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OPEN CONJECTURES ON CONGRUENCES

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Abstract. We collect here 100 open conjectures on congruences made by the author, some of which have never been published. This is a new edition of the author's preprint `arXiv:0911.5665` with those confirmed conjectures removed and some new conjectures added. Many congruences here are related to representations of primes by binary quadratic forms or series for powers of π; for example, we mention two new conjectural identities

$$\sum_{n=0}^\infty \frac{12n+1}{100^n}\binom{2n}{n}\sum_{k=0}^n\binom{2k}{k}^2\binom{2(n-k)}{n-k}\left(\frac{9}{4}\right)^{n-k}=\frac{75}{4\pi}$$

and

$$\sum_{k=1}^\infty \frac{3H_{k-1}^2+4H_{k-1}/k}{k^2\binom{2k}{k}}=\frac{\pi^4}{360}\ \text{ with } H_{k-1}:=\sum_{0<j\leqslant k-1}\frac{1}{j},$$

and include related congruences. We hope that this paper will interest number theorists and stimulate further research.

1 Introduction

Congruences modulo primes have been widely investigated since the time of Fermat. However, we find that there are still lots of new challenging congruences that cannot be easily solved. They appeal for new powerful tools or advanced theory.

Here we collect 100 conjectures of the author on congruences. Many of them can be found in the author's papers available from `arxiv` or his homepage, but some are first published here. Most of the congruences here are *supercongruences* in the sense that they happen to hold modulo some higher power of a prime. The topic of supercongruences is related to the p-adic Γ-function, Gauss and Jacobi sums, hypergeometric series, modular forms, Calabi-Yau manifolds, and some sophisticated combinatorial identities involving harmonic numbers (cf. [1, 54]). The recent theory of super congruences also involves Bernoulli and Euler numbers (see [72, 75, 88, 101]) and various series related to π or the Riemann zeta function (cf. [106, 73, 92, 93, 104, 114]); for van Hamme's philosophy to find p-adic congruences from series and the author's philosophy to find series from

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congruences, see van Hamme [106] and Z.-W. Sun [84]. In particular, the author's previous papers [84, 90, 92] contains many conjectural congruences related to series for powers of π. Many congruences collected here are about $\sum_{k=0}^{p-1} a_k/m^k$ modulo powers of a prime p, where m is an integer not divisible by p and the quantity a_k is a sum or a product of some binomial coefficients which usually arises from enumerative combinatorics.

For the sake of clarity, we often state the prime version of a conjecture instead of the general version. We do not exhaust all congruences conjectured by the author but select some typical ones. For many new conjectures, we add the exact dates when the author discovered them.

Now we introduce some basic notation in this paper.

As usual, we set

$$\mathbb{N} = \{0, 1, 2, \ldots\} \text{ and } \mathbb{Z}^+ = \{1, 2, 3, \ldots\}.$$

For an integer m and a positive odd number n, the notation $(\frac{m}{n})$ stands for the Jacobi symbol. For a prime p and an integer $a \not\equiv 0 \pmod p$, we call

$$q_p(a) := \frac{a^{p-1} - 1}{p} \in \mathbb{Z}$$

a Fermat quotient. For a polynomial or a power series $P(x)$, we write $[x^n]P(x)$ for the coefficient of x^n in the expansion of $P(x)$. For $k_1, \ldots, k_n \in \mathbb{N}$, we define the multinomial coefficient

$$\binom{k_1 + \cdots + k_n}{k_1, \ldots, k_n} := \frac{(k_1 + \cdots + k_n)!}{k_1! \cdots k_n!}.$$

The harmonic numbers are given by

$$H_0 = 0 \text{ and } H_n = \sum_{k=1}^{n} \frac{1}{k} \quad (n = 1, 2, 3, \ldots).$$

For each $m = 2, 3, \ldots$, the harmonic numbers of order m are given by

$$H_n^{(m)} := \sum_{0<k\leqslant n} \frac{1}{k^m} \quad (n \in \mathbb{N}).$$

A classical theorem of J. Wolstenholme asserts that

$$H_{p-1} \equiv 0 \pmod{p^2} \text{ and } H_{p-1}^{(2)} \equiv 0 \pmod p$$

for any prime $p > 3$. Another useful result of E. Lehmer [38] states that

$$H_{(p-1)/2} \equiv -2q_p(2) + p\, q_p(2)^2 \pmod{p^2}$$

for each odd prime p. The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } \operatorname{Re}(s) > 1.$$

The Catalan numbers are those integers

$$C_n := \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} \quad (n \in \mathbb{N}).$$

Note that if p is an odd prime then

$$\binom{2k}{k}=\frac{(2k)!}{(k!)^2}\equiv 0 \pmod{p}\quad \text{for every } k=\frac{p+1}{2},\ldots,p-1.$$

The Bernoulli numbers $B_0,B_1,B_2,\ldots$ are rational numbers given by

$$B_0=1,\quad \text{and}\quad \sum_{k=0}^{n}\binom{n+1}{k}B_k=0\quad \text{for } n\in\mathbb{Z}^+.$$

It is well known that $B_{2n+1}=0$ for all $n\in\mathbb{Z}^+$ and

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}\quad (0<|x|<2\pi).$$

The Euler numbers $E_0,E_1,E_2,\ldots$ are integers defined by

$$E_0=1,\quad \text{and}\quad \sum_{\substack{k=0\\ 2\mid k}}^{n}\binom{n}{k}E_{n-k}=0\quad \text{for } n\in\mathbb{Z}^+.$$

It is well known that $E_{2n+1}=0$ for all $n\in\mathbb{N}$ and

$$\sec x=\sum_{n=0}^{\infty}(-1)^nE_{2n}\frac{x^{2n}}{(2n)!}\quad \left(|x|<\frac{\pi}{2}\right).$$

The Bernoulli polynomials and the Euler polynomials are given by

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k}\ \text{ and }\ E_n(x)=\sum_{k=0}^{n}\binom{n}{k}\frac{E_k}{2^k}\left(x-\frac{1}{2}\right)^{n-k}\quad (n\in\mathbb{N}).$$

For $A,B\in\mathbb{Z}$, we define the Lucas sequences $u_n=u_n(A,B)$ $(n\in\mathbb{N})$ and $v_n=v_n(A,B)$ $(n\in\mathbb{N})$ as follows:

$$u_0=0,\ u_1=1,\ \text{and } u_{n+1}=Au_n-Bu_{n-1}\ (n=1,2,3,\ldots);$$

$$v_0=2,\ v_1=A,\ \text{and } v_{n+1}=Av_n-Bv_{n-1}\ (n=1,2,3,\ldots).$$

The sequence $F_n=u_n(1,-1)$ $(n\in\mathbb{N})$ is called the Fibonacci sequence, and those numbers $L_n=v_n(1,-1)$ $(n\in\mathbb{N})$ are called Lucas numbers. It is well known that $u_{p-(\frac{A^2-4B}{p})}(A,B)\equiv 0 \pmod{p}$ for any prime $p\nmid 2B$ (see, e.g., [70]).

Let p be a prime. As usual we let $\mathbb{Z}_p$ denote the ring of all p-adic integers. For $x\in\mathbb{Z}_p$, we use $\langle x\rangle_p$ to denote the unique $r\in\{0,\ldots,p-1\}$ with $x\equiv r \pmod{p}$. For a nonzero integer m, its p-adic valuation (or p-adic order) is given by

$$\nu_p(m):=\max\{n\in\mathbb{N}:\ p^n\mid m\}.$$

We consider $\nu_p(0)$ as $+\infty$. For a rational number $x=a/b$ with $a\in\mathbb{Z}$ and $b\in\mathbb{Z}^+$, we define $\nu_p(x)=\nu_p(a)-\nu_p(b)$.

2 Congruences mainly Involving Binomial Coefficients

Conjecture 1. (2009-11-02) *If $n>1$ is an odd integer satisfying the Morley congruence*

$$\binom{n-1}{(n-1)/2}\equiv(-1)^{(n-1)/2}4^{n-1}\pmod{n^3},$$

then n must be a prime.

Remark 1. In 1895 F. Morley [48] showed that

$$\binom{p-1}{(p-1)/2}\equiv(-1)^{(p-1)/2}4^{p-1}\pmod{p^3}$$

for any prime $p>3$. In 2009 the author verified the conjecture for all odd numbers $1<n<10^4$. If Conjecture 1 indeed holds, then we have a new characterization of primes $p>3$ via Morley's congruence. In 1953, L. Carlitz [5] showed that

$$(-1)^{(p-1)/2}\binom{p-1}{(p-1)/2}\equiv 4^{p-1}+\frac{p^3}{12}B_{p-3}\pmod{p^4}$$

for any prime $p>3$. Note that $B_{p-3}\equiv 0\pmod p$ for the prime $p=16843$. For the odd composite number $n=16843^2$, the integer $\binom{n-1}{(n-1)/2}-(-1)^{(n-1)/2}4^{n-1}$ is divisible by n^2 but not divisible by n^3.

Conjecture 2. *Let p be an odd prime and let $n\in\mathbb{Z}^+$.*

(i) (Z.-W. Sun [77]) *The number*

$$\frac{1}{n\binom{2n}{n}}\sum_{k=0}^{n-1}\binom{(p-1)k}{k,\ldots,k}$$

is always a p-adic integer, where $\binom{(p-1)k}{k,\ldots,k}$ is the multi-nomial coefficient $((p-1)k)!/(k!)^{p-1}$.

(ii) *The number*

$$\frac{1}{n\binom{(p-1)n}{\frac{p-1}{2}n}}\sum_{k=0}^{n-1}\binom{(p-1)k}{k,\ldots,k}$$

is always a p-adic integer.

(iii) *If $2<n\leqslant p$ and $2\nmid n$, then*

$$\frac{1}{\binom{(p-1)n}{(p-1)n/2}}\sum_{k=0}^{n-1}\binom{(p-1)k}{k,\ldots,k}\equiv 0\pmod p.$$

If $p+1<n\leqslant 2p$ and $2\mid n$, then

$$\frac{1}{p\binom{(p-1)n}{(p-1)n/2}}\sum_{k=0}^{n-1}\binom{(p-1)k}{k,\ldots,k}\equiv(n-1)a_p\pmod p$$

for some p-adic integer a_p not depending on n. If $2<n<p$ and $2\mid n$, then

$$\frac{1}{\binom{(p-1)n}{(p-1)n/2}}\sum_{k=0}^{n-1}\binom{(p-1)k}{k,\ldots,k}\equiv 2-\left(B_{p-1}+\frac{1}{p}\right)\pmod p.$$

Remark 2. (a) Let p be a prime. By the von Staudt-Clausen theorem (cf. [35, p. 233]), $B_{p-1}+1/p$ is p-adic integral. The author [77, Theorem 1.2] showed that

$$\sum_{k=0}^{p-1}\binom{(p-1)k}{k,\ldots,k}\equiv pB_{p-1}+(-1)^{p-1}-2p\pmod{p^2}$$

and determined $\sum_{k=0}^{n-1}\binom{(p-1)k}{k,\ldots,k}$ modulo p for any $n\in\mathbb{Z}^+$. Sun [77] also proved that an integer $n>1$ is a prime if and only if

$$\sum_{k=0}^{n-1}\binom{(n-1)k}{k,\ldots,k}\equiv 0\pmod{n}.$$

(b) In 1992 N. Strauss, J. Shallit, D. Zagier [63] showed that for any $n\in\mathbb{Z}^+$ we have

$$\nu_3\left(\sum_{k=0}^{n-1}\binom{2k}{k}\right)=\nu_3\left(n\binom{2n}{n}\right).$$

So, parts (i) and (ii) of Conjecture 2 hold for $p=3$. V.J.W. Guo and J. Zeng [24] conjectured that $\nu_5(\sum_{k=0}^{n-1}\binom{4k}{2k}\binom{2k}{k}^2)\geqslant\nu_5(n)$ for all $n\in\mathbb{Z}^+$.

Conjecture 3. (Sun [96]) *Let p be an odd prime and let $n\in\mathbb{Z}^+$.*

(i) *For any integer $m\not\equiv 0\pmod p$, we have*

$$\frac{1}{n\binom{2n-1}{n-1}}\left(\sum_{k=0}^{pn-1}\frac{\binom{2k}{k}}{m^k}-\left(\frac{\Delta}{p}\right)\sum_{r=0}^{n-1}\frac{\binom{2r}{r}}{m^r}\right)\equiv\frac{u_{p-(\frac{\Delta}{p})}(m-2,1)}{m^{n-1}}\pmod{p^2},\tag{2.1}$$

where $\Delta=m(m-4)$.

