

The above corollary solves the first part of [9, Conjecture 1.1] and is a generalization of [9, Theorem 1.2].

3. On Natural Numbers Which Can Be Represented as a Sum of a Square and a Practical Number

We now prove that every natural number of the form $8k + 1$ can be expressed as a sum of a practical number and a square. We will prove this using some lemmas.

Lemma 3.1. *If $n = 2^k m$ with $k \geq 1$ and $m \leq 2^{k+1}$, then n is a practical number.*

Proof. As 2^k is a practical number and $m \leq 2^{k+1} = \sigma(2^k) + 1$, we can conclude that $2^k m$ is a practical number. $\square$

Lemma 3.2. *Let m be a natural number such that $m \equiv 1 \mod 8$ and let k be a positive integer. There exists a natural number x, with $1 \leq x \leq 2^k - 1$, satisfying $x^2 \equiv m \pmod{2^{k+2}}$.*

Proof. Let m be any natural number such that $m \equiv 1 \mod 8$. Using mathematical induction, we prove that for all positive integers k, there exists a natural number x, with $1 \leq x \leq 2^k - 1$, such that $x^2 \equiv m \mod 2^{k+2}$. The statement is true for $k = 1$ as $1^2 \equiv m \mod 2^3$. As $m \equiv 1 \mod 8$ implies $m \equiv 1^2 \mod 2^4$ or $m \equiv 3^2 \mod 2^4$, the statement is true for $k = 2$.

Assume that the statement is true for $k = s \geq 2$. Then there exists a natural number x_s, with $1 \leq x_s \leq 2^s - 1$, such that $x_s^2 \equiv m \mod 2^{s+2}$. Now, $x_s^2 \equiv m \mod 2^{s+3}$ or $x_s^2 \equiv m + 2^{s+2} \mod 2^{s+3}$. Let $x_{s+1} = x_s$ if $x_s^2 \equiv m \mod 2^{s+3}$, and $x_{s+1} = 2^{s+1} - x_s$ if $x_s^2 \equiv m + 2^{s+2} \mod 2^{s+3}$. Note that $1 \leq x_{s+1} \leq 2^{s+1} - 1$. We claim $x_{s+1}^2 \equiv m \mod 2^{s+3}$. If $x_s^2 \equiv m \mod 2^{s+3}$ then, as $x_{s+1} = x_s$, we have $x_{s+1}^2 \equiv m \mod 2^{s+3}$. If $x_s^2 \equiv m + 2^{s+2} \mod 2^{s+3}$ then

$$
\begin{aligned}
x_{s+1}^2 &\equiv (2^{s+1} - x_s)^2 \mod 2^{s+3} \\
&\equiv 2^{2s+2} + x_s^2 - 2^{s+2}x_s \mod 2^{s+3} \\
&\equiv m + 2^{s+2} - 2^{s+2}x_s \mod 2^{s+3} \\
&\equiv m + 2^{s+2}(1 - x_s) \mod 2^{s+3} \\
&\equiv m \mod 2^{s+3} \text{ (as } (1 - x_s) \text{ is even, } 2^{s+2}(1 - x_s) \text{ is divisible by } 2^{s+3}).
\end{aligned}
$$

Hence the statement is true for $k = s + 1$ and the lemma follows from the principle of mathematical induction. □

Now we are ready to prove the main result of this section.

Theorem 3.3. *Every natural number of the form $8k + 1$ can be expressed as a sum of a square and a practical number. Also, for every $j \in \{0, \dots, 7\} \setminus \{1\}$ there exist infinitely many natural numbers k such that $8k + j$ cannot be written as a sum of a practical number and a square.*

Proof. Let $n > 1$ be a natural number of the form $8k + 1$. Let $m = \left\lfloor \log_2 \sqrt{8k + 1} \right\rfloor$. This implies $2^{2m} \leq 8k + 1 < 2^{2m+2}$. As $m \geq 1$, from Lemma 3.2 there exists a natural number x, with $1 \leq x \leq 2^m - 1$, such that $x^2 \equiv 8k + 1 \pmod{2^{m+2}}$. Therefore, $8k + 1 - x^2 = 2^{m+2}s$ for some positive integer s (observe that s is a positive integer as $x^2 < 2^{2m} \leq 8k+1$). As $2^{m+2}s \leq 8k+1 \leq 2^{2m+2}$, we have $s \leq 2^m$. Hence from Lemma 3.1, $2^{m+2}s$ is a practical number and $8k + 1 = x^2 + 2^{m+2}s$ is a sum of a square and a practical number. This proves the first part of the theorem.

Let us prove the second part by considering each $j \in \{0, \dots, 7\} \setminus \{1\}$ separately.

