DIFFERENTIAL OF HOLONOMY FOR TORUS BUNDLES

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Ref. http://math.huji.ac.il/~piz/documents/DBlog-EX-DOHFTB.pdf

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We explicit the Holonomy function on a torus bundle on a diffeological space. Here the word torus denotes any quotient $T = \mathbf{R}/\Gamma$, where Γ is a strict subgroup of $\mathbf{R}$. Then, we explicit the differential of the holonomy in terms of the Chain-Homotopy Operator and the curvature.

In this note we will consider a principal fiber bundle $\pi : Y \to X$, with X and Y two diffeological spaces, and with structure group a *torus* $T = \mathbf{R}/\Gamma$, where Γ is any strict subgroup of $\mathbf{R}$. As we know, if $\Gamma = a\mathbf{Z}$ then the torus T is a manifold isomorphic to the circle S^1, and if Γ has more than 1 generator, T is said to be *irrational*, and it's not anymore a manifold. We assume that there exists on Y a connexion form λ, that is, a differential 1-from satisfying the two conditions

(1) λ is **invariant** by the action of T:

$$\text{For all } \tau \in T, \quad \tau_Y^*(\lambda) = \lambda,$$

where τ_Y denotes the action of τ on Y.

(2) λ is **calibrated**:

$$\text{For all } y \in Y, \quad \hat{y}^*(\lambda) = \theta,$$

where $\hat{y} : T \to Y$ is the orbit map $\hat{y}(\tau) = \tau_Y(y)$, and θ is the canonical 1-form on T, pushforward of the canonical 1-form dt on $\mathbf{R}$.

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The question of the differential of the holonomy has been raised, in a larger context, by Jean-Paul Mohsen during our weekly work group.

Then, the connexion is defined by the *horizontal paths* in Y, and they are

$$\mathrm{Hor}(Y) = \{\gamma \in \mathrm{Paths}(Y) \mid \lambda(\gamma)_t = 0 \text{ for all } t \in \mathbf{R}\}.$$

See [PIZ13] for the details of these constructions, in particular §8.37.

Loops Bundles

✎ Exercise (I). Show that the pushforward

$$\pi_* : \mathrm{Loops}(Y) \to \mathrm{Loops}(X), \text{ defined by } \pi_*(\tilde{\ell}) = \pi \circ \tilde{\ell},$$

is surjective.

Solution — Consider a loop ℓ in X, that is, $\ell \in \mathcal{C}^\infty(\mathbf{R}, X)$ and $\ell(0) = \ell(1)$. The pullback of Y by ℓ is a principal bundle on $\mathbf{R}$ with a connexion, therefore it is trivial [PIZ13, §8.34]: there exists an equivariant diffeomorphism $\varphi : \mathbf{R} \times T \to \ell^*(Y)$. Let $\varphi(t, \tau) = (t, f(t, \tau))$, with $f(t, \tau) \in Y_{\ell(t)}$.

$$\begin{array}{ccccc} \mathbf{R} \times T & \xrightarrow{\varphi} & \ell^*(Y) & \xrightarrow{\mathrm{pr}_2} & Y \\ & \searrow^{\mathrm{pr}_1} & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \pi} \\ & & \mathbf{R} & \xrightarrow[\ell]{} & X \end{array}$$

Then, by equivariance, $\varphi(t, \tau) = (t, \tau_Y(f(t)))$, where $f \in \mathrm{Paths}(Y)$ and $\pi \circ f = \ell$. There exists a unique $a \in T$ such that $f(1) = a_Y(f(0))$. Now, since T is connected there exists $\tau \in \mathrm{Paths}(T)$ such that $\tau(0) = 1$ and $\tau(1) = a^{-1}$. Then, $\tilde{\ell} : t \mapsto \tau(t)_Y(f(t))$ is a path in Y such that $\pi \circ \tilde{\ell} = \ell$, and $\tilde{\ell}(0) = \tilde{\ell}(1) = f(0)$. Therefore, $\tilde{\ell} \in \mathrm{Paths}(Y)$ and $\pi \circ \tilde{\ell} = \ell$. □

Now, there is more than π_* being surjective. Using the same ideas than in exercise I:

✎ Exercise (II). Show that the pushforward $\pi_* : \mathrm{Loops}(Y) \to \mathrm{Loops}(X)$ is not only surjective but even a subduction.