(ii) *We have*

$$\frac{\sum_{k=0}^{pn-1}\binom{2k}{k}-(\frac{p}{3})\sum_{r=0}^{n-1}\binom{2r}{r}}{n^2\binom{2n-1}{n-1}}\equiv\sum_{k=0}^{p-1}\binom{2k}{k}-\left(\frac{p}{3}\right)\pmod{p^4}.$$

(iii) *If $p>3$, $m\in\{2,3\}$ and $\Delta=m(m-4)$, then there is a p-adic integer $c_p^{(m)}$ only depending on p and m such that for any $n\in\mathbb{Z}^+$ we have*

$$\frac{m^{n-1}}{n^2\binom{2n-1}{n-1}}\left(\sum_{k=0}^{pn-1}\frac{\binom{2k}{k}}{m^k}-\left(\frac{\Delta}{p}\right)\sum_{r=0}^{n-1}\frac{\binom{2r}{r}}{m^r}\right)$$
$$\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{m^k}-\left(\frac{\Delta}{p}\right)+p^3c_p^{(m)}(n-1)\pmod{p^4}.$$

Remark 3. The author [96] showed that if we multiply both sides of (2.1) by $\binom{2n-1}{n-1}$ then the new version of (2.1) is true. Sun and R. Tauraso [99] proved that $\sum_{k=0}^{p-1}\binom{2k}{k}\equiv(\frac{p}{3})\pmod{p^2}$ for any prime p. The author [75] obtained that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{2^k}\equiv\left(\frac{-1}{p}\right)-p^2E_{p-3}\pmod{p^3}\ \text{ and }\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{3^k}\equiv\left(\frac{p}{3}\right)\pmod{p^2}$$

for any prime $p>3$.

Conjecture 4. (Sun [96]) *Let p be an odd prime and let $n\in\mathbb{Z}^+$.*

(i) *If $p>3$ then*

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\binom{pn-1}{k}\frac{\binom{2k}{k}}{(-3)^k}-\left(\frac{p}{3}\right)\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\binom{2r}{r}}{(-3)^r}\right)\in\mathbb{Z}_p.$$

(ii) *If n is odd, then*

$$\frac{1}{(pn)^2\binom{n-1}{(n-1)/2}}\left(\sum_{k=0}^{(pn-1)/2}\frac{\binom{2k}{k}}{8^k}-\left(\frac{2}{p}\right)\sum_{r=0}^{(n-1)/2}\frac{\binom{2r}{r}}{8^r}\right)\in\mathbb{Z}_p$$

and

$$\frac{1}{(pn)^2\binom{n-1}{(n-1)/2}}\left(\sum_{k=0}^{(pn-1)/2}\frac{\binom{2k}{k}}{16^k}-\left(\frac{3}{p}\right)\sum_{r=0}^{(n-1)/2}\frac{\binom{2r}{r}}{16^r}\right)\in\mathbb{Z}_p.$$

Remark 4. Let $p>2$ be a prime. Sun [80] determined $\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{k}/(-m)^k$ modulo p^2 for any integer $m\not\equiv0\pmod p$; in particular, he showed that

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\frac{\binom{2k}{k}}{(-3)^k}\equiv\left(\frac{p}{3}\right)\pmod{p^2}$$

if $p>3$. The author [82] also determined $\sum_{k=0}^{(p-1)/2}\binom{2k}{k}/m^k$ modulo p^2 for any integer $m\not\equiv0\pmod p$; in particular, he proved that

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{8^k}\equiv\left(\frac{2}{p}\right)\pmod{p^2}\ \text{ and }\ \sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{16^k}\equiv\left(\frac{3}{p}\right)\pmod{p^2}.$$

Conjecture 5. (Sun [82]) *Let p be an odd prime and let $a\in\mathbb{Z}^+$.*

(i) *If $p\equiv1\pmod 3$ or $a>1$, then*

$$\sum_{k=0}^{\lfloor\frac{5}{6}p^a\rfloor}\frac{\binom{2k}{k}}{16^k}\equiv\left(\frac{3}{p^a}\right)\pmod{p^2}.$$

For any $n\in\mathbb{N}$ we have

$$\frac{1}{(2n+1)^2\binom{2n}{n}}\sum_{k=0}^n\frac{\binom{2k}{k}}{16^k}\equiv\begin{cases}1\pmod 9&\text{if }3\mid n,\\ 4\pmod 9&\text{if }3\nmid n.\end{cases}$$

Also,

$$\frac{1}{3^{2a}}\sum_{k=0}^{(3^a-1)/2}\frac{\binom{2k}{k}}{16^k}\equiv(-1)^a10\pmod{27}.$$

(ii) *Suppose $p\neq5$. If $p^a\equiv1,2\pmod 5$ or $p\equiv2\pmod 5$ or $a>2$, then*

$$\sum_{k=0}^{\lfloor\frac45p^a\rfloor}(-1)^k\binom{2k}{k}\equiv\left(\frac{5}{p^a}\right)\pmod{p^2}.$$

If $p^a\equiv1,3\pmod 5$ or $p\equiv3\pmod 5$ or $a>2$, then

$$\sum_{k=0}^{\lfloor\frac35p^a\rfloor}(-1)^k\binom{2k}{k}\equiv\left(\frac{5}{p^a}\right)\pmod{p^2}.$$

(iii) *If $p^a\equiv1,2\pmod 5$ or $p\equiv2\pmod 5$ or $a>2$, then*

$$\sum_{k=0}^{\lfloor\frac{7}{10}p^a\rfloor}\frac{\binom{2k}{k}}{(-16)^k}\equiv\left(\frac{5}{p^a}\right)\pmod{p^2}.$$

If $p^a\equiv1,3\pmod 5$ or $p\equiv3\pmod 5$ or $a>2$, then

$$\sum_{k=0}^{\lfloor\frac{9}{10}p^a\rfloor}\frac{\binom{2k}{k}}{(-16)^k}\equiv\left(\frac{5}{p^a}\right)\pmod{p^2}.$$

Remark 5. Let $(F_n)_{n\geqslant0}$ be the Fibonacci sequence. For any prime $p\neq2,5$ and $a\in\mathbb{Z}^+$, Pan and the author [57] proved that

$$\sum_{k=0}^{p^a-1}(-1)^k\binom{2k}{k}\equiv\left(\frac{p^a}{5}\right)\left(1-2F_{p^a-(\frac{p^a}{5})}\right)\pmod{p^3}$$

which is [98, Conjecture 3.1], and Sun [82] proved that

$$\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{(-16)^k}\equiv\left(\frac{p^a}{5}\right)\left(1+\frac{F_{p^a-(\frac{p^a}{5})}}{2}\right)\pmod{p^3}.$$

Conjecture 6. (Sun [72, 92]) *Let p be an odd prime. Then*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k2^k}\equiv-\frac{H_{(p-1)/2}}{2}+\frac{7}{16}p^2B_{p-3}\pmod{p^3}.$$

If $p>3$, then
$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k3^k}\equiv-2\sum_{\substack{k=1\\ k\not\equiv p\ (\mathrm{mod}\ 3)}}^{p-1}\frac{1}{k}\pmod{p^3}.$$
If $p>5$, then
$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^24^k}\equiv-\frac{H_{(p-1)/2}^2}{2}-\frac{7}{4}\cdot\frac{H_{p-1}}{p}\pmod{p^3}.$$

Remark 6. The congruences in Conjecture 6 were motivated by the following known identities:
$$\sum_{k=1}^{\infty}\frac{2^k}{k^2\binom{2k}{k}}=\frac{\pi^2}{8},\quad\sum_{k=1}^{\infty}\frac{3^k}{k^2\binom{2k}{k}}=\frac{2}{9}\pi^2,\quad\sum_{k=1}^{\infty}\frac{\binom{2k}{k}}{k^24^k}=\frac{\pi^2-3\log^2 4}{6}.$$
The author and Tauraso [98] showed that for any prime $p>3$ we have
$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv\frac{8}{9}p^2B_{p-3}\pmod{p^3}.$$
Sun [75] also proved that
$$-\frac{1}{2p}\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}}{k}\equiv\sum_{k=1}^{(p-1)/2}\frac{1}{k^2\binom{2k}{k}}\equiv\left(\frac{-1}{p}\right)\frac{4}{3}E_{p-3}\pmod{p}$$
and
$$\sum_{k=1}^{(p-1)/2}\frac{4^k}{k^2\binom{2k}{k}}\equiv\left(\frac{-1}{p}\right)4E_{p-3}\pmod{p}$$
for all primes $p>3$. Tauraso [103] showed that $\sum_{k=1}^{p-1}\binom{2k}{k}/(k4^k)\equiv-H_{(p-1)/2}\pmod{p^3}$ for any prime $p>5$. Via computation the author recently observed that
$$\sum_{k=1}^{(p-1)/2}\frac{2^k}{k^2\binom{2k}{k}}\equiv\left(\frac{-2}{p}\right)\frac{1}{4}E_{p-3}\left(\frac{1}{4}\right)\pmod{p}$$
and
$$\sum_{k=1}^{(p-1)/2}\frac{3^k}{k^2\binom{2k}{k}}\equiv\left(\frac{p}{3}\right)\frac{5}{6}B_{p-2}\left(\frac{1}{3}\right)\pmod{p}$$
for any odd prime p.

Conjecture 7. (Sun [75]) *Let p be an odd prime. If $p>7$ then*
$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^3}\equiv-\frac{2}{p^2}H_{p-1}-\frac{13}{27}H_{p-1}^{(3)}\pmod{p^4}.$$

If $p>5$ then

$$\sum_{k=1}^{p-1}\frac{1}{k^4\binom{2k}{k}}-\frac{H_{p-1}}{p^3}\equiv-\frac{7}{45}pB_{p-5}\pmod{p^2}.$$

Remark 7. It is known that $H_{p-1}/p^2\equiv-B_{p-3}/3\pmod{p}$ for any prime $p>3$ and $H_{p-1}^{(3)}\equiv-\frac{6}{5}p^2B_{p-5}\pmod{p^3}$ for each prime $p>5$. Also,

$$\sum_{k=1}^{\infty}\frac{1}{k^4\binom{2k}{k}}=\frac{17}{36}\zeta(4).$$

The two congruences in Conjecture 7 modulo p have been confirmed by K. Hessami Pilehrood and T. Hessami Pilehrood [29].

Conjecture 8. (Sun [72]) *For any prime $p>5$, we have*

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}}{(2k+1)^3 16^k}\equiv\left(\frac{-1}{p}\right)\left(\frac{H_{p-1}}{4p^2}+\frac{p^2}{36}B_{p-5}\right)\pmod{p^3}. \tag{2.2}$$

Remark 8. By Sun [93, (2.10)], for any prime $p>3$ we have

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}}{(2k+1)^3 16^k}\equiv-\left(\frac{-1}{p}\right)\frac{B_{p-3}}{12}\pmod{p}.$$

By I. J. Zucker [113, (2.23)], we have the identity

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)^3 16^k}=\frac{7\pi^3}{216}.$$

It is also known that

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)16^k}=\frac{\pi}{3}\quad\text{and}\quad\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)^2(-16)^k}=\frac{\pi^2}{10}.$$

The author [72] proved that $\sum_{k=0}^{(p-3)/2}\binom{2k}{k}/((2k+1)16^k)\equiv0\pmod{p^2}$ for any prime $p>3$, and his conjecture that

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}}{(2k+1)^2(-16)^k}\equiv\frac{H_{p-1}}{5p^2}\pmod{p^3}$$

for each prime $p>5$, was later confirmed by K. Hessami Pilehrood, T. Hessami Pilehrood and Tauraso [30].

Conjecture 9. (Sun [79]) *Let p be an odd prime. Then*

$$\sum_{k=1}^{p-1}\frac{2^k}{k^2}\binom{3k}{k}\equiv\left(\frac{-1}{p}\right)6E_{p-3}\pmod{p}$$

and

$$\sum_{k=1}^{p-1}\frac{2^k}{k}\binom{3k}{k}\equiv -3p\,q_p(2)^2 \pmod{p^2}.$$

Also,

$$p\sum_{k=1}^{p-1}\frac{1}{k2^k\binom{3k}{k}}\equiv\begin{cases}0 \pmod{p^2} & \text{if } p\equiv 1 \pmod{4},\\ -3/5 \pmod{p^2} & \text{if } p\equiv 3 \pmod{4}.\end{cases}$$

When $p>3$ we have

$$p\sum_{k=1}^{p-1}\frac{1}{k^2 2^k\binom{3k}{k}}\equiv -\frac{q_p(2)}{2}-\frac{p}{4}q_p(2)^2 \pmod{p^2}.$$

Remark 9. L.-L. Zhao, Pan and Sun [110] proved that $\sum_{k=1}^{p-1}\frac{2^k}{k}\binom{3k}{k}\equiv 0 \pmod{p}$ for any odd prime p. The author [69] determined $\sum_{k=0}^{p-1}\binom{3k}{k}/m^k$ modulo an odd prime p for any integer $m\not\equiv 0 \pmod{p}$.

Conjecture 10. (2019) *Let a and b be integers with $0<a<b$ and $\gcd(a,b)=1$. Let $p>3$ be a prime with $p\equiv\pm1 \pmod{b}$. Then, for each $n\in\mathbb{Z}^+$, the number*

$$w_{p,n}(a,b):=\frac{\sum_{k=0}^{pn-1}\binom{-a/b}{k}\binom{(a-b)/b}{k}-(-1)^{\langle -a/b\rangle_p}\sum_{r=0}^{n-1}\binom{-a/b}{r}\binom{(a-b)/b}{r}}{p^2n^2\binom{-a/b}{n}\binom{(a-b)/b}{n}} \tag{2.3}$$

is a p-adic integer, and furthermore

$$w_{p,n}(a,b)\equiv w_{p,1}(a,b)+(n-1)pc_p \pmod{p^2}$$

for some p-adic integer c_p depending only on p.