($j = 0$): Consider numbers m such that $m \equiv 24 \pmod{32}$, $m \equiv 2 \pmod 3$, $m \equiv 2 \pmod 5$, $m \equiv -1 \pmod 7$, $m \equiv -1 \pmod{11}$, and $m \equiv 2 \pmod{13}$. Then m is of the form $8k$ and $m \equiv x^2 \pmod{16}$, $m \equiv x^2 \mod 3$, $m \equiv x^2 \pmod 5$, $m \equiv x^2 \pmod 7$, $m \equiv x^2 \pmod{11}$, and $m \equiv x^2 \pmod{13}$ have no solutions. Hence if $m = n^2 + P$ then $16 \nmid P$, $3 \nmid P$, $5 \nmid P$,

$7 \nmid P$, $11 \nmid P$, and $13 \nmid P$. The highest power of 2 dividing P is less than or equal to 8 and P is not divisible by any prime less than or equal to $\sigma(8) + 1 = 16$. Hence P cannot be a practical number and m cannot be written as a sum of a square and a practical number. Hence there are infinitely many numbers of the form $8k$ which cannot be written as a sum of a square and a practical number.

($j = 2$): Consider numbers m such that $m - 1$ is not a perfect square and m is of the form $24n + 2$. Then $m \equiv x^2 \pmod 4$ and $m \equiv x^2 \pmod 3$ have no solutions. The highest power of 2 dividing $m - x^2$ for any x is less than or equal to 2. As $m - x^2 \neq 1$ is not divisible by any prime less than or equal to $\sigma(2) + 1 = 4$, we can conclude that $m - x^2$ is not a practical number. Hence there are infinitely many numbers of the form $8k + 2$ which cannot be written as a sum of a square and a practical number.

($j = 3$): Consider numbers m of the form $24n + 11$. Then $m \equiv x^2 \pmod 4$ and $m \equiv x^2 \pmod 3$ have no solutions. The highest power of 2 dividing $m - x^2$ for any x is less than or equal to 2. As $m - x^2$ is not divisible by any prime less than or equal to $\sigma(2) + 1 = 4$, $m - x^2$ is not a practical number. Hence there are infinitely many numbers of the form $8k + 3$ which cannot be written as a sum of a square and a practical number.

($j = 4$): Consider numbers m such that $m \equiv 12 \pmod{16}$, $m \equiv 2 \pmod 3$, $m \equiv 2 \pmod 5$, $m \equiv -1 \pmod 7$, $m \equiv -1 \pmod{11}$, and $m \equiv 2 \pmod{13}$. Then m is of the form $8k+4$ and $m \equiv x^2 \pmod{16}$, $m \equiv x^2 \pmod 3$, $m \equiv x^2 \pmod 5$, $m \equiv x^2 \pmod 7$, $m \equiv x^2 \pmod{11}$, and $m \equiv x^2 \pmod{13}$ have no solutions. Hence if $m = n^2 + P$ then $16 \nmid P$, $3 \nmid P$, $5 \nmid P$, $7 \nmid P$, $11 \nmid P$ and $13 \nmid P$. The highest power of 2 dividing P is less than or equal to 8 and P is not divisible by any prime less than or equal to $\sigma(8) + 1 = 16$. Hence P cannot be a practical number and m cannot be written as a sum of a square and a practical number. Hence there are infinitely many numbers of the form $8k + 4$ which cannot be written as a sum of a square and a practical number.

($j = 5$): Consider numbers m such that $m - 1$ is not a perfect square, $m \equiv 5 \pmod 8$, $m \equiv 2 \pmod 3$, $m \equiv 2 \pmod 5$, and $m \equiv -1 \pmod 7$. Then m is of the form $8k + 5$ and $m \equiv x^2 \pmod 8$, $m \equiv x^2 \pmod 3$, $m \equiv x^2 \pmod 5$, and $m \equiv x^2 \pmod 7$ have no solutions. Hence if $m = n^2 + P$ then $8 \nmid P$, $3 \nmid P$, $5 \nmid P$ and $7 \nmid P$. The highest power of 2 dividing P is less than or equal to 4 and P is not divisible by any prime less than or equal to $\sigma(4) + 1 = 8$. Hence P cannot be a practical number and m cannot be written as a sum of a square and a practical number. Hence there are infinitely many numbers of the form $8k+5$ which cannot be written as a sum of a square and a practical number.

($j = 6$): Consider numbers m of the form $24n + 14$. Then $m \equiv x^2 \pmod 4$ and $m \equiv x^2 \pmod 3$ have no solutions. The highest power of 2 dividing $m-x^2$ for any x is less than or equal to 2, and since $m-x^2$ is not divisible by any prime less than or equal to $\sigma(2)+1 = 4$, we have $m-x^2$ is not a practical number. Hence there are infinitely many numbers of the form $8k+6$ which cannot be written as a sum of a square and a practical number.

($j = 7$): Consider numbers m of the form $24n + 23$. Then $m \equiv x^2 \pmod 4$ and $m \equiv x^2 \pmod 3$ have no solutions. The highest power of 2 dividing $m-x^2$ for any x is less than or equal to 2, and since $m-x^2$ is not divisible by any prime less than or equal to $\sigma(2)+1 = 4$, we have $m-x^2$ is not a practical number. Hence there are infinitely many numbers of the form $8k+7$ which cannot be written as a sum of a square and a practical number.

□

4. Future Prospects

In this paper, we have given necessary and sufficient conditions for quadratic polynomials to represent infinitely many practical numbers (Theorem 2.3). It would be worthwhile to explore such necessary and sufficient conditions for cubic and biquadratic polynomials with integer coefficients.

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