# Algorithms for the Multiplication Table Problem

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# Abstract

Let $M(n)$ be the number of distinct entries in the multiplication table for integers $< n$. More precisely, we use a slightly non-standard definition

$$M(n) := \#\{i \times j \mid 0 \le i, j < n\}.$$

The order of magnitude of $M(n)$ was established in a series of papers, starting with Erdős (1950) and ending with Ford (2008), but an asymptotic formula is still unknown. After describing some of the history of $M(n)$ we consider some algorithms for calculating/approximating $M(n)$ for large $n$. This naturally leads to consideration of algorithms, due to Bach (1985–88) and Kalai (2003), for generating random factored integers.

The talk describes work in progress with Carl Pomerance (Dartmouth, New Hampshire) and Jonathan Webster (Butler, Indiana).

# Outline

- History
- Two algorithms for exact computation - naive and incremental
- Two approximate (Monte Carlo) algorithms - Bernoulli and product trials
- Avoiding factoring - algorithms of Bach and Kalai
- Counting divisors
- Numerical results

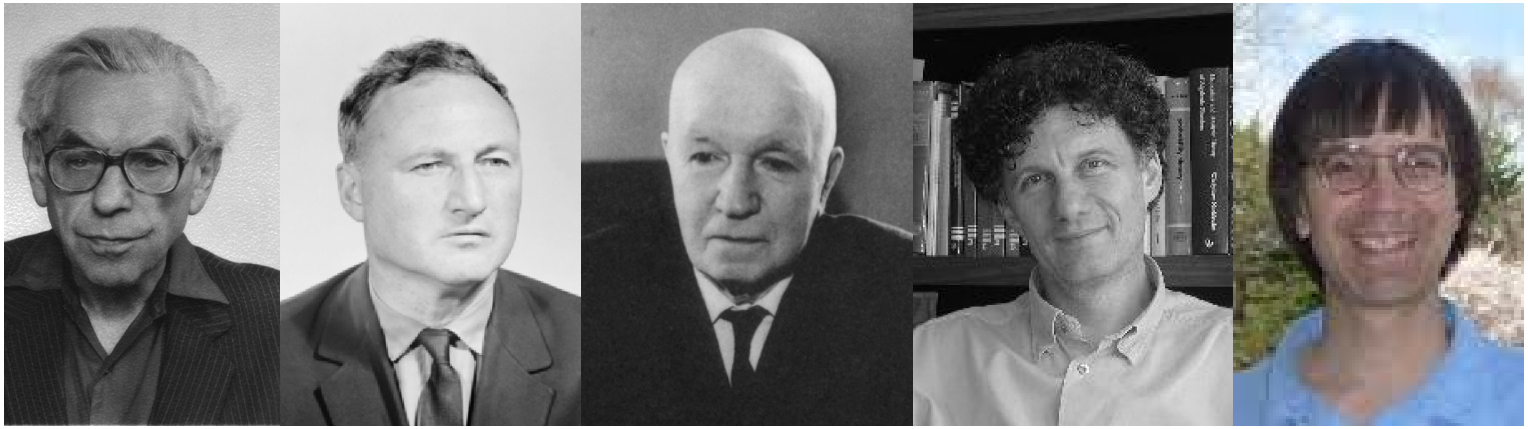
# The multiplication table for $n = 8$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
| 0 | 3 | 6 | 9 | 12 | 15 | 18 | 21 |
| 0 | 4 | 8 | 12 | 16 | 20 | 24 | 28 |
| 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 |
| 0 | 6 | 12 | 18 | 24 | 30 | 36 | 42 |
| 0 | 7 | 14 | 21 | 28 | 35 | 42 | 49 |

We've included in gray borders of zeroes but not the row and column corresponding to multiplication by $n = 8$ (often the opposite convention is used). We could write rows in the reverse order. A set of distinct entries is shown in blue.

The function $M(n)$ is the number of distinct entries in the $n \times n$ multiplication table: $M(8) = 26$ in the example shown. We say that $M(n)$ is the *size* of the table.

# The cast



Erdős Linnik Vinogradov Tenenbaum Ford

Quiz: which of these mathematicians fully understood $M(n)$?

Answer: none!

At least, none could give an asymptotic formula, though Ford at least knew the correct order of magnitude.

# Notation

$f \sim g$ if $\lim_{x\to\infty} f(x)/g(x) = 1$.