Solution — Let $r \mapsto \ell_r$ be a plot in Loops(X) defined on a small open ball B centered at 0 in some $\mathbf{R}^n$. Since $r \mapsto \ell_r$ is smooth, the map $Q : (r, t) \mapsto \ell_r(t)$ defined on $B \times \mathbf{R}$ is a plot of X. Since we have a connection on $\pi : Y \to X$, we have a connection, by pullback, on $\mathrm{pr}_1 : Q^*(Y) \to B \times \mathbf{R}$. And since $B \times \mathbf{R}$ is contractible, the fibration $\mathrm{pr}_1 : \mathrm{Q}^*(\mathrm{Y}) \to \mathrm{B} \times \mathbf{R}$ is trivial, that is, there exists an equivariant diffeomorphism $\varphi : \mathrm{B} \times \mathbf{R} \times \mathrm{T} \to \mathrm{Q}^*(\mathrm{Y})$. By equivariance, $\varphi(r, t, \tau) = (r, t, \tau_{\mathrm{Y}}(f_r(t)))$, where $(r, t) \mapsto f_r(t)$ is a plot of Y. That is equivalent to say that $r \mapsto f_r$ is a plot of Paths(Y). We have then $\pi \circ f_r(t) = \ell_r(t)$.

$$\begin{array}{ccccc} \mathrm{B} \times \mathbf{R} \times \mathrm{T} & \xrightarrow{\varphi} & \ell^*(\mathrm{Y}) & \xrightarrow{\mathrm{pr}_3} & \mathrm{Y} \\ & \searrow^{\mathrm{pr}_{1,2}} & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \pi} \\ & & \mathrm{B} \times \mathbf{R} & \xrightarrow[\ell]{} & \mathrm{X} \end{array}$$

Thus, there exists $r \mapsto \mathrm{a}_r \in \mathrm{T}$ such that $f_r(1) = (\mathrm{a}_r)_{\mathrm{Y}}(f_r(0)$. Let $\alpha : r \mapsto f_r(0)$ and $\beta : r \mapsto f_r(1)$. Since α and β are homotopic, the pullbacks $\alpha^*(\mathrm{Y})$ and $\beta^*(\mathrm{Y})$ are equivalent (actually trivial) [PIZ13, §8.34]. Therefore, $r \mapsto \mathrm{a}_r$ is a plot of T, as well as $r \mapsto \mathrm{a}_r^{-1}$. Now, the projection class : $\mathbf{R} \to \mathrm{T}$ is the universal covering. The ball B being contractible, there exist a unique lifting $r \mapsto \bar{\mathrm{a}}_r$ of $r \mapsto \mathrm{a}_r^{-1}$ such that $\bar{\mathrm{a}}_0 = 0$ [PIZ13, §8.25]. Consider then $(r, t) \mapsto \tau_r(t) = \mathrm{class}(t\bar{\mathrm{a}}_r)$. The parametrization $r \mapsto \tau_r$ is a plot of Paths(T) such that $\bar{\mathrm{a}}_r(0) = 1$ and $\bar{\mathrm{a}}_r(1) = \mathrm{a}_r^{-1}$. Then, let $\tilde{\ell}_r(t) = \tau_r(t)_{\mathrm{Y}}(f_r(t)))$, it is a plot of Y such that $\tilde{\ell}_r(0) = f_r(0)$ and $\tilde{\ell}_r(1) = \tau_r(1)_{\mathrm{Y}}(f_r(1)) = (\mathrm{a}_r^{-1})_{\mathrm{Y}}((\mathrm{a}_r)_{\mathrm{Y}}(f_r(0)) = f_r(0)$. Thus, $r \mapsto \tilde{\ell}_r$ is a plot of Loops(Y), and a lifting of $r \mapsto \ell_r$, that is, such that $\pi \circ \tilde{\ell}_r = \ell_r$, for all $r \in \mathrm{B}$. Hence, every plot in Loops(X) has a local smooth lifting in Loops(Y) everywhere. Therefore, $\pi_* : \mathrm{Loops}(\mathrm{Y}) \to \mathrm{Loops}(\mathrm{X})$ is a subduction. ☐