Remark 10. Let $p>3$ be a prime. In 2003, E. Morterson [49, 50] proved the congruences

$$\sum_{k=0}^{p-1}\binom{-1/2}{k}^2-(-1)^{\langle -1/2\rangle_p}=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}-\left(\frac{-1}{p}\right)\equiv 0 \pmod{p^2},$$

$$\sum_{k=0}^{p-1}\binom{-1/3}{k}\binom{-2/3}{k}-(-1)^{\langle -1/3\rangle_p}=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}-\left(\frac{p}{3}\right)\equiv 0 \pmod{p^2},$$

$$\sum_{k=0}^{p-1}\binom{-1/4}{k}\binom{-3/4}{k}-(-1)^{\langle -1/4\rangle_p}=\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{64^k}-\left(\frac{-2}{p}\right)\equiv 0 \pmod{p^2},$$

$$\sum_{k=0}^{p-1}\binom{-1/6}{k}\binom{-5/6}{k}-(-1)^{\langle -1/6\rangle_p}=\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}-\left(\frac{-1}{p}\right)\equiv 0 \pmod{p^2},$$

which were first conjectured by F. Rodriguez-Villegas [60]. In 2014 Z.-H. Sun [66] extended this by showing that

$$\sum_{k=0}^{p-1}\binom{-x}{k}\binom{x-1}{k}\equiv(-1)^{\langle -x\rangle_p}\pmod{p^2}$$

for any p-adic integer x; another extension given by J.-C. Liu [39] in 2017 states that for any $x\in\{1/2,1/3,1/4,1/6\}$ we have

$$\sum_{k=0}^{pn-1}\binom{-x}{k}\binom{x-1}{k}\equiv(-1)^{\langle -x\rangle_p}\sum_{r=0}^{n-1}\binom{-x}{k}\binom{x-1}{k}\pmod{p^2}.$$

In 2011, the author [75] showed that

$$\frac14 w_{p,1}(1,2)=\frac{\sum_{k=0}^{p-1}\binom{2k}{k}^2/16^k-(\frac{-1}{p})}{p^2}\equiv -E_{p-3}\pmod p.$$

The author's conjectural congruences (cf. [75, Conjecture 5.12])

$$\frac29 w_{p,1}(1,3)=\frac{\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}/27^k-(\frac{p}{3})}{p^2}\equiv-\frac13B_{p-2}\left(\frac13\right)\pmod p,$$

$$\frac{3}{16} w_{p,1}(1,4)=\frac{\sum_{k=0}^{p-1}\binom{4k}{2k}\binom{2k}{k}/64^k-(\frac{-2}{p})}{p^2}\equiv-\frac{3}{16}E_{p-3}\left(\frac14\right)\pmod p,$$

$$\frac{5}{36} w_{p,1}(1,6)=\frac{\sum_{k=0}^{p-1}\binom{6k}{3k}\binom{3k}{k}/432^k-(\frac{-1}{p})}{p^2}\equiv-\frac{25}{9}E_{p-3}\pmod p,$$

were confirmed by Z.-H. Sun [68] in 2016.

Conjecture 11. (i) *For any prime $p>3$ and positive odd integer n, we have*

$$\frac{4^{n-1}}{n^2\binom{n-1}{(n-1)/2}^2}\left(\sum_{k=0}^{(pn-1)/2}\frac{\binom{2k}{k}^2}{16^k}-\left(\frac{-1}{p}\right)\sum_{r=0}^{(n-1)/2}\frac{\binom{2r}{r}^2}{16^r}\right)\equiv p^2E_{p-3}\pmod{p^3}.$$

(ii) (Sun [75]) *Let $p>3$ be a prime and let $a\in\mathbb{Z}^+$. If $p\equiv 1,3\pmod 8$ or $a>1$, then*

$$\sum_{k=0}^{\lfloor\frac58p^a\rfloor}\frac{\binom{2k}{k}^2}{16^k}\equiv\sum_{k=0}^{\lfloor\frac78p^a\rfloor}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p^a}\right)\pmod{p^3}.$$

(iii) (2014-11-19) *For any prime $p>3$, we have*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{4k}{2k}\binom{2k}{k}}{(2k+1)64^k}\equiv\left(\frac{-1}{p}\right)\left(2^{p-1}-(2^{p-1}-1)^2\right)\pmod{p^3}.$$

Remark 11. Let $p>3$ be a prime. The author [75] showed that

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p}\right)+p^2E_{p-3}\pmod{p^3}.$$

G.-S. Mao and the author [45] determined $\sum_{k=0}^{\lfloor 3p^a/4\rfloor}\binom{2k}{k}^2/16^k$ modulo p^3 for any $a\in\mathbb{Z}^+$, and proved that

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{4k}{2k}\binom{2k}{k}}{(2k+1)64^k}\equiv\left(\frac{-1}{p}\right)2^{p-1}\pmod{p^2}.$$

Conjecture 12. *Let $p>3$ be a prime and let $n\in\mathbb{Z}^+$. Then*

$$\begin{aligned}&\frac{27^n}{(pn)^2\binom{2n}{n}\binom{3n}{n}}\left(\sum_{k=0}^{pn-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(2k+1)27^k}-\left(\frac{p}{3}\right)\sum_{r=0}^{n-1}\frac{\binom{2r}{r}\binom{3r}{r}}{(2r+1)27^r}\right)\\&\equiv-3B_{p-2}\left(\frac13\right)\pmod p,\end{aligned}\tag{2.4}$$

$$\begin{aligned}&\frac{64^n}{(pn)^2\binom{4n}{2n}\binom{2n}{n}}\left(\sum_{k=0}^{pn-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{(2k+1)64^k}-\left(\frac{-1}{p}\right)\sum_{r=0}^{n-1}\frac{\binom{4r}{2r}\binom{2r}{r}}{(2r+1)64^r}\right)\\&\equiv-16E_{p-3}\pmod p\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}&\frac{432^n}{(pn)^2\binom{6n}{3n}\binom{3n}{n}}\left(\sum_{k=0}^{pn-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{(2k+1)432^k}-\left(\frac{p}{3}\right)\sum_{r=0}^{n-1}\frac{\binom{6r}{3r}\binom{3r}{r}}{(2r+1)432^r}\right)\\&\equiv-\frac{15}{2}B_{p-2}\left(\frac13\right)\pmod p.\end{aligned}\tag{2.6}$$

Remark 12. Those integers

$$C_k^{(2)}:=\frac{\binom{3k}{k}}{2k+1}=\binom{3k}{k}-2\binom{3k}{k-1}\quad(k=1,2,3,\ldots)$$

are called the second-order Catalan numbers. In the case $n=1$, (2.4) and (2.5), as well as the fact that the left-hand side of (2.6) is p-adic integral, were originally conjectured by the author [75, Conjecture 5.12]. In 2016 Z.-H. Sun [67] confirmed (2.4), (2.5) and (2.6) in the case $n=1$.

Conjecture 13. *Let $p>3$ be a prime and let $n\in\mathbb{Z}^+$. Then*

$$t_p(n):=\frac{27^n}{(pn)^4\binom{2n}{n}\binom{3n}{n}}\left(\sum_{k=0}^{pn-1}\frac{4k+1}{2k+1}\cdot\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}-\left(\frac{p}{3}\right)\sum_{r=0}^{n-1}\frac{4r+1}{2r+1}\cdot\frac{\binom{2r}{r}\binom{3r}{r}}{27^r}\right)$$

is a p-adic integer. Moreover, when $p>5$ we have

$$t_p(n)\equiv a_p+(n-1)pb_p\pmod{p^2}$$

for some $a_p,b_p\in\mathbb{Z}_p$ not depending on n.

Remark 13. That

$$\frac{2}{9}t_p(1)=\frac{1}{p^4}\bigg(\sum_{k=0}^{p-1}\frac{4k+1}{2k+1}\cdot\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}-\left(\frac{p}{3}\right)\bigg)\in\mathbb{Z}_p$$

for any prime $p>3$ was first conjectured in [75, Conjecture 5.12(iii)], and it still remains open.

Conjecture 14. (Sun [88]) *Let $p>5$ be a prime. Then*

$$\sum_{p/2<k<p}\frac{\binom{2k}{k}^2}{k16^k}\equiv-\frac{21}{2}H_{p-1}\pmod{p^4}\tag{2.7}$$

and

$$\sum_{k=0}^{(p-3)/2}\frac{(-16)^k}{(2k+1)^3\binom{2k}{k}}\equiv-\frac{3}{4}\cdot\frac{H_{p-1}}{p^2}-\frac{47}{400}p^2B_{p-5}\pmod{p^3}.$$

Remark 14. The author [88] proved (2.7) modulo p^3. Tauraso [105] showed that

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^2}{k16^k}\equiv-2H_{(p-1)/2}\pmod{p^3}$$

for each prime $p>3$. `Mathematica 9` yields

$$\sum_{k=1}^{\infty}\frac{\binom{2k}{k}^2}{k16^k}=4\log 2-\frac{8G}{\pi},$$

where $G=\sum_{k=0}^{\infty}(-1)^k/(2k+1)^2$ is the Catalan constant.

Conjecture 15. (i) *For any odd prime p, we have*

$$\sum_{k=1}^{(p-1)/2}\frac{(2k^2-4k+1)8^k}{k^2\binom{2k}{k}^2}\equiv 2-2\left(\frac{2}{p}\right)+\left(\frac{2}{p}\right)5p\,q_p(2)\pmod{p^2}.$$

(ii) (Sun [72]) *For each prime $p\equiv 3\pmod 4$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}\equiv-\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k}\pmod{p^3}.$$

(iii) (2009-11-10) *For any prime $p\equiv 3\pmod 4$, $m\in\{8,-16,32\}$ and $n\in\mathbb{Z}^+$, we have*

$$\nu_p\bigg(\sum_{k=0}^{n-1}\frac{\binom{2k}{k}^2}{m^k}\bigg)\geqslant\left\lfloor\frac{\nu_p(n)+1}{2}\right\rfloor\quad\textit{and}\quad\sum_{k=0}^{p^{2n}-1}\frac{\binom{2k}{k}^2}{m^k}\equiv(-p)^n\pmod{p^{n+2}}.$$

(iv) (Sun [72]) *If p is a prime with $p\equiv 1\pmod 3$, then*

$$\sum_{k=0}^{(p-1)/2}\frac{kC_k^3}{16^k}\equiv 2p-2\pmod{p^2}.$$

Remark 15. For any prime $p\equiv 1\ (\mathrm{mod}\ 4)$, the author [72] had a conjecture on $\sum_{k=0}^{p-1}\binom{2k}{k}^2/m^k$ modulo p^2 with $m=8,-16,32$ which was confirmed by Z.-H. Sun [64]. By induction, for any $n\in\mathbb{N}$ we have the identity

$$\sum_{k=0}^{n}\frac{2k^2+4k+1}{8^k}\binom{2k}{k}^2=\frac{(2n+1)^2}{8^n}\binom{2n}{n}^2.$$

Conjecture 16. *Let $p>3$ be a prime.*

(i) (Sun [75]) *We have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{24^k}\equiv\begin{cases}\binom{2(p-1)/3}{(p-1)/3}\pmod{p^2} & \text{if } p\equiv 1\pmod 3,\\ p/\binom{2(p+1)/3}{(p+1)/3}\pmod{p^2} & \text{if } p\equiv 2\pmod 3.\end{cases}$$

(ii) (Sun [75]) *When $p\equiv 1\ (\mathrm{mod}\ 3)$ and $4p=x^2+27y^2$ with $x\equiv 2\ (\mathrm{mod}\ 3)$, we may determine x mod p^2 in the following way:*

$$\sum_{k=0}^{p-1}\frac{k+2}{24^k}\binom{2k}{k}\binom{3k}{k}\equiv x\ (\mathrm{mod}\ p^2).$$

(iii) (2009-11-10) *Suppose that $p\equiv 2\ (\mathrm{mod}\ 3)$ and $n\in\mathbb{Z}^+$. Then*

$$\nu_p\left(\sum_{k=0}^{n-1}\frac{\binom{2k}{k}\binom{3k}{k}}{24^k}\right)\geqslant\left\lfloor\frac{\nu_p(n)+1}{2}\right\rfloor\quad\text{and}\quad\sum_{k=0}^{p^{2n}-1}\frac{\binom{2k}{k}\binom{3k}{k}}{24^k}\equiv(-p)^n\ (\mathrm{mod}\ p^{n+2}).$$

Remark 16. See [75] for more such conjectures. It is known (cf. [34]) that for any prime $p\equiv 1\ (\mathrm{mod}\ 3)$ with $4p=x^2+27y^2$ $(x,y\in\mathbb{Z})$ we have $\binom{2(p-1)/3}{(p-1)/3}\equiv(\frac{x}{3})(\frac{p}{x}-x)\ (\mathrm{mod}\ p^2)$. The author [83] showed that for any prime $p>3$ we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{24^k}&\equiv\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-216)^k}\pmod{p^2},\\ \sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{3k}{k}}{24^k}&\equiv 9\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{3k}{k}}{(-216)^k}\pmod{p^2}.\end{aligned}$$