$f = o(g)$ if $\lim_{x\to\infty} f(x)/g(x) = 0$.

$f = O(g)$ if, for some constant $K$ and all sufficiently large $x$, $|f(x)| \le K|g(x)|$.

$f \ll g$ means that $f = O(g)$.

$f \gg g$ means that $g \ll f$.

$f \asymp g$ means that $f \ll g$ and $f \gg g$.

$\log(x)$ means a logarithm to any base.

$\ln(x)$ means the natural logarithm.

$\lg(x)$ means $\ln(x)/\ln(2)$, the logarithm to base 2.

# History

There is an easy lower bound

$$M(n) \geq \sum_{\text{prime } p < n} p \gg \frac{n^2}{\ln n}.$$

Erdős (1955, in Hebrew) gave an upper bound $M(n) = o(n^2)$ as $n \to \infty$. After some encouragement by Linnik and Vinogradov, he proved (1960, in Russian) that

$$M(n) = \frac{n^2}{(\ln n)^{c+o(1)}} \text{ as } n \to \infty,$$

where $$c = \int_1^{1/\ln 2} \ln t \, \mathrm{d}t = 1 - \frac{1 + \ln\ln 2}{\ln 2} \approx 0.0861.$$

Tenenbaum (1984, in French) partially clarified the "error term" $(\ln n)^{o(1)}$ but did not give its exact order of magnitude.

# Recent history

Ford (2008, in English) got the exact order-of-magnitude

$$M(n) \asymp \frac{n^2}{(\ln n)^c (\ln \ln n)^{3/2}}, \tag{1}$$

where $c \approx 0.0861$ is as in Erdős's result.

In other words, there exist positive constants $c_1$, $c_2$ such that

$$c_1 g(n) \le M(n) \le c_2 g(n)$$

for all sufficiently large $n$, where $g(n)$ is the RHS of (1).

Ford did not give explicit values for $c_1, c_2$.

# Asymptotic behaviour still unknown

We still do not know if there exists

$$K = \lim_{n \to \infty} \frac{M(n)(\ln n)^c (\ln \ln n)^{3/2}}{n^2},$$

or have any good estimate of the value of $K$ (if it exists).

Ford's result only shows that the lim inf and lim sup are positive and finite.

# Area-time complexity of multiplication

In 1981, RB and H. T. Kung considered how much area $A$ and time $T$ are needed to perform $n$-bit binary multiplication in a model of computation that was meant to be realistic for VLSI circuits. We proved an "area-time" lower bound

$$AT \gg n^{3/2},$$

or more generally, for all $\alpha \in [0, 1]$,

$$AT^{2\alpha} \gg n^{1+\alpha}.$$

To prove this we needed a lower bound on $M(n)$. The easy bound $M(n) \gg n^2 / \ln n$ was sufficient. However, we could have improved the constants in our result if $M(n) \gg n^2 / \ln \ln n$, and we made this conjecture based on numerical evidence (next slide). Erdős wrote to me pointing out that he had disproved our conjecture (in a paper written in Russian).

# Excerpt from Table II of Brent and Kung (1981)

$$n = 2^w, \quad M^*(n) = \frac{n^2}{0.71 + \lg\lg n}, \quad g(n) = \frac{n^2}{(\ln n)^c (\ln\ln n)^{3/2}}.$$

| $w$ | $M(n)$ | $M(n)/M^*(n)$ | $M(n)/g(n)$ |
|---|---|---|---|
| 12 | 3,902,357 | 0.999002 | 0.8606 |
| 13 | 15,202,050 | 0.999089 | 0.8922 |
| 14 | 59,410,557 | 0.999788 | 0.9220 |
| 15 | 232,483,840 | 0.999637 | 0.9490 |
| 16 | 911,689,012 | 0.999788 | 0.9745 |
| 17 | 3,581,049,040 | 1.000005 | 0.9986 |
| 28 | 13,023,772,682,665,849 | 0.997213 | 1.1915 |

On the basis of this table (excluding more recent gray entries), B&K conjectured that $M(n) \sim n^2/\lg\lg n$. This contradicts the result of Erdős.

# Moral

It is hard to tell the difference between $\ln\ln n$ and $(\ln n)^c$ numerically. For $2^6 < n < 2^{30}$, we have $(\ln n)^c \in (1.1, 1.3)$ and $\ln\ln n \in (1.4, 3.1)$.