Holonomy Function

In this section we shall explicit the holonomy H of the connection defined by λ, and compute its differential.

✎ Exercise (III). Show that there exists a smooth map

$$\mathrm{H} : \mathrm{Loops}(\mathrm{X}) \to \mathrm{T} \quad \text{defined by} \quad \mathrm{H}(\ell) = \mathrm{class}\left(\int_{\tilde{\ell}} \lambda\right),$$

for all $\tilde{\ell} \in \mathrm{Loops}(\mathrm{Y})$ such that $\pi \circ \tilde{\ell} = \ell$. And where class denotes the canonical projection from $\mathbf{R}$ to T.

Solution — Let $\tilde{\ell}$ and $\tilde{\ell}'$ be two loops in Y projecting on ℓ. Since $\pi : \mathrm{Y} \to \mathrm{X}$ is a diffeological fiber bundle, there exists a loop

τ in T such that $\tilde\ell'(t) = \tau(t)_Y(\tilde\ell(t))$. According to [PIZ13, §8.37], we have

$$\lambda\big(t \mapsto \tau(t)_Y(\tilde\ell(t))\big)_t(1) = \tau^*(\theta)_t(1) + \lambda(\tilde\ell)_t(1).$$

Then,

$$\begin{aligned}
\int_{\tilde\ell'} \lambda &= \int_0^1 \lambda(\tilde\ell')_t(1)\ dt \\
&= \int_0^1 \lambda(t \mapsto \tau(t)_Y(\tilde\ell(t)))_t(1)\ dt \\
&= \int_0^1 \tau^*(\theta)_t(1)\ dt + \int_0^1 \lambda(t \mapsto \tilde\ell(t))_t(1)\ dt \\
&= \int_\tau \theta + \int_{\tilde\ell} \lambda.
\end{aligned}$$

But since τ is a loop in T and θ is the canonical 1-form on $T = \mathbf{R}/\Gamma$, $\int_\tau \theta \in \Gamma$. Therefore, $\mathrm{class}\big(\int_{\tilde\ell'} \lambda\big) = \mathrm{class}\big(\int_{\tilde\ell} \lambda\big)$. We get the map H, which is well defined: $H(\ell) = \mathrm{class}\big(\int_{\tilde\ell} \lambda\big)$. Let us denote $\tilde H$ the integral of λ on the loops in Y. We have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Loops}(Y) & \xrightarrow{\ \tilde H\ } & \mathbf{R} \\
{\scriptstyle \pi_*}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{class}} \\
\mathrm{Loops}(X) & \xrightarrow[\ H\ ]{} & T
\end{array}$$

Now, since π_* is a subduction, according to the previous exercise, the map H is smooth. ☐

The values of H is exactly the group of holonomy of the connection defined by λ, see [PIZ13, §8.35]. It is a subgroup of T, it is either discrete or the whole T. A way to check if it is discrete is to compute its "differential". We will define it as:

$$d_T H = H^*(\theta).$$

✎ Exercise (IV). Show that

$$d_T H + \mathcal{K}\omega = 0,$$

where $\mathcal{K}$ is the Chain-Homotopy Operator [PIZ13, §6.83].