Conjecture 17. *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{p-1}\frac{\binom{4k}{2k+1}\binom{2k}{k}}{48^k}\equiv\frac{5}{12}p^2B_{p-2}\left(\frac{1}{3}\right)\ (\mathrm{mod}\ p^3)\tag{2.8}$$

and

$$p^2\sum_{k=1}^{p-1}\frac{48^k}{k(2k-1)\binom{4k}{2k}\binom{2k}{k}}\equiv 4\left(\frac{p}{3}\right)+4p \pmod{p^2}.$$

Moreover, there is a p-adic integer a_p depending only on p such that for any $n\in\mathbb{Z}^+$ we have

$$\begin{aligned}&\frac{48^n}{(pn)^2\binom{4n}{2n}\binom{2n}{n}}\left(\sum_{k=0}^{pn-1}\frac{\binom{4k}{2k+1}\binom{2k}{k}}{48^k}-\left(\frac{p}{3}\right)\sum_{r=0}^{n-1}\frac{\binom{4r}{2r+1}\binom{2r}{r}}{48^r}\right)\\ &\equiv\frac{4}{p^2}\sum_{k=0}^{p-1}\frac{\binom{4k}{2k+1}\binom{2k}{k}}{48^k}+(n-1)pa_p \pmod{p^2}.\end{aligned}\tag{2.9}$$

Remark 17. The first and the second congruences in Conjecture 17 appeared as [93, (1.24) and (1.25)]. We even don't know how to prove (2.8) modulo p, and the congruence modulo p^2 was first conjectured in [75, Conjecture 5.14(i)]. Conjecture 17 is related to the author's conjectural identity

$$\sum_{k=1}^{\infty}\frac{48^k}{k(2k-1)\binom{4k}{2k}\binom{2k}{k}}=\frac{15}{2}\sum_{k=1}^{\infty}\frac{\left(\frac{k}{3}\right)}{k^2}\tag{2.10}$$

(cf. [93, (1.23)]) which looks quite difficult, the author would like to offer 480 US dollars as the prize for the first proof of the curious identity (2.10). In view of the conjecture, it is interesting to investigate what primes $p>3$ satisfy the congruence $B_{p-2}(1/3)\equiv 0 \pmod p$. In 2015 the author [93] reported that 205129 is the unique prime below 2×10^7 with that property.

Conjecture 18. *Let $p>3$ be a prime and let $n\in\mathbb{Z}^+$. Then*

$$\frac{1}{n^2}\left(\sum_{k=0}^{pn-1}\left(\sum_{j=0}^{k}\binom{k}{j}\frac{\binom{2j}{j}}{2^j}\right)^2-\left(\frac{-1}{p}\right)\sum_{k=0}^{n-1}\left(\sum_{j=0}^{k}\binom{k}{j}\frac{\binom{2j}{j}}{2^j}\right)^2\right)\equiv 0 \pmod{p^2}$$

and

$$\frac{1}{n^2}\left(\sum_{k=0}^{pn-1}\left(\sum_{j=0}^{k}\binom{k}{j}\frac{\binom{2j}{j}}{(-6)^j}\right)^2-\left(\frac{-1}{p}\right)\sum_{k=0}^{n-1}\left(\sum_{j=0}^{k}\binom{k}{j}\frac{\binom{2j}{j}}{(-6)^j}\right)^2\right)\equiv 0 \pmod{p^2}.$$

Remark 18. The two congruences with $n=1$ were posed by the author [95, Conjecture 6.7(i)]. As pointed out in [95, Remark 6.2],

$$\sum_{k=0}^{n}\binom{n}{k}\frac{\binom{2k}{k}}{2^k}=3^n\sum_{k=0}^{n}\binom{n}{k}\frac{\binom{2k}{k}}{(-6)^k}\quad\text{for all } n\in\mathbb{N}.$$

Conjecture 19. *Let $b,n\in\mathbb{Z}^+$ and let p be a prime with $p\equiv\pm1\pmod b$ and $\langle -1/b\rangle_p\equiv 0\pmod 2$. Then*
$$\frac{1}{n^2\binom{-1/b}{n}\binom{1/b-1}{n}}\sum_{k=0}^{pn-1}(b^2k+b-1)\frac{\binom{2k}{k}}{4^k}\binom{-1/b}{k}\binom{1/b-1}{k}\equiv 0\pmod{p^2}. \tag{2.11}$$

Remark 19. The conjecture with $n=1$ and $b\in\{2,3,4,6\}$ was first stated by the author in [75, Conjecture 5.9]; for example, when $b=3$ and $n=1$, the conjecture says that for any prime $p\equiv 1\pmod 3$ we have
$$\sum_{k=0}^{p-1}\frac{9k+2}{108^k}\binom{2k}{k}^2\binom{3k}{k}\equiv 0\pmod{p^2}.$$
A related conjecture of Rodriguez-Villegas [60] confirmed by Mortenson [50] and the author [77] together states that for any prime $p>3$ we have
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{p}{3})=1\ \&\ p=x^2+3y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } p\equiv 2\pmod 3,\end{cases}$$
$$\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{256^k}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-2}{p})=1\ \&\ p=x^2+2y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{-2}{p})=-1,\ \text{i.e., } p\equiv 5,7\pmod 8,\end{cases}$$
and
$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k,k,k}}{12^{3k}}=\sum_{k=0}^{p-1}\frac{(6k)!}{(3k)!(k!)^3}1728^{-k}$$
$$\equiv\begin{cases}(\frac{p}{3})(4x^2-2p)\pmod{p^2} & \text{if } 4\mid p-1\ \&\ p=x^2+4y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4.\end{cases}$$

Conjecture 20. (2010-01-22) *Let p be an odd prime.*

(i) *If $p\equiv 1\pmod 4$ then*
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\equiv\left(\frac{2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-512)^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{648^k}\pmod{p^3};$$
if $p\equiv 1\pmod 3$ then
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k}\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{256^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{(-144)^k}\pmod{p^3}.$$

(ii) *If $p\equiv 1,2,4\pmod 7$, then*
$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\binom{2k}{k}^3\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{4096^k}\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{81^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{(-3969)^k}\pmod{p^3}.$$

Remark 20. Let p be an odd prime. The author's conjectures on $\sum_{k=0}^{p-1}\binom{2k}{k}^3/m^k$ modulo p^2 with $m\in\{1,-8,16,-64,256,-512,4096\}$ (cf. [75]) were confirmed by J. Kibelbek, L. Long, K. Moss, B. Sheller and H. Yuan [36] as well as Z.-H. Sun [65]. See also Z.-H. Sun [64] for his conjectures on $\sum_{k=0}^{p-1}\binom{4k}{k,k,k,k}/m^k$ mod p^2 with $m=-144,\ 648,\ -3969$ motivated by the author's papers [72, 75]. Most of the congruences in Conjecture 20 were contained in Sun [75, Conjectures 5.2 and 5.3]. The author [77] noted that MacMahon's identity

$$\sum_{k=0}^{n}\binom{n}{k}^3x^k=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\binom{n-k}{k}x^k(1+x)^k$$

with $n=(p-1)/2$ implies that

$$\left(\frac{x}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64x)^k}\equiv\left(\frac{x+1}{p}\right)\sum_{k=0}^{p-1}\binom{4k}{k,k,k,k}\left(\frac{x}{64(x+1)^2}\right)^k\ (\mathrm{mod}\ \ p)$$

for any p-adic integer $x\not\equiv 0,-1\ (\mathrm{mod}\ \ p)$. The author also conjectured that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{256^k}\ (\mathrm{mod}\ \ p^3)$$

for any prime $p\equiv 1,3\ (\mathrm{mod}\ 8)$, but this was recently confirmed by Pan, Tauraso and C. Wang [58].

Conjecture 21. (i) (Sun [96]) *Let $p\neq 2,5$ be a prime and let $n\in\mathbb{Z}^+$. Then*

$$\frac{\sum_{k=0}^{pn-1}(21k+8)\binom{2k}{k}^3-p\sum_{r=0}^{n-1}(21r+8)\binom{2r}{r}^3}{(pn)^4\binom{2n}{n}^3}\equiv -6\frac{H_{p-1}}{p^2}\ (\mathrm{mod}\ p^2).$$

(ii) *For any prime $p>3$ and positive odd integer n, we have*

$$\frac{\sum_{k=0}^{(pn-1)/2}(21k+8)\binom{2k}{k}^3-p\sum_{r=0}^{(n-1)/2}(21r+8)\binom{2r}{r}^3}{(pn)^3\binom{n-1}{(n-1)/2}^3}$$
$$\equiv\left(\frac{-1}{p}\right)32E_{p-3}\ (\mathrm{mod}\ p).$$

(iii) (Sun [75]) *If p is a prime and a is a positive integer with $p^a\equiv 1\ (\mathrm{mod}\ \ 3)$, then*

$$\sum_{k=0}^{\lfloor\frac23 p^a\rfloor}(21k+8)\binom{2k}{k}^3\equiv 8p^a\ (\mathrm{mod}\ \ p^{a+5+(-1)^p}).$$

Remark 21. (a) The author [75] proved that for any odd prime p and $a\in\mathbb{Z}^+$ we have

$$\frac{1}{p^a}\sum_{k=0}^{p^a-1}(21k+8)\binom{2k}{k}^3\equiv 8+16p^3B_{p-3}\ (\mathrm{mod}\ \ p^4).$$

The author [96] guessed that all those Ramanujan-type supercongruences should have extensions involving $n\in\mathbb{Z}^+$ similar to parts (i) and (ii) of Conjecture 21.

(b) The author [92] proved that for any prime $p>3$ we have

$$\sum_{k=0}^{(p-1)/2}(21k+8)\binom{2k}{k}^3\equiv 8p+\left(\frac{-1}{p}\right)32p^3E_{p-3}\pmod{p^4},$$

which has the following equivalent form:

$$\sum_{k=1}^{(p-1)/2}\frac{21k-8}{k^3\binom{2k}{k}^3}\equiv(-1)^{(p+1)/2}4E_{p-3}\pmod{p}.$$

Note that

$$\sum_{k=1}^{\infty}\frac{21k-8}{k^3\binom{2k}{k}^3}=\zeta(2)=\frac{\pi^2}{6}$$

by D. Zeilberger [111] (see also [27, (7)]).

Conjecture 22. *Let $p>3$ be a prime and let $n\in\mathbb{Z}^+$.*

(i) *We have*

$$\frac{(-8)^n}{(pn)^3\binom{2n}{n}^3}\left(\sum_{k=0}^{pn-1}\frac{3k+1}{(-8)^k}\binom{2k}{k}^3-\left(\frac{-1}{p}\right)p\sum_{r=0}^{n-1}\frac{3r+1}{(-8)^r}\binom{2r}{r}^3\right)\equiv -E_{p-3}\pmod{p},\quad(2.12)$$

$$\frac{16^n}{(pn)^4\binom{2n}{n}^3}\left(\sum_{k=0}^{pn-1}\frac{3k+1}{16^k}\binom{2k}{k}^3-p\sum_{r=0}^{n-1}\frac{3r+1}{16^r}\binom{2r}{r}^3\right)\equiv\frac{7}{3}B_{p-3}\pmod{p},\quad(2.13)$$

$$\begin{aligned}&\frac{(-64)^{n-1}}{(pn)^3\binom{2n-1}{n-1}^3}\left(\sum_{k=0}^{pn-1}\frac{4k+1}{(-64)^k}\binom{2k}{k}^3-\left(\frac{-1}{p}\right)p\sum_{r=0}^{n-1}\frac{4r+1}{(-64)^r}\binom{2r}{r}^3\right)\\&\quad\equiv E_{p-3}\pmod{p},\end{aligned}\quad(2.14)$$

$$\begin{aligned}&\frac{256^{n-1}}{(pn)^3\binom{2n-1}{n-1}^3}\left(\sum_{k=0}^{pn-1}\frac{6k+1}{256^k}\binom{2k}{k}^3-\left(\frac{-1}{p}\right)p\sum_{r=0}^{n-1}\frac{6r+1}{256^r}\binom{2r}{r}^3\right)\\&\quad\equiv -E_{p-3}\pmod{p},\end{aligned}\quad(2.15)$$

$$\begin{aligned}&\frac{(-512)^n}{(pn)^3\binom{2n}{n}^3}\left(\sum_{k=0}^{pn-1}\frac{6k+1}{(-512)^k}\binom{2k}{k}^3-\left(\frac{-2}{p}\right)p\sum_{r=0}^{n-1}\frac{6r+1}{(-512)^r}\binom{2r}{r}^3\right)\\&\quad\equiv -4E_{p-3}\left(\frac{1}{4}\right)\pmod{p},\end{aligned}\quad(2.16)$$

and

$$\frac{4096^{n-1}}{(pn)^3\binom{2n-1}{n-1}^3}\left(\sum_{k=0}^{pn-1}\frac{42k+5}{4096^k}\binom{2k}{k}^3-\left(\frac{-1}{p}\right)p\sum_{r=0}^{n-1}\frac{42r+5}{4096^r}\binom{2r}{r}^3\right)\equiv -E_{p-3}\pmod p. \tag{2.17}$$