We find numerically that

$$(\ln n)^c < \ln\ln n \quad \text{for} \quad 20 < n < 10^{5\times 10^{18}},$$

even though

$$(\ln n)^c \gg \ln\ln n.$$

# Exact computation of $M(n)$ – the naive algorithm

It is easy to write a program to compute $M(n)$ for small values of $n$. We need an array $A$ of size $n^2$ bits, indexed $0$ to $n^2 - 1$, which is initially cleared. Then, using two nested loops, set

$$A[i \times j] \leftarrow 1 \quad \text{for} \quad 0 \le i \le j < n.$$

Finally, count the number of bits in the array that are $1$ (or sum the elements of the array). The time and space requirements are both of order $n^2$.

The inner loop of the above program is essentially the same as the inner loop of a program that is sieving for primes. Thus, the same tricks can be used to speed it up. For example, multiplications can be avoided as the inner loop sets bits indexed by an arithmetic progression.

# Segmenting the sieve

If the memory requirement of $n^2$ bits is too large, the problem can be split into pieces. For given $[a_k, a_{k+1}) \subseteq [0, n^2)$, we can count the products $ij \in [a_k, a_{k+1})$. Covering $[0, n^2)$ by a set of disjoint intervals $[a_0, a_1), [a_1, a_2), \ldots$, we can split the problem into as many pieces as desired.

There is a certain startup cost associated with each interval, so the method becomes inefficient if the interval lengths are smaller than $n$.

A parallel program can easily be implemented if each parallel process handles a separate interval $[a_k, a_{k+1})$.

# Exact computation - the incremental algorithm

The naive algorithm takes time $\asymp n^2$ to compute one value $M(n)$. If we want to tabulate $M(k)$ for $1 \le k \le n$, the time required is $\asymp n^3$. A more efficient approach might be to compute the differences $D(n) := M(n+1) - M(n)$.

Assuming we know $M(n)$, we need to consider the products $m \times n$, for $1 \le m \le n$. Let $\delta(n)$ denote the number of these products that have already occurred in the table. The number of elements that have *not* already appeared is $D(n) = n - \delta(n)$. Thus, it is sufficient to compute $\delta(n)$ in order to compute $D(n)$ and then $M(n+1)$.

# Computing $\delta(n)$

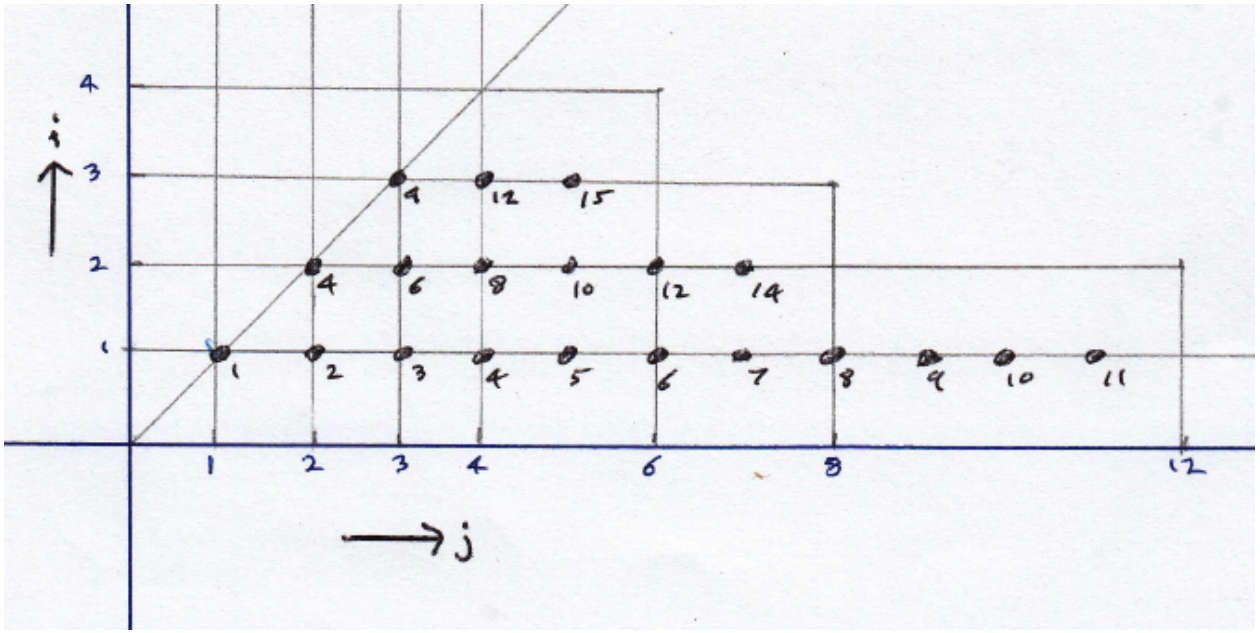
Assume we know the divisors of $n$ (these can be computed by trial division up to $\sqrt{n}$). Suppose that $g|n$, and let $h = n/g$. By symmetry we can assume that $g \leq \sqrt{n} \leq h$.