☞ Solution — Remark that

$$\tilde{\mathrm{H}} = \left[\tilde{\ell} \mapsto \int_{\tilde{\ell}} \lambda\right] = \mathcal{K}\lambda.$$

Then, apply the fundamental property of the Chain-Homotopy Operator, restricted to the space of loops of Y. That gives

$$d(\mathcal{K}\lambda) + \mathcal{K}(d\lambda) = [(\hat{1}^* - \hat{0}^*) \upharpoonright \mathrm{Loops}(\mathrm{Y})](\lambda) = 0.$$

Recalling that $d\lambda = \pi^*(\omega)$, we get

$$d\tilde{\mathrm{H}} + \mathcal{K}(\pi^*(\omega)) = 0.$$

The variance of the Chain-Homotopy Operator states that the following diagram is commutative, see [PIZ13, §6.84].

$$\begin{array}{ccc} \Omega^k(\mathrm{X}) & \xrightarrow{\mathcal{K}_\mathrm{X}} & \Omega^{k-1}(\mathrm{Paths}(\mathrm{X})) \\ \pi^* \downarrow & & \downarrow (\pi_*)^* \\ \Omega^k(\mathrm{Y}) & \xrightarrow{\mathcal{K}_\mathrm{Y}} & \Omega^{k-1}(\mathrm{Paths}(\mathrm{Y})) \end{array}$$

Thus (forgetting the indices on $\mathcal{K}$),

$$d\tilde{\mathrm{H}} + (\pi_*)^*(\mathcal{K}\omega) = 0.$$

But $d\tilde{\mathrm{H}} = \tilde{\mathrm{H}}^*(dt)$, and $dt = \mathrm{class}^*(\theta)$, thus $d\tilde{\mathrm{H}} = \tilde{\mathrm{H}}^*(\mathrm{class}^*(\theta)) = (\mathrm{class} \circ \tilde{\mathrm{H}})^*(\theta) = (\mathrm{H} \circ \pi_*)^*(\theta) = (\pi_*)^*(d_\mathrm{T}\mathrm{H})$. Hence,

$$(\pi_*)^*(d_\mathrm{T}\mathrm{H}) + (\pi_*)^*(\mathcal{K}\omega) = (\pi_*)^*(d_\mathrm{T}\mathrm{H} + \mathcal{K}\omega) = 0.$$

Now, since π_* is a subduction, according to the previous exercise, and thanks to [PIZ13, §6.39],

$$d_\mathrm{T}\mathrm{H} + \mathcal{K}\omega = 0.$$

And that is the expression of the differential of the holonomy. We get from this identity that if the curvature ω vanishes, then the holonomy is discrete and the fiber bundle reduces to a covering. ☐

By definition [PIZ13, §6.83], the Chain-Homotopy Operator is defined by

$$\mathcal{K}\omega = i_\tau \circ \Phi(\omega),$$

where $\tau \in \mathrm{Hom}^\infty(\mathbf{R}, \mathrm{Diff}(\mathrm{Loops}(\mathrm{X})))$ is the action of reparametrization of paths:

$$\tau(\varepsilon)(\gamma) = [t \mapsto \gamma(t + \varepsilon)],$$

and $\Phi(\omega)$ is the mean value of the "time-pullbacks":

$$\Phi(\omega) = \int_0^1 \hat{t}^*(\omega)\ dt,$$

with $\hat{t}(\gamma) = \gamma(t)$. That way, the differential of the holonomy writes:

$$d_{\mathrm{T}}\mathrm{H} + i_\tau \mathrm{F} = 0 \quad \text{with} \quad \mathrm{F} = \Phi(\omega).$$

That is the (infinitesimal) equivariant cohomology way of expressing this differential: H is the moment map associated with the reparametrization group action on Loops(X), with respect to the 2-form F.

References

[PIZ13] Patrick Iglesias-Zemmour. *Diffeology,* Mathematical Surveys and Monographs, vol. 185. Am. Math. Soc., Providence R.I. 2013. http://www.ams.org/bookstore-getitem/item=SURV-185

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