(ii) *Suppose that n is odd. Then*

$$\frac{(-8)^{(n-1)/2}}{(pn)^3\binom{n-1}{(n-1)/2}^3}\left(\sum_{k=0}^{(pn-1)/2}\frac{3k+1}{(-8)^k}\binom{2k}{k}^3-\left(\frac{-1}{p}\right)p\sum_{r=0}^{(n-1)/2}\frac{3r+1}{(-8)^r}\binom{2r}{r}^3\right)\equiv\left(\frac{2}{p}\right)\frac{1}{4}E_{p-3}\left(\frac14\right)\pmod p, \tag{2.18}$$

$$\frac{4^{n-1}}{(pn)^3\binom{n-1}{(n-1)/2}^3}\left(\sum_{k=0}^{(pn-1)/2}\frac{3k+1}{16^k}\binom{2k}{k}^3-p\sum_{r=0}^{(n-1)/2}\frac{3r+1}{16^r}\binom{2r}{r}^3\right)\equiv\left(\frac{-1}{p}\right)2E_{p-3}\pmod p, \tag{2.19}$$

$$\frac{(-64)^{n-1}}{(pn)^3\binom{n-1}{(n-1)/2}^3}\left(\sum_{k=0}^{(pn-1)/2}\frac{4k+1}{(-64)^k}\binom{2k}{k}^3-\left(\frac{-1}{p}\right)p\sum_{r=0}^{(n-1)/2}\frac{4r+1}{(-64)^r}\binom{2r}{r}^3\right)\equiv E_{p-3}\pmod p, \tag{2.20}$$

$$\frac{16^{n-1}}{(pn)^4\binom{n-1}{(n-1)/2}^3}\left(\sum_{k=0}^{(pn-1)/2}\frac{6k+1}{256^k}\binom{2k}{k}^3-\left(\frac{-1}{p}\right)p\sum_{r=0}^{(n-1)/2}\frac{6r+1}{256^r}\binom{2r}{r}^3\right)\equiv\left(\frac{-1}{p}\right)\frac{7}{24}B_{p-3}\pmod p \tag{2.21}$$

and

$$\frac{(-512)^{(n-1)/2}}{(pn)^3\binom{n-1}{(n-1)/2}^3}\left(\sum_{k=0}^{(pn-1)/2}\frac{6k+1}{(-512)^k}\binom{2k}{k}^3-\left(\frac{-2}{p}\right)p\sum_{r=0}^{(n-1)/2}\frac{6r+1}{(-512)^r}\binom{2r}{r}^3\right)\equiv\left(\frac{2}{p}\right)\frac{E_{p-3}}{4}\pmod p. \tag{2.22}$$

If $p>5$, then

$$\frac{64^{n-1}}{(pn)^4\binom{n-1}{(n-1)/2}^3}\left(\sum_{k=0}^{(pn-1)/2}\frac{42k+5}{4096^k}\binom{2k}{k}^3-\left(\frac{-1}{p}\right)p\sum_{r=0}^{(n-1)/2}\frac{42r+5}{4096^r}\binom{2r}{r}^3\right)\equiv-\frac{3}{4}\left(\frac{-1}{p}\right)\frac{H_{p-1}}{p^2}\pmod{p^2}. \tag{2.23}$$

Remark 22. (a) These congruences with $n=1$ correspond to the two Zeilberger-type series

$$\sum_{k=1}^\infty\frac{(3k-1)(-8)^k}{k^3\binom{2k}{k}^3}=-2G\quad\text{and}\quad\sum_{k=1}^\infty\frac{(3k-1)16^k}{k^3\binom{2k}{k}^3}=\frac{\pi^2}{2}$$

given in [15, Identities 1 and 3 with $a=1/2$], and the four Ramanujan-type series (cf. [3, 4, 10, 59])

$$\sum_{k=0}^\infty\frac{4k+1}{(-64)^k}\binom{2k}{k}^3=\frac{2}{\pi},\quad\sum_{k=0}^\infty\frac{6k+1}{256^k}\binom{2k}{k}^3=\frac{4}{\pi},$$
$$\sum_{k=0}^\infty\frac{6k+1}{(-512)^k}\binom{2k}{k}^3=\frac{2\sqrt2}{\pi},\quad\sum_{k=0}^\infty\frac{42k+5}{4096^k}\binom{2k}{k}^3=\frac{16}{\pi}.$$

Two q-analogues of the identity $\sum_{k=1}^\infty(3k-1)16^k/(k\binom{2k}{k})^3=\pi^2/2$ were given by Q.-H. Hou, C. Krattenthaler and Sun [31].

(b) Let $p>3$ be a prime. In 1997 van Hamme [106] conjectured that

$$\sum_{k=0}^{p-1}\frac{4k+1}{(-64)^k}\binom{2k}{k}^3\equiv\left(\frac{-1}{p}\right)p\pmod{p^3},$$
$$\sum_{k=0}^{(p-1)/2}\frac{6k+1}{256^k}\binom{2k}{k}^3\equiv\left(\frac{-1}{p}\right)p\pmod{p^4},$$
$$\sum_{k=0}^{(p-1)/2}\frac{6k+1}{(-512)^k}\binom{2k}{k}^3\equiv\left(\frac{-2}{p}\right)p\pmod{p^3},$$
$$\sum_{k=0}^{(p-1)/2}\frac{42k+5}{4096^k}\binom{2k}{k}^3\equiv\left(\frac{-2}{p}\right)5p\pmod{p^4},$$

and these were confirmed by E. Mortenson [52], L. Long [41], H. Swisher [102], and R. Osburn and W. Zudilin [56] respectively. J. Guillera and Zudilin [20] proved the congruences

$$\sum_{k=0}^{p-1}\frac{3k+1}{(-8)^k}\binom{2k}{k}^3\equiv\left(\frac{-1}{p}\right)p\pmod{p^3}\ \text{and}\ \sum_{k=0}^{p-1}\frac{3k+1}{16^k}\binom{2k}{k}^3\equiv p\pmod{p^3}.$$

(c) The author [79] proved the congruences (2.14) and (2.20) for $n=1$. The congruences (2.12), (2.13), (2.15), (2.17), (2.19), (2.22) and (2.23) with $n=1$ were conjectured by the author [75]; later, in the case $n=1$, (2.12), (2.15), (2.17), (2.19) and (2.23) were confirmed by Y. G. Chen, X. Y. Xie and B. He [9], G.-S. Mao and C.-W. Wen (quite recently), D.-W. Hu and Mao [33], Mao and T. Zhang [46], and Hu [32], respectively. In his PhD thesis, Hu [32] proved that the left-hand side of (2.13) with $n=1$ is a p-adic integer. Motivated by the author's comments in the paper [96], Guo also realized that the left-hand sides of all the congruences in Conjecture 22 with the binomial coefficients in the denominators removed should be p-adic integers if n is a power of p, this is of course much weaker than Conjecture 22.

Conjecture 23. *Let $p>3$ be a prime and let $n\in\mathbb{Z}^+$. Then*

$$\frac{648^{n-1}}{(pn)^3\binom{4n}{n,n,n,n}}a_{p,n}\equiv-\frac{5}{72}E_{p-3}\pmod{p},\tag{2.24}$$

$$\frac{(-1024)^{n-1}}{(pn)^3\binom{4n}{n,n,n,n}}b_{p,n}\equiv\frac{1}{8}E_{p-3}\pmod{p},\tag{2.25}$$

$$\frac{(-2^{10}3^4)^{n-1}}{(pn)^3\binom{4n}{n,n,n,n}}c_{p,n}\equiv\frac{5}{72}E_{p-3}\pmod{p},\tag{2.26}$$

where

$$a_{p,n}:=\sum_{k=0}^{pn-1}\frac{7k+1}{648^k}\binom{4k}{k,k,k,k}-\left(\frac{-1}{p}\right)p\sum_{r=0}^{n-1}\frac{7r+1}{648^r}\binom{4r}{r,r,r,r},$$

$$b_{p,n}:=\sum_{k=0}^{pn-1}\frac{20k+3}{(-1024)^k}\binom{4k}{k,k,k,k}-\left(\frac{-1}{p}\right)p\sum_{r=0}^{n-1}\frac{20r+3}{(-1024)^r}\binom{4r}{r,r,r,r},$$

$$c_{p,n}:=\sum_{k=0}^{pn-1}\frac{260k+23}{(-2^{10}3^4)^k}\binom{4k}{k,k,k,k}-\left(\frac{-1}{p}\right)p\sum_{r=0}^{n-1}\frac{260r+23}{(-2^{10}3^4)^r}\binom{4r}{r,r,r,r}.$$

Remark 23. (a) The congruences in this conjecture correspond to the Ramanujan series (cf. [3], [17] and [59])

$$\sum_{k=0}^{\infty}\frac{7k+1}{648^k}\binom{4k}{k,k,k,k}=\frac{9}{2\pi},\quad\sum_{k=0}^{\infty}\frac{20k+3}{(-1024)^k}\binom{4k}{k,k,k,k}=\frac{8}{\pi}$$

and

$$\sum_{k=0}^{\infty}\frac{260k+23}{(-2^{10}3^4)^k}\binom{4k}{k,k,k,k}=\frac{72}{\pi}.$$

(b) In the case $n=1$, the congruences (2.24) and (2.25), and (2.26) were conjectured by the author in [79] and [84] respectively. For any prime $p>3$, the sum $\sum_{k=0}^{(p-1)/2}\frac{20k+3}{(-1024)^k}\binom{4k}{k,k,k,k}$ modulo p^3 and p^4 were determined by Zudilin [114] and Sun [79] respectively.

Conjecture 24. (i) (Sun [75]) *For any prime $p>3$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}$$

$$\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } p\equiv 1 \pmod 3 \ \&\ 4p=x^2+27y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

(ii) *For any prime $p>3$ and $n\in\mathbb{Z}^+$, we have*

$$\frac{(-192)^n}{(pn)^3\binom{2n}{n}^2\binom{3n}{n}}a_{p,n}\equiv-\frac{40}{9}B_{p-2}\left(\frac13\right) \pmod p,$$

where

$$a_{p,n}:=\sum_{k=0}^{pn-1}\frac{5k+1}{(-192)^k}\binom{2k}{k}^2\binom{3k}{k}-\left(\frac{p}{3}\right)p\sum_{r=0}^{n-1}\frac{5r+1}{(-192)^r}\binom{2r}{r}^2\binom{3r}{r}.$$

(iii) *For $n\in\mathbb{Z}^+$ set*

$$a_n:=\frac{1}{n(2n+1)\binom{2n}{n}}\sum_{k=0}^{n-1}(5k+1)\binom{2k}{k}^2\binom{3k}{k}(-192)^{n-1-k}.$$

Then $a_n\in\mathbb{Z}$ for $n=2,3,4,\ldots$ unless $2n+1$ is a power of 3 in which case $3a_n\in\mathbb{Z}\setminus 3\mathbb{Z}$.

Remark 24. It is well known that for any prime $p\equiv 1 \pmod 3$ there are unique $x,y\in\mathbb{Z}^+$ such that $4p=x^2+27y^2$ (see, e.g., [13]). Also, Ramanujan [59] found that

$$\sum_{k=0}^{\infty}\frac{5k+1}{(-192)^k}\binom{2k}{k}^2\binom{3k}{k}=\frac{4\sqrt3}{\pi}.$$

Part (ii) of Conjecture 24 with $n=1$ appeared in Sun [75, Conjecture 5.6]. The author [75, 84] had many other conjectures similar to Conjecture 24.

Conjecture 25. (i) *For $n\in\mathbb{Z}^+$ set*

$$a_n:=\frac{1}{2n(2n+1)\binom{2n}{n}}\sum_{k=0}^{n-1}(20k+3)\binom{4k}{k,k,k,k}(-2^{10})^{n-1-k}.$$

Then $(-1)^{n-1}a_n\in\mathbb{Z}^+$ for all $n=2,3,4,\ldots$.

(ii) *For $n\in\mathbb{Z}^+$ set*

$$b_n:=\frac{1}{2n(2n+1)\binom{2n}{n}}\sum_{k=0}^{n-1}(28k+3)\binom{4k}{k,k,k,k}(-3\times 2^{12})^{n-1-k}.$$

Then we have $(-1)^{n-1}b_n\in\mathbb{Z}^+$ for all $n=2,3,4,\ldots$.

(iii) *For* $n\in\mathbb{Z}^+$ *set*

$$c_n:=\frac{1}{2n(2n+1)\binom{2n}{n}}\sum_{k=0}^{n-1}(10k+1)\binom{4k}{k,k,k,k}12^{4(n-1-k)}.$$

Given an integer $n>1$, *we have* $c_n\in\mathbb{Z}$ *unless* $2n+1$ *is a power of* 3 *in which case* $3c_n\in\mathbb{Z}\setminus3\mathbb{Z}$.