Can we express $m \times n$ as a product that already occurred in the table? If $m = i \times j$ and $n = g \times h$, then $m \times n = ij \times gh = ih \times jg$. If $ih < n$ and $jg < n$, then the product $ih \times jg$ has already occurred. Observe that $ih < n$ iff $i < g$ and $jg < n$ iff $j < h$.

Thus, to compute $\delta(n)$, we need to count the unique products $ij$ with $0 < i < g$ and $0 < j < n/g$, for each divisor $g \leq \sqrt{n}$ of $n$. This can be done by sieving in an array of size $n$ (the naive algorithm required size $n^2$).

If this is implemented so that work is not duplicated where rectangles of size $g_1 \times n/g_1$ and $g_2 \times n/g_2$ overlap, then the work is bounded by the number of lattice points under the hyperbola $xy = n$. Thus, we can compute $\delta(n)$ in (worst case) time $O(n \log n)$ and space $O(n)$.

# Example: $n = 24$




If $n = 24$, the relevant divisors are $g \in \{2, 3, 4\}$.
If $g = 2$, we sieve over $(i, j) \in \{1\} \times \{1, 2, \ldots, 11\}$.
If $g = 3$, we sieve over $(i, j) \in \{1^*, 2\} \times \{1^*, 2, 3, 4, 5, 6, 7\}$.
If $g = 4$, we sieve over $(i, j) \in \{1^*, 2^*, 3\} \times \{1^*, 2^*, 3, 4, 5\}$.
Starred entries give duplicates so may be omitted.
Conclude that $m \times n$ is already in the table iff
$m \in \{1, 2, \ldots, 11, 12, 14, 15\}$, so $\delta(n) = 14$.
e.g. $12 \times n = 16 \times 18$, $14 \times n = 16 \times 21$, $15 \times n = 20 \times 18$.

# Tabulating $M(1), \ldots, M(n)$

Using the algorithm that we just described to compute the differences $D(n) = M(n+1) - M(n)$ we can compute *all* of $M(1), \ldots, M(n)$ in time $O(n^2 \log n)$ and space $O(n)$ (not counting space for the output).

This is much better than time $O(n^3)$ and space $O(n^2)$ using the naive algorithm!

## Space requirements

In practice, reducing space often reduces time, because of the memory hierarchy built into modern computers.

The space requirement for the incremental algorithm can be reduced to $O(\sqrt{n})$ without changing the time bound, by splitting the sieve into $O(\sqrt{n})$ segments. It could be reduced to $O(n^{1/3}(\log n)^{2/3})$, using ideas due to Harald Helfgott, and maybe even further using ideas due to Oliveira e Silva. What we have implemented is $O(\sqrt{n})$, which is fine for $n \leq 10^9$.

# The distribution of divisors

Let $\tau(n)$ be the number of positive divisors of $n$. ($\tau$ stands for the German *Teiler*. The notation $d(n)$ is often used.)

The mean value of $\tau(k)$ for $1 \le k \le n$ is $\sim \ln n$ [Dirichlet], but *usually* $\tau(n)$ is about $(\ln n)^{\ln 2}$.

More precisely, the *normal order* of $\ln \tau(n)$ is $\ln(2) \ln \ln n$, which is a way of saying that, for all positive $\varepsilon$, $\tau(n)$ is almost always in $[(\ln n)^{\ln(2)-\varepsilon}, (\ln n)^{\ln(2)+\varepsilon}]$.