(iv) *For* $n\in\mathbb{Z}^+$ *set*

$$d_n:=\frac{1}{10n(2n+1)\binom{2n}{n}}\sum_{k=0}^{n-1}(154k+15)\binom{6k}{3k}\binom{3k}{k,k,k}(-2^{15})^{n-1-k}.$$

Given an integer $n>1$, *we have* $(-1)^{n-1}d_n\in\mathbb{Z}^+$ *unless* $2n+1$ *is a power of* 5 *in which case* $5d_n\in\mathbb{Z}\setminus5\mathbb{Z}$.

Remark 25. Recall the Ramanujan series (cf. [17, 59])

$$\sum_{k=0}^{\infty}\frac{20k+3}{(-2^{10})^k}\binom{4k}{k,k,k,k}=\frac{8}{\pi},\quad \sum_{k=0}^{\infty}\frac{28k+3}{(-3\times2^{12})^k}\binom{4k}{k,k,k,k}=\frac{16\sqrt3}{3\pi},$$

$$\sum_{k=0}^{\infty}\frac{10k+1}{12^{4k}}\binom{4k}{k,k,k,k}=\frac{9\sqrt2}{4\pi},\quad \sum_{k=0}^{\infty}\frac{154k+15}{(-2^{15})^k}\binom{6k}{3k}\binom{3k}{k,k,k}=\frac{32\sqrt2}{\pi}.$$

Actually, for each Ramanujan-type series for $1/\pi$ we have a conjecture similar to Conjecture 25.

Conjecture 26. *Let* p *be an odd prime.*

(i) (Sun [75]) *We have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{8^k}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{-2}{p})=1\ \&\ p=x^2+2y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{-2}{p})=-1.\end{cases}$$

(ii) *For any* $n\in\mathbb{Z}^+$, *we have*

$$\frac{8^n}{(pn)^4\binom{2n}{n}^2\binom{3n}{n}}\bigg(\sum_{k=0}^{pn-1}\frac{10k+3}{8^k}\binom{2k}{k}^2\binom{3k}{k}-p\sum_{r=0}^{n-1}\frac{10r+3}{8^r}\binom{2r}{r}^2\binom{3r}{r}\bigg)\equiv-\frac{49H_{p-1}}{4p^2}\pmod p. \tag{2.27}$$

Remark 26. The congruence (2.27) with $n=1$, and the conjectural identity

$$\sum_{k=0}^{\infty}\frac{(10k-3)8^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{\pi^2}{2}$$

were posed by the author [75]. The last identity was later confirmed by J. Guillera and M. Rogers [19].

Conjecture 27. *Let $p>3$ be a prime.*

(i) (Sun [75]) *We have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k}\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if } p\equiv 1,4\pmod{15}\ \&\ p=x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 20x^2-2p\pmod{p^2}&\text{if } p\equiv 2,8\pmod{15}\ \&\ p=5x^2+3y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2}&\text{if } (\frac{p}{15})=-1.\end{cases}$$

(ii) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{(-27)^n}{(pn)^3\binom{2n}{n}^2\binom{3n}{n}}a_{p,n}\equiv -3B_{p-2}\left(\frac{1}{3}\right)\pmod{p},\tag{2.28}$$

where

$$a_{p,n}:=\sum_{k=0}^{pn-1}\frac{15k+4}{(-27)^k}\binom{2k}{k}^2\binom{3k}{k}-\left(\frac{p}{3}\right)p\sum_{r=0}^{n-1}\frac{15r+4}{(-27)^r}\binom{2r}{r}^2\binom{3r}{r}.$$

Remark 27. (a) Let $p>5$ be a prime. By the theory of binary quadratic forms (cf. [13]), if $p\equiv 1,4\pmod{15}$ then $p=x^2+15y^2$ for some $x,y\in\mathbb{Z}$; if $p\equiv 2,8\pmod{15}$ then $p=5x^2+3y^2$ for some $x,y\in\mathbb{Z}$.

(b) The congruence (2.28) with $n=1$, and the conjectural identity

$$\sum_{k=1}^{\infty}\frac{(15k-4)(-27)^{k-1}}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\sum_{k=1}^{\infty}\frac{(\frac{k}{3})}{k^2},$$

were proposed by the author [75]. K. Hessami Pilehrood and T. Hessami Pilehrood [28] confirmed the last identity and proved the congruence

$$\sum_{k=0}^{p-1}\frac{15k+4}{(-27)^k}\binom{2k}{k}^2\binom{3k}{k}\equiv\left(\frac{p}{3}\right)p\pmod{p^2}$$

for any prime $p>3$.

Conjecture 28. *Let p be an odd prime.*

(i) (Sun [72]) *We have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k}\equiv\begin{cases}x^2-2p\pmod{p^2}&\text{if } (\frac{p}{11})=1\ \&\ 4p=x^2+11y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2}&\text{if } (\frac{p}{11})=-1,\ \text{i.e., } p\equiv 2,6,7,8,10\pmod{11}.\end{cases}$$

(ii) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{64^n}{(pn)^4\binom{2n}{n}^2\binom{3n}{n}}\bigg(\sum_{k=0}^{pn-1}\frac{11k+3}{64^k}\binom{2k}{k}^2\binom{3k}{k}-p\sum_{r=0}^{n-1}\frac{11r+3}{64^r}\binom{2r}{r}^2\binom{3r}{r}\bigg)\equiv -56\frac{H_{p-1}}{p^2}\pmod p. \tag{2.29}$$

Remark 28. It is well-known that the quadratic field $\mathbb{Q}(\sqrt{-11})$ has class number one and hence for any odd prime p with $(\frac{p}{11})=1$ we can write $4p=x^2+11y^2$ with $x,y\in\mathbb{Z}$. Concerning the parameters in the representation $4p=x^2+11y^2$, Jacobi (see, e.g., [34]) proved the following result: If $p=11f+1$ is a prime (with $f\in\mathbb{N}$) and $4p=x^2+11y^2$ $(x,y\in\mathbb{Z})$ with $x\equiv 2\pmod{11}$, then $x\equiv\binom{6f}{3f}\binom{3f}{f}/\binom{4f}{2f}\pmod p$. The congruence (2.29) with $n=1$, and the conjectural identity

$$\sum_{k=0}^{\infty}\frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=8\pi^2,$$

were posed by the author [75]. The last identity was later confirmed by Guillera [18].

Conjecture 29. *Let $p>3$ be a prime.*

(i) (Sun [75]) *We have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{81^k}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,2,4\pmod 7\ \&\ p=x^2+7y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{p}{7})=-1,\ \text{i.e., } p\equiv 3,5,6\pmod 7.\end{cases}$$

(ii) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{81^n}{(pn)^4\binom{2n}{n}^2\binom{4n}{2n}}\bigg(\sum_{k=0}^{pn-1}\frac{35k+8}{81^k}\binom{2k}{k}^2\binom{4k}{2k}-p\sum_{r=0}^{n-1}\frac{35r+8}{81^r}\binom{2r}{r}^2\binom{4r}{2r}\bigg)\equiv 52B_{p-3}\pmod p. \tag{2.30}$$

Also, the number

$$\sum_{k=0}^{(pn-1)/2}\frac{35k+8}{81^k}\binom{2k}{k}^2\binom{4k}{2k}-p\sum_{r=0}^{(n-1)/2}\frac{35r+8}{81^r}\binom{2r}{r}^2\binom{4r}{2r}$$

divided by $p^2n\binom{n-1}{(n-1)/2}^2\binom{2n-2}{n-1}$ is a p-adic integer for each positive odd integer n, and

$$\frac{1}{p^a}\sum_{k=0}^{(p^a-1)/2}\frac{35k+8}{81^k}\binom{2k}{k}^2\binom{4k}{2k}\equiv 8\times 3^{p-1}\pmod{p^2}$$

for any $a\in\mathbb{Z}^+$.

Remark 29. The congruence (2.30) with $n=1$, and the last congruence, as well as the conjectural identity
$$\sum_{k=1}^\infty\frac{(35k-8)81^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=12\pi^2$$
were proposed by the author [75]. The last identity were later confirmed by Guillera and Rogers [19].

Conjecture 30. (i) *For any prime* $p>3$ *and* $n\in\mathbb{Z}^+$, *the number*
$$\sum_{k=0}^{pn-1}\frac{28k^2+18k+3}{(-64)^k}\binom{2k}{k}^4\binom{3k}{k}-p^2\sum_{r=0}^{n-1}\frac{28r^2+18r+3}{(-64)^r}\binom{2r}{r}^4\binom{3r}{r}$$
divided by $(pn)^5\binom{2n}{n}^4\binom{3n}{n}$ *is a* p*-adic integer. For any odd prime* p *and positive odd integer* n, *the number*
$$\sum_{k=0}^{(pn-1)/2}\frac{28k^2+18k+3}{(-64)^k}\binom{2k}{k}^4\binom{3k}{k}-p^2\sum_{r=0}^{(n-1)/2}\frac{28r^2+18r+3}{(-64)^r}\binom{2r}{r}^4\binom{3r}{r}$$
divided by $p^4n^3\binom{n-1}{(n-1)/2}^4\binom{3(n-1)/2}{(n-1)/2}$ *is a* p*-adic integer.*

(ii) (2010-04-05) *For any odd prime* p, *we have*
$$\sum_{k=0}^{p-1}\frac{28k^2+18k+3}{(-64)^k}\binom{2k}{k}^4\binom{3k}{k}\equiv 3p^2-\frac{7}{2}p^5B_{p-3}\pmod{p^6}$$
and
$$\sum_{k=0}^{(p-1)/2}\frac{28k^2+18k+3}{(-64)^k}\binom{2k}{k}^4\binom{3k}{k}\equiv 3p^2+\left(\frac{-1}{p}\right)6p^4E_{p-3}\pmod{p^5}.$$

(ii) (2010-04-05) *For any integer* $n>1$, *we have*
$$\sum_{k=0}^{n-1}(28k^2+18k+3)\binom{2k}{k}^4\binom{3k}{k}(-64)^{n-1-k}\equiv 0\ \left(\bmod\ (2n+1)n^2\binom{2n}{n}^2\right).$$

Remark 30. Parts (ii) and (iii), as well as the author's conjectural identity
$$\sum_{k=1}^\infty\frac{(28k^2-18k+3)(-64)^k}{k^5\binom{2k}{k}^4\binom{3k}{k}}=-14\zeta(3),$$
appeared in [81, Conjecture 8].

Conjecture 31. *Let p be an odd prime.*

(i) *If $p>3$ and $n\in\mathbb{Z}^+$, then the number*

$$\sum_{k=0}^{pn-1}\frac{10k^2+6k+1}{(-256)^k}\binom{2k}{k}^5-p^2\sum_{r=0}^{n-1}\frac{10r^2+6r+1}{(-256)^r}\binom{2r}{r}^5$$

divided by $(pn)^5\binom{2n}{n}^5$, and the number

$$\sum_{k=0}^{pn-1}\frac{74k^2+27k+3}{4096^k}\binom{2k}{k}^4\binom{3k}{k}-p^2\sum_{r=0}^{n-1}\frac{74r^2+27r+3}{4096^r}\binom{2r}{r}^4\binom{3r}{r}$$

divided by $(pn)^5\binom{2n}{n}^4\binom{3n}{n}$, are both p-adic integers.