Heuristically, this is because the number $\omega(n)$ of distinct *prime* divisors of $n$ is asymptotically normal with mean and variance $\ln \ln n$ [Erdős-Kac theorem], but the number of (not necessarily prime) divisors is $2^{\omega(n)}$, assuming that $n$ is square-free. Thus, the distribution of divisors has a long tail.

The situation is slightly more complicated if $n$ is not square-free, but the conclusion remains the same.

# Improved time bound for tabulation

The time bound $O(n^2 \log n)$ that we gave for tabulating $M(1), \ldots, M(n)$ is not best possible. The worst case $\asymp n \log n$ for computing $\delta(n)$ occurs only when $n$ has $\gg \log n$ divisors. This is rare, so on average the incremental algorithm runs in time $o(n \log n)$, so tabulation takes time $T(n) = o(n^2 \log n)$.

Recall that, by a result of Kevin Ford,

$$M(n) \asymp \frac{n^2}{(\log n)^c (\log \log n)^{3/2}} .$$

We can show, using a (different, but related) result of Ford, that

$$T(n) \asymp n^2 \frac{(\log n)^{1-c}}{(\log \log n)^{3/2}} ,$$

so

$$T(n) \asymp M(n) \log n.$$

No "simple" proof that does not require Ford's results is known. It is a curious case where the run-time of an algorithm depends in a nontrivial way on the output of the algorithm!

# OEIS A027417

The OEIS sequence A027417 is defined by $a_n = M(2^n)$. Until recently only $a_0, \ldots, a_{25}$ were listed.

Using parallel implementations of the naive and incremental algorithms, we (RB and JW) have extended the computation to $a_{28}$.

| $n$ | $a_n$ | $4^n/a_n$ |
|---|---|---|
| 1 | 2 | 2.0000 |
| 2 | 7 | 2.2857 |
| 3 | 26 | 2.4615 |
| 4 | 90 | 2.8444 |
| ... | ... | ... |
| 25 | 209962593513292 | 5.3624 |
| 26 | 830751566970327 | 5.4211 |
| 27 | 3288580294256953 | 5.4779 |
| 28 | 13023772682665849 | 5.5328 |

# Some details

Results up to $a_{27} = M(2^{27})$ were computed by JW and RB using different programs (based on essentially the same algorithm, but implemented independently in different languages). They were also verified by RB using a parallel implementation of the naive algorithm.

The computation to $a_{28} = M(2^{28})$ took about one week on 208 xeon3 and xeon4 cores, using the University of Newcastle's ARCS (Academic Research Computing Support) facility. Although not yet verified by an independent computation, the result is consistent with estimates obtained by a Monte Carlo method (more on such methods later).

The runtime for our program is $\approx 10^{-13} M(n) \lg(n)$ in units of xeon3 processor-hours.

# The scaled sequence

Since $M(n) \asymp \dfrac{n^2}{(\ln n)^c (\ln \ln n)^{3/2}}$, replace $n$ by $N = 2^n$ and define

$$b_n := \frac{N^2}{M(N)} \asymp n^c (\ln n)^{3/2} \text{ and } f_n := \frac{b_n}{n^c (\ln n)^{3/2}} \asymp 1.$$

| $n$ | $b_n$ | $f_n$ |
|---|---|---|
| 5 | 3.0118 | 1.284 |
| 10 | 4.0366 | 0.948 |
| 15 | 4.6186 | 0.821 |
| 20 | 5.0331 | 0.750 |
| 25 | 5.3624 | 0.704 |

$f_n$ appears to be monotonic decreasing. It is not clear from the table what $\lim_{n\to\infty} f_n$ is, or even if the limit exists. An estimate, using much larger values of $n$, is $\lim_{n\to\infty} f_n \approx 0.116$.

# Monte Carlo computation

We can estimate $M(n)$ using two different Monte Carlo methods. Recall that

$$M(n) \;=\; \#S_n, \quad S_n = \{ij : 0 \le i < n,\ 0 \le j < n\}.$$

**Bernoulli trials**

We can generate a random integer $x \in [0, n^2)$, and count a *success* if $x \in S_n$. Repeat several times and estimate

$$\frac{M(n)}{n^2} \approx \frac{\#\text{successes}}{\#\text{trials}}.$$

To check if $x \in S_n$ we need to find some of the divisors of $x$, which probably requires the prime factorisation of $x$. There is no obvious algorithm that is much more efficient than finding the prime factors of $x$.