(ii) (2010-04-05) *If $p>3$, then*

$$\sum_{k=0}^{p-1}\frac{10k^2+6k+1}{(-256)^k}\binom{2k}{k}^5\equiv p^2-\frac{7}{6}p^5B_{p-3}\pmod{p^6}$$

and

$$\sum_{k=0}^{(p-1)/2}\frac{10k^2+6k+1}{(-256)^k}\binom{2k}{k}^5\equiv p^2+\frac{7}{3}p^5B_{p-3}\pmod{p^6}.$$

(iii) (2010-04-05) *If $p\neq 5$, then*

$$\sum_{k=0}^{p-1}\frac{74k^2+27k+3}{4096^k}\binom{2k}{k}^4\binom{3k}{k}\equiv 3p^2+7p^5B_{p-3}\pmod{p^6}$$

and

$$\sum_{k=0}^{(p-1)/2}\frac{74k^2+27k+3}{4096^k}\binom{2k}{k}^4\binom{3k}{k}\equiv 3p^2-\frac{9}{4}p^3H_{p-1}\pmod{p^7}.$$

Remark 31. By [15, Identity 8] and [16], we have

$$\sum_{k=1}^{\infty}\frac{(10k^2-6k+1)(-256)^k}{k^5\binom{2k}{k}^5}=-28\zeta(3)$$

and

$$\sum_{k=0}^{\infty}\frac{74k^2+27k+3}{4096^k}\binom{2k}{k}^4\binom{3k}{k}=\frac{48}{\pi^2}.$$

Conjecture 32. (i) *Let $p\neq 2,5$ be a prime. Then the number*

$$\sum_{k=0}^{pn-1}\frac{21k^3+22k^2+8k+1}{256^k}\binom{2k}{k}^7-p^3\sum_{r=0}^{n-1}\frac{21r^3+22r^2+8r+1}{256^r}\binom{2r}{r}^7$$

divided by $(pn)^8\binom{2n}{n}^7$ *is a p-adic integer for every* $n\in\mathbb{Z}^+$, *and also the number*

$$\sum_{k=0}^{(pn-1)/2}\frac{168k^3+76k^2+14k+1}{2^{20k}}\binom{2k}{k}^7 - \left(\frac{-1}{p}\right)p^3\sum_{r=0}^{(n-1)/2}\frac{168r^3+76r^2+14r+1}{2^{20r}}\binom{2r}{r}^7$$

divided by $(pn)^8\binom{n-1}{(n-1)/2}^7$ *is a p-adic integer for any positive odd integer n.*

(ii) (2010-04-06) *For any integer* $n>1$, *we have*

$$\sum_{k=0}^{n-1}(21k^3+22k^2+8k+1)\binom{2k}{k}^7 256^{n-1-k}\equiv 0 \pmod{2n^3\binom{2n}{n}^3}$$

and

$$\sum_{k=0}^{n-1}(168k^3+76k^2+14k+1)\binom{2k}{k}^7 2^{20(n-1-k)}\equiv 0 \pmod{2n^3\binom{2n}{n}^3}.$$

Remark 32. B. Gourevich and Guillera (see [14, Section 4]) conjectured

$$\sum_{k=0}^{\infty}\frac{168k^3+76k^2+14k+1}{2^{20k}}\binom{2k}{k}^7=\frac{32}{\pi^3}$$

and

$$\sum_{k=1}^{\infty}\frac{(21k^3-22k^2+8k-1)256^k}{k^7\binom{2k}{k}^7}=\frac{\pi^4}{8}$$

respectively. Zudilin [114, (31)] suggested that for any odd prime p we might have

$$\sum_{k=0}^{p-1}\frac{168k^3+76k^2+14k+1}{2^{20k}}\binom{2k}{k}^7\equiv\left(\frac{-1}{p}\right)p^3 \pmod{p^7},$$

which is much weaker than the second assertion in part (i).

Conjecture 33. (i) (2014–07-22) *Let* p *be an odd prime and let* $m\in\mathbb{Z}$ *with* $m\neq 1$ *and* $p\nmid m$. *Then*

$$\sum_{n=0}^{p-1}\frac{1}{m^n}\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\left(\frac{m-1}{4}\right)^k\equiv p+2p\frac{1-(\frac{m}{p})}{m-1} \pmod{p^2}.$$

(ii) (Sun [92]) *For any prime* $p>3$, *we have*

$$\sum_{n=0}^{p-1}\frac{n+1}{8^n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}^2\equiv(-1)^{(p-1)/2}p+5p^3E_{p-3} \pmod{p^4},$$

$$\sum_{n=0}^{p-1}\frac{2n+1}{(-16)^n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}^2\equiv(-1)^{(p-1)/2}p+3p^3E_{p-3} \pmod{p^4}.$$

Remark 33. We are able to show the congruence in part (i) modulo p. The two congruences in part (ii) modulo p^3 were proved by the author [77, Lemma 3.2]. Note also that

$$16^n\sum_{k=0}^n\binom{-1/2}{k}^2\binom{-1/2}{n-k}^2=\sum_{k=0}^n\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2$$
$$=\sum_{k=0}^n\binom{2k}{k}^3\binom{k}{n-k}(-16)^{n-k}$$

by [77, (3.1)].

Conjecture 34. *Let*

$$a_n:=\sum_{k=0}^n\binom{-1/4}{k}^2\binom{-3/4}{n-k}^2\qquad \textit{for all } n\in\mathbb{N}.$$

(i) (2013-08-21) *For any odd prime p, we have*

$$\left(\frac{2}{p}\right)\det[a_{i+j}]_{0\leqslant i,j\leqslant (p-1)/2}\equiv\begin{cases}4x^2\pmod p & \textit{if } p=x^2+4y^2\ (x,y\in\mathbb{Z}),\\ -p\pmod{p^2} & \textit{if } p\equiv 3\pmod 4.\end{cases}$$

(ii) *For any odd prime p and $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{(pn)^3\binom{2n}{n}^2}\left(\sum_{k=0}^{pn-1}\frac{18k^2+7k+1}{(-128)^k}\binom{2k}{k}^2a_k-\left(\frac{2}{p}\right)p^2\sum_{k=0}^{n-1}\frac{18k^2+7k+1}{(-128)^k}\binom{2k}{k}^2a_k\right)\in\mathbb{Z}_p. \tag{2.31}$$

Remark 34. Part (ii) corresponds to the author's conjectural series

$$\sum_{k=0}^\infty\frac{18k^2+7k+1}{(-128)^k}\binom{2k}{k}^2\sum_{j=0}^k\binom{-1/4}{j}^2\binom{-3/4}{k-j}^2=\frac{4\sqrt2}{\pi^2}$$

(cf. [75, (1.22)]), and (2.31) with $n=1$ was first posed by the author in [75, Conjecture 5.15(i)]. By Sun [91, (3.1)], for all $n\in\mathbb{N}$ we have

$$64^n\sum_{k=0}^n\binom{-1/4}{k}^2\binom{-3/4}{n-k}^2=\sum_{k=0}^n\binom{2k}{k}^3\binom{2(n-k)}{n-k}16^{n-k}.$$

Recall that a polynomial $P(x)\in\mathbb{Q}[x]$ is called *integer-valued* if $P(m)\in\mathbb{Z}$ for all $m\in\mathbb{Z}$.

Conjecture 35. (i) *For any $\varepsilon\in\{\pm1\}$ and $l,m,n\in\mathbb{Z}^+$, the polynomial*

$$\frac1n\sum_{k=0}^{n-1}\varepsilon^k(2k+1)^{2l-1}\sum_{j=0}^k\binom{-x}{j}^m\binom{x-1}{k-j}^m$$

is integer-valued.

(ii) *For any $l,n\in\mathbb{Z}^+$, the polynomial*

$$\frac{(2l-1)!!}{n^2}\sum_{k=0}^{n-1}(2k+1)^{2l-1}\sum_{j=0}^{k}\binom{-x}{j}^2\binom{x-1}{k-j}^2$$

is integer-valued.

(iii) *Let $p>3$ be a prime. For any $x\in\mathbb{Z}_p$ with $3x\not\equiv 1,2 \pmod p$, we have*

$$\sum_{k=0}^{p-1}(-1)^k(2k+1)\sum_{j=0}^{k}\binom{-x}{j}^3\binom{x-1}{k-j}^3\equiv 0 \pmod{p^2}.$$

For any $x\in\mathbb{Z}_p$ with $x\equiv 1/3 \pmod p$, we have

$$\sum_{k=0}^{p-1}(-1)^k(2k+1)\sum_{j=0}^{k}\binom{-x}{j}^3\binom{x-1}{k-j}^3\equiv x+\frac{p(\frac{p}{3})-1}{3} \pmod{p^2}.$$

Moreover,

$$\sum_{k=0}^{p-1}(-1)^k(2k+1)\sum_{j=0}^{k}\binom{-1/2}{j}^3\binom{-1/2}{k-j}^3\equiv p^2+\frac{7}{2}p^5B_{p-3} \pmod{p^6}$$

and

$$\sum_{k=0}^{p-1}(-1)^k(2k+1)\sum_{j=0}^{k}\binom{-1/3}{j}^3\binom{-2/3}{k-j}^3\equiv \frac{p}{3}\left(\left(\frac{p}{3}\right)+2p\right) \pmod{p^4}.$$

(iv) *Let p be an odd prime. For any $x\in\mathbb{Z}_p$ with $3x\not\equiv \pm1,2,4 \pmod p$, we have*

$$\sum_{k=0}^{p-1}(-1)^k(2k+1)^3\sum_{j=0}^{k}\binom{-x}{j}^3\binom{x-1}{k-j}^3\equiv 0 \pmod{p^2}.$$

Moreover,

$$\sum_{k=0}^{p-1}(-1)^k(2k+1)^3\sum_{j=0}^{k}\binom{-1/2}{j}^3\binom{-1/2}{k-j}^3\equiv -\frac{3}{5}p^2 \pmod{p^5}.$$

If $p>5$, then for $x=\pm(p-(\frac{p}{3}))/3$ we have

$$\sum_{k=0}^{p-1}(-1)^k(2k+1)^3\sum_{j=0}^{k}\binom{-x}{j}^3\binom{x-1}{k-j}^3\equiv 0 \pmod{p^2}.$$

(v) *Let p be an odd prime. If $p\equiv 5,7 \pmod 8$, then*

$$\sum_{k=0}^{p-1}(2k+1)\sum_{j=0}^{k}\binom{-1/2}{j}^3\binom{-1/2}{k-j}^3\equiv 0 \pmod{p^3}.$$

If $p\equiv 2 \pmod 3)$*, then*

$$\sum_{k=0}^{p-1}(2k+1)\sum_{j=0}^{k}\binom{-1/3}{j}^3\binom{-2/3}{k-j}^3\equiv 0 \pmod{p^3}.$$

Also,

$$\sum_{k=0}^{p-1}(-1)^k(2k+1)\sum_{j=0}^{k}\binom{-1/4}{j}^3\binom{-3/4}{k-j}^3\equiv p^2 \pmod{p^3}$$

if $p\not\equiv 5 \pmod 8$*, and*

$$\sum_{k=0}^{p-1}(-1)^k(2k+1)\sum_{j=0}^{k}\binom{-1/6}{j}^3\binom{-5/6}{k-j}^3\equiv p^2 \pmod{p^4}$$

if $p\equiv \pm1 \pmod{12}$.

Remark 35. Those congruences modulo p^2 in this conjecture might not be very difficult.

Conjecture 36. (i) *For each prime* $p>3$*, we have*

$$\sum_{k=0}^{p-1}(2k+1)\sum_{j=0}^{k}\binom{-1/6}{j}^4\binom{-5/6}{k-j}^4\equiv 0 \pmod{p^2}.$$

For any prime $p>7$*, we have*

$$\sum_{k=0}^{p-1}(2k+1)^3\sum_{j=0}^{k}\binom{-1/6}{j}^4\binom{-5/6}{k-j}^4\equiv 0 \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}(-1)^k(2k+1)\sum_{j=0}^{k}\binom{-1/6}{j}^5\binom{-5/6}{k-j}^5\equiv 0 \pmod{p^2}.$$

Also, for each prime $p>20$ *we have*

$$\sum_{k=0}^{p-1}(2k+1)^3\binom{-1/6}{j}^5\binom{-5/6}{k-j}^5\equiv 0 \pmod{p^2}.$$

(ii) *Let* p *be an odd prime. If* $p\equiv 3 \pmod 4$*, then*

$$\sum_{k=0}^{p-1}(2k+1)\sum_{j=0}^{k}\binom{-1/4}{j}^4\binom{-3/4}{k-j}^4\equiv 0 \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}(2k+1)\sum_{j=0}^{k}\binom{-1/2}{j}^5\binom{-1/2}{k-j}^5\equiv 0 \pmod{p^3}.$$

If $p\equiv 5 \pmod 6$, *then*

$$\sum_{k=0}^{p-1}(2k+1)\sum_{j=0}^{k}\binom{-1/6}{j}^6\binom{-5/6}{k-j}^6\equiv 0 \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}(2k+1)^3\sum_{j=0}^{k}\binom{-1/6}{j}^6\binom{-5/6}{k-j}^6\equiv 0 \pmod{p^2}.$$

(iii) *For any prime* $p>11$, *we have*

$$\sum_{k=0}^{p-1}(2k+1)^3\sum_{j=0}^{k}\binom{-1/5}{j}^4\binom{-4/5}{k-j}^4\equiv 0 \pmod{p^2}$$

if $p\equiv \pm 1 \pmod 5$, *and*

$$\sum_{k=0}^{p-1}(2k+1)^3\sum_{j=0}^{k}\binom{-2/5}{j}^4\binom{-3/5}{k-j}^4\equiv 0 \pmod{p^2}$$

if $p\equiv \pm 2 \pmod 5$. *Also,*

$$\sum_{k=0}^{p-1}(-1)^k(2k+1)\sum_{j=0}^{k}\binom{-1/5}{j}^5\binom{-4/5}{k-j}^5\equiv 0 \pmod{p^2}$$

for any prime $p\equiv 4 \pmod 5$, *and*

$$\sum_{k=0}^{p-1}(-1)^k(2k+1)\sum_{j=0}^{k}\binom{-2/5}{j}^5\binom{-3/5}{k-j}^5\equiv 0 \pmod{p^2}$$

for each prime $p\equiv 3 \pmod 5$.

Remark 36. It is interesting to compare part (iii) of this conjecture with the classical Rogers-Ramanujan identities.

Conjecture 37. (i) *Let* p *be an odd prime. Then*

$$\sum_{k=0}^{p-1}(64k^2+64k+23)\sum_{j=0}^{k}\binom{-1/4}{j}^3\binom{-3/4}{k-j}^3$$
$$\equiv 23p^2+174\left(\frac{-1}{p}\right)p^4E_{p-3} \pmod{p^5}.$$

Also, for each $n\in\mathbb{Z}^+$ the number

$$\sum_{k=0}^{pn-1}(64k^2+64k+23)\sum_{j=0}^{k}\binom{-1/4}{j}^3\binom{-3/4}{k-j}^3 - p^2\sum_{k=0}^{n-1}(64k^2+64k+23)\sum_{j=0}^{k}\binom{-1/4}{j}^3\binom{-3/4}{k-j}^3$$

divided by $(pn)^4$ is a p-adic integer.