# Monte Carlo computation - alternative method

There is another Monte Carlo algorithm, using what we call *product trials*.

Generate random integers $x, y \in [0, n)$. Count the number $\nu = \nu(xy)$ of ways that we can write $xy = ij$ with $i < n$, $j < n$. Repeat several times, and estimate

$$\frac{M(n)}{n^2} \approx \frac{\sum 1/\nu}{\#\text{trials}} .$$

This works because $z \in S_n$ is sampled at each trial with probability $\nu(z)/n^2$, so the weight $1/\nu(z)$ is necessary to give an unbiased estimate of $M(n)/n^2$.

To compute $\nu(xy)$ we need to find the divisors of $xy$.

Note that $x, y < n$, whereas for Bernoulli trials $x < n^2$, so the integers considered in product trials are generally smaller than those considered in Bernoulli trials.

# Comparison

For Bernoulli trials, $p = M(n)/n^2$ is the probability of a success, and the distribution after $T$ trials has mean $pT$, variance $p(1-p)T \approx pT$.

For product trials, we know $\mathbb{E}(1/\nu) = M(n)/n^2 = p$, but we do not know $\mathbb{E}(1/\nu^2)$ theoretically. We can estimate it from the sample variance.

It turns out that, for a given number $T$ of trials, the product method has smaller expected error (by a factor of 2 to 3 in typical cases).

This is not the whole story, because we also need to factor $x$ (for Bernoulli trials) or $xy$ (for product trials), and then find (some of) their divisors.

For large $n$, the most expensive step is factoring, which is easier for product trials because the numbers involved are smaller.

# Avoiding factoring large integers

We can avoid the factoring steps by generating random integers together with their factorisations, using algorithms due to Bach (1988) or Kalai (2003).

Bach's algorithm is more efficient than Kalai's, but also much more complicated, so I will describe Kalai's algorithm and refer you to Bach's paper, or the book *Algorithmic Number Theory* (by Bach and Shallit), for a description of Bach's algorithm.

# Kalai's algorithm

*Input:* Positive integer $n$.

*Output:* A random integer $r$, uniformly distributed in $[1, n]$, and its prime power factorisation.

1. Generate a sequence $n = s_0 \geq s_1 \geq \cdots \geq s_\ell = 1$ by choosing $s_{i+1}$ uniformly in $[1, s_i]$ until reaching 1.
2. Let $r$ be the product of all the prime $s_i$, $i > 0$.
3. If $r \leq n$, output $r$ and its prime factorisation with probability $r/n$, otherwise restart at step 1.

Kalai's algorithm clearly outputs an integer $r \in [1, n]$ and its prime factorisation, but why is $r$ uniformly distributed in $[1, n]$? The answer is the *acceptance-rejection* step 3.

# Correctness of Kalai's algorithm

Kalai shows that, if $1 \le R \le n$, then (after step 2)

$$\mathbb{P}\mathrm{rob}[r = R] = \frac{\mu_n}{R}\,,$$

where

$$\mu_n = \prod_{p \text{ prime},\, p \le n} (1 - 1/p).$$

If $1 \le r \le n$, then step 3 accepts $r$ with probability $r/n$, so the probability of outputting $r$ at step 3 is proportional to

$$\frac{\mu_n}{r}\frac{r}{n} = \frac{\mu_n}{n}\,,$$

which is independent of $r$. Thus, the output is uniformly distributed in $[1, n]$.

# The expected running time

The running time is dominated by the time for primality tests. The expected number of primality tests is $H_n/\mu_n$, where

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + O\left(\frac{1}{n}\right).$$

By a theorem of Mertens (1874),

$$\frac{1}{\mu_n} \sim e^{\gamma} \ln n,$$

so the expected number of primality tests is

$$\sim e^{\gamma} (\ln n)^2.$$

# Bach's algorithm

Bach's algorithm requires prime power tests which are (slightly) more expensive than primality tests. However, it is possible to modify the algorithm so that only primality tests are required. (This is what we implemented.)

Bach's algorithm is more efficient than Kalai's – the expected number of primality tests is of order $\log n$. The reason is that Bach's algorithm generates factored integers uniform in $(n/2, n]$ rather than $[1, n]$, which makes the acceptance/rejection process more efficient.

We can generate integers in $[1, n]$ by calling Bach's algorithm appropriately. First choose an interval $(m/2, m] \subseteq [1, n]$ with the correct probability $\lceil m/2 \rceil / n$, then call Bach's algorithm.

# Primality testing

For large $n$, the main cost of both Bach's algorithm and Kalai's algorithm is the primality tests.

Since we are using Monte Carlo algorithms, it seems reasonable to use the Miller-Rabin probabilistic primality test, which has a nonzero (but tiny) probability of error, rather than a much slower "guaranteed" test such as the polynomial-time deterministic test of Agrawal, Kayal and Saxena (AKS), or the randomised but error-free ("Las Vegas") elliptic curve test (ECPP) of Atkin and Morain.

The Miller-Rabin test makes it feasible to use Bach's or Kalai's algorithm for $n$ up to say $2^{1000}$.

# Divisors (again)

An integer

$$x = \prod p_i^{\alpha_i}$$

has

$$\tau(x) = \prod(\alpha_i + 1)$$

distinct divisors, each of the form $\prod p_i^{\beta_i}$ for $0 \le \beta_i \le \alpha_i$.

We do not need all the divisors of the the random integers $x, y$ that occur in our Monte Carlo computation. We only need the divisors in a certain interval.

We'll consider the algorithms using Bernoulli and product trials separately.

# Bernoulli and product trials

## Bernoulli trials

For Bernoulli trials, we merely need to know if a given $x < n^2$ has a divisor $d < n$ such that $x/d < n$, i.e. $x/n < d < n$. Thus, given $n$ and $x \in [1, n)$, it is enough to compute the divisors of $x$ in the interval $(x/n, n)$ until we find one, or show that there are none.

## Product trials

For product trials we generate random (factored) $x, y < n$ and need (some of) the divisors of $xy$. We can easily compute the prime-power factorisation of $z := xy$ from the factorisations of $x$ and $y$. We then need to count the divisors of $z$ in the interval $(z/n, n)$.

# Cost of Bernoulli and product trials

An integer $x \in [1, n^2)$ has on average

$$\sim \ln n^2 = 2 \ln n$$

divisors [Dirichlet]. This is relevant for Bernoulli trials.

However, for product trials, our numbers $z = xy$ have on average $\gg (\ln n)^2$ divisors, because $x$ and $y$ have on average $\sim \ln n$ divisors.

Thus, the divisor computation for product trials is more expensive than that for Bernoulli trials.

# Counting divisors in an interval

We can count the divisors of $x$ in a given interval $[a, b]$ faster than actually computing all the divisors in this interval, by using a "divide and conquer" approach.

Here is an outline. Write $x = uv$ where $(u, v) = 1$ and $u$, $v$ have about equal numbers of divisors. Find all the divisors of $v$ and sort them. Then, for each divisor $d$ of $u$, compute bounds $a \le d' \le b$ for relevant divisors $d'$ of $v$, and search for $a$ and $b$ in the sorted list, using a binary search.

The expected running time is roughly (ignoring $\ln \ln n$ factors) proportional to the mean value of $\tau(x)^{1/2}$ over $x \le n$. By a result of Ramanujan [Montgomery and Vaughan, (2.27)], this is $\asymp (\ln n)^{\alpha}$, where $\alpha = \sqrt{2} - 1 \approx 0.4142$.

Thus, for Bernoulli trials the cost is $O(\log^{\alpha} n)$ and for product trials the cost is $O(\log^{2\alpha} n)$.

# Avoiding some primality testing

When $n$ is very large, say greater than $2^{1000}$, even Miller-Rabin probabilistic primality testing becomes very expensive. We can avoid primality tests on large numbers by making a plausible assumption about the density of primes in short intervals (but not so short that Maier's theorem applies). There is no time to discuss this in detail today. If you are interested, see the slides from my Hong Kong talk (February 2015), which are on my website.

Using the density assumption, we have estimated $M(n)$ with an accuracy of three or more significant digits for $n$ up to $2^{5\times 10^8}$.

# Numerical results

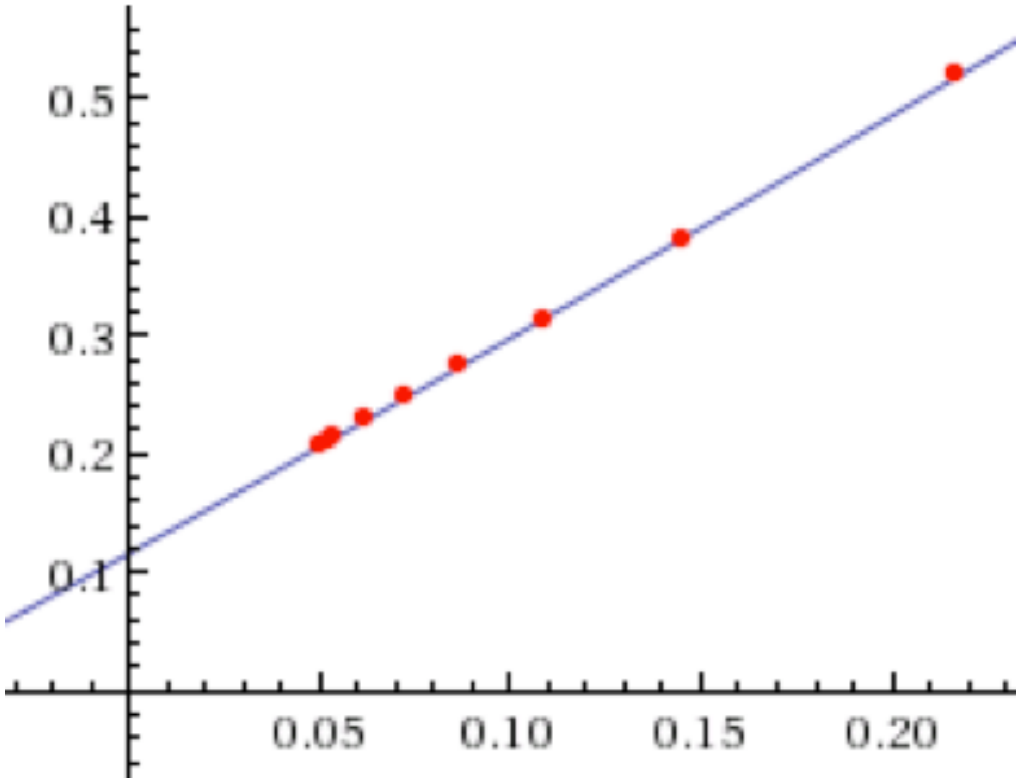
$$N = 2^n, \quad b_n := \frac{N^2}{M(N)}, \quad f_n := \frac{b_n}{n^c (\ln n)^{3/2}} \asymp 1.$$

| $n$ | $b_n$ | $f_n$ |
|---|---|---|
| 10 | 4.0366 | 0.948 |
| $10^2$ | 7.6272 | 0.519 |
| $10^3$ | 12.526 | 0.381 |
| $10^4$ | 19.343 | 0.313 |
| $10^5$ | 28.74 | 0.273 |
| $10^6$ | 41.6 | 0.247 |
| $10^7$ | 59.0 | 0.228 |
| $10^8$ | 82.7 | 0.214 |
| $2\times10^8$ | 90.9 | 0.210 |
| $5\times10^8$ | 103.4 | 0.206 |

The value of $\lim_{n\to\infty} f_n$ is not obvious from this table!

# Least squares quadratic fit

Fitting $f_n$ by a quadratic in $x = (\ln n)^{-1}$ to the data for $n = 10^2, 10^3, \ldots, 5 \times 10^8$ (as in the previous table) gives $f_n \approx 0.1157 + 1.7894x + 0.2993x^2$.



# Conclusion

On the basis of the numerical results, a plausible conjecture is

$$b_n = N^2/M(N) \sim c_0 n^c (\ln n)^{3/2}, \quad c_0 \approx 0.1157,$$

which suggests

$$M(N) \sim K \frac{N^2}{(\ln N)^c (\ln \ln N)^{3/2}}$$

with

$$K = \frac{(\ln 2)^c}{c_0} \approx 8.4.$$

This estimate of $K$ might be inaccurate, since we have only taken three terms in a plausible (but not proved) asymptotic series

$$f_n \sim c_0 + c_1/\ln n + c_2/(\ln n)^2 + \cdots,$$

and the first two terms are of the same order of magnitude in the region where we can estimate $M(N) = M(2^n)$.

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