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}(3k^2+3k+1)\sum_{j=0}^{k}\binom{-1/3}{j}^3\binom{-2/3}{k-j}^3 \equiv p^2+\left(\frac{p}{3}\right)\frac{2}{3}p^4B_{p-2}\left(\frac{1}{3}\right) \pmod{p^5}$$

and

$$\sum_{k=0}^{p-1}(48k^2+48k+19)\sum_{k=0}^{n}\binom{-1/6}{j}^3\binom{-5/6}{k-j}^3 \equiv 19p^2+\left(\frac{p}{3}\right)\frac{335}{3}p^4B_{p-2}\left(\frac{1}{3}\right) \pmod{p^5}.$$

Also, for any $n\in\mathbb{Z}^+$, the number

$$\sum_{k=0}^{pn-1}(3k^2+3k+1)\sum_{j=0}^{k}\binom{-1/3}{j}^3\binom{-2/3}{k-j}^3 - p^2\sum_{k=0}^{n-1}(3k^2+3k+1)\sum_{j=0}^{k}\binom{-1/3}{j}^3\binom{-2/3}{k-j}^3$$

divided by $(pn)^4$, and the number

$$\sum_{k=0}^{pn-1}(48k^2+48k+19)\sum_{j=0}^{k}\binom{-1/6}{j}^3\binom{-5/6}{k-j}^3 - p^2\sum_{k=0}^{n-1}(48k^2+48k+19)\sum_{j=0}^{k}\binom{-1/6}{j}^3\binom{-5/6}{k-j}^3$$

divided by $(pn)^4$, are both p-adic integers.

Remark 37. The first congruence in Conjecture 37 with $n=1$ was discovered by the author on March 14, 2013.

Conjecture 38. (i) *For any prime $p>5$, we have*

$$\left(\frac{-1}{p}\right)\sum_{n=0}^{p-1}\frac{\binom{2n}{n}}{576^n}\sum_{k=0}^n\frac{5^k\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2}{\binom nk}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac 2p)=(\frac p3)=(\frac p5)=1\ \&\ p=x^2+30y^2,\\8x^2-2p\pmod{p^2}&\text{if }(\frac 2p)=1,\ (\frac p3)=(\frac p5)=-1\ \&\ p=2x^2+15y^2,\\2p-12x^2\pmod{p^2}&\text{if }(\frac p3)=1,\ (\frac 2p)=(\frac p5)=-1\ \&\ p=3x^2+10y^2,\\20x^2-2p\pmod{p^2}&\text{if }(\frac p5)=1,\ (\frac 2p)=(\frac p3)=-1\ \&\ p=5x^2+6y^2,\\0\pmod{p^2}&\text{if }(\frac{-30}{p})=-1,\end{cases}$$

where x and y are integers.

(ii) *For any $n\in\mathbb{Z}^+$, we have*

$$n\binom{2n-1}{n-1}\ \Big|\ \sum_{k=0}^{n-1}(28k+5)576^{n-1-k}\binom{2k}{k}\sum_{j=0}^k 5^j\frac{\binom{2j}{j}\binom{2(k-j)}{k-j}}{\binom kj}.$$

Remark 38. This is related to the author's conjectural identity (cf. [90, (8)])

$$\sum_{n=0}^\infty\frac{28n+5}{576^n}\binom{2n}{n}\sum_{k=0}^n\frac{5^k\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2}{\binom nk}=\frac 9\pi(2+\sqrt2)$$

and corresponding conjectural congruence (cf. [90])

$$\sum_{n=0}^{p-1}\frac{28n+5}{576^n}\binom{2n}{n}\sum_{k=0}^n\frac{5^k\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2}{\binom nk}\equiv p\left(3\left(\frac{-1}{p}\right)+2\left(\frac{-2}{p}\right)\right)\pmod{p^2}$$

for any prime $p>3$. Those numbers

$$P_n:=\sum_{k=0}^n\frac{\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2}{\binom nk}=2^n\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}^2 4^{n-2k}\quad(n\in\mathbb{N})\tag{2.32}$$

are usually called Catalan-Larcombe-French numbers; for some series for $1/\pi$ involving Catalan-Larcombe-French numbers one may consult [7].

Conjecture 39. (i) (Sun [84, Conjecture 1.7]) *For any prime $p>5$, we have*

$$\sum_{n=0}^{p-1}\frac{357n+103}{2160^n}\binom{2n}{n}\sum_{k=0}^n\binom nk\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}$$
$$\equiv p\left(\frac{-1}{p}\right)\left(54+49\left(\frac{p}{15}\right)\right)\pmod{p^2},$$

and

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}}{2160^n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{7})=1,\ p=x^2+105y^2,\\ 2x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{7})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1,\ 2p=x^2+105y^2,\\ 2p-12x^2\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{7})=-1,\ p=3x^2+35y^2,\\ 2p-6x^2\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{7})=-1,\ (\frac{p}{3})=(\frac{p}{5})=1,\ 2p=3x^2+35y^2,\\ 20x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{5})=1,\ (\frac{p}{3})=(\frac{p}{7})=-1,\ p=5x^2+21y^2,\\ 10x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=1,\ (\frac{p}{5})=(\frac{p}{7})=-1,\ 2p=5x^2+21y^2,\\ 28x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{5})=-1,\ (\frac{p}{3})=(\frac{p}{7})=1,\ p=7x^2+15y^2,\\ 14x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=-1,\ (\frac{p}{5})=(\frac{p}{7})=1,\ 2p=7x^2+15y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-105}{p})=-1,\end{cases}$$

where x and y are integers.

(ii) *For any $n\in\mathbb{Z}^+$, we have*

$$n\binom{2n-1}{n-1}\ \Big|\ \sum_{k=0}^{n-1}(357k+103)2160^{n-1-k}\binom{2k}{k}\sum_{j=0}^{k}\binom{k}{j}\binom{k+2j}{2j}\binom{2j}{j}(-324)^{k-j}.$$

Remark 39. The quadratic field $\mathbb{Q}(\sqrt{-105})$ has class number eight. The author would like to offer 105 US dollars for the first correct proof of Conjecture 39, and 90 US dollars for the first rigorous proof of the author's conjectural identity (cf. [84])

$$\sum_{n=0}^{\infty}\frac{357n+103}{2160^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}=\frac{90}{\pi}.$$

For the author's another similar conjectural series (cf. [91, (4.35)])

$$\sum_{n=0}^{\infty}\frac{n}{3645^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}486^{n-k}=\frac{10}{3\pi},$$

we also conjecture that

$$2n\binom{2n}{n}\ \Big|\ \sum_{k=0}^{n-1}k\binom{2k}{k}3645^{n-1-k}\sum_{j=0}^{k}\binom{k}{j}\binom{k+2j}{2j}\binom{2j}{j}486^{k-j}.$$

Conjecture 40. (2007) *Let p be a prime and let $l,n\in\mathbb{N}$ and $r\in\mathbb{Z}$. If n or r is not divisible by p, then we have*

$$\begin{aligned}&\nu_p\bigg(\sum_{k\equiv r\,(\mathrm{mod}\ p)}\binom{n}{k}(-1)^k\binom{(k-r)/p}{l}\bigg)\\&\geqslant\left\lfloor\frac{n-lp-1}{p-1}\right\rfloor+\nu_p\bigg(\binom{\lfloor(n-l-1)/(p-1)\rfloor}{l}\bigg).\end{aligned}$$

Remark 40. D. Wan [107] proved that the inequality holds if the last term on the right-hand side is omitted (see also Sun and Wan [100]).

3 Congruences Involving the Polynomials $p_n(x)=\sum_{k=0}^n\binom{2k}{k}^2\binom{2(n-k)}{n-k}x^{n-k}$ or $S_n(x)=\sum_{k=0}^n\binom{n}{k}^4x^k$

In 2011 the author (cf. [76, 91]) posed many conjectural series for $1/\pi$ involving a new kind of polynomials

$$p_n(x):=\sum_{k=0}^n\binom{2k}{k}^2\binom{2n-2k}{n-k}x^{n-k}\quad(n\in\mathbb{N}).\tag{3.1}$$

Here we present some related conjectures mainly made by the author in 2011.

Conjecture 41. (i) *For any prime $p\neq 2,5$, we have*

$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{100^k}p_k\left(\frac{9}{4}\right)\\&\equiv\begin{cases}(\frac{-1}{p})(4x^2-2p)\pmod{p^2} & \text{if } (\frac{p}{7})=1\ \&\ p=x^2+7y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{p}{7})=-1,\ \text{i.e., } p\equiv 3,5,6\pmod 7,\end{cases}\end{aligned}$$

and

$$\frac{1}{(pn)^2\binom{2n}{n}}\bigg(\sum_{k=0}^{pn-1}\frac{12k+1}{100^k}\binom{2k}{k}p_k\left(\frac{9}{4}\right)-\left(\frac{-1}{p}\right)p\sum_{r=0}^{n-1}\frac{12r+1}{100^r}\binom{2r}{r}p_r\left(\frac{9}{4}\right)\bigg)\in\mathbb{Z}_p$$

for all $n\in\mathbb{Z}^+$. Moreover, we have

$$\sum_{k=0}^{\infty}\frac{12k+1}{100^k}\binom{2k}{k}p_k\left(\frac{9}{4}\right)=\frac{75}{4\pi}.\tag{3.2}$$

(ii) *For any integer $n>1$, we have*

$$\frac{4^{n-1}}{n\binom{2n}{n}}\sum_{k=0}^{n-1}(12k+1)100^{n-1-k}\binom{2k}{k}p_k\left(\frac{9}{4}\right)\in\mathbb{Z}^+.$$

Remark 41. This conjecture was formulated by the author in 2019. We found the identity (3.2) by using the Philosophy about Series for $1/\pi$ stated by the author [84].

Conjecture 42. (i) *For any prime $p>3$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{(-192)^k}p_k(4)$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1 \pmod 3 \ \&\ p=x^2+3y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

and

$$\frac{1}{(pn)^2\binom{2n}{n}}\left(\sum_{k=0}^{pn-1}\frac{4k+1}{(-192)^k}\binom{2k}{k}p_k(4)-\left(\frac{p}{3}\right)p\sum_{r=0}^{n-1}\frac{4r+1}{(-192)^r}\binom{2r}{r}p_r(4)\right)\in\mathbb{Z}_p$$

for all $n\in\mathbb{Z}^+$.

(ii) *For any $n\in\mathbb{Z}^+$, we have*

$$n\binom{2n-1}{n-1}\ \Big|\ \sum_{k=0}^{n-1}(4k+1)(-192)^{n-1-k}\binom{2k}{k}p_k(4).$$

Remark 42. This is related to the author's following conjectural series (cf. [76, 91] discovered in 2011:

$$\sum_{k=0}^{\infty}\frac{4k+1}{(-192)^k}\binom{2k}{k}p_k(4)=\frac{\sqrt{3}}{\pi}.$$

By [91, Lemma 2.2], for any $n\in\mathbb{N}$ we have

$$\binom{2n}{n}p_n(4)=\sum_{k=0}^{n}\binom{2k}{k}^2\binom{4k}{2k}\binom{k}{n-k}(-64)^{n-k}.$$

Conjecture 43. (i) *For any prime $p>5$, we have*

$$\sum_{k=0}^{p-1}\frac{17k-224}{(-225)^k}\binom{2k}{k}p_k(-14)\equiv 32p\left(2\left(\frac{-1}{p}\right)-9\right) \pmod{p^2},$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{(-225)^k}p_k(-14)$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=1\ \&\ p=x^2+7y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{p}{7})=-1.\ \text{i.e., } p\equiv 3,5,6 \pmod 7).\end{cases}$$

(ii) *Let p be an odd prime with $p \neq 17$. Then*

$$\sum_{k=0}^{p-1}\frac{15k-256}{17^{2k}}\binom{2k}{k}p_k(18)\equiv -64p\left(3+\left(\frac{-1}{p}\right)\right)\pmod{p^2}.$$

Also,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{17^{2k}}p_k(18)=\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{p}{7})=1\ \&\ p=x^2+7y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{p}{7})=-1,\ \text{i.e., } p\equiv 3,5,6\pmod 7.\end{cases}$$

(iii) *For any $n\in\mathbb{Z}^+$, we have*

$$4n\binom{2n}{n}\ \Big|\ \sum_{k=0}^{n-1}(17k-224)(-225)^{n-1-k}\binom{2k}{k}p_k(-14),$$
$$4n\binom{2n}{n}\ \Big|\ \sum_{k=0}^{n-1}(15k-256)289^{n-1-k}\binom{2k}{k}p_k(18).$$

Remark 43. Two related conjectural series posed by the author are

$$\sum_{k=0}^{\infty}\frac{17k-224}{(-225)^k}\binom{2k}{k}p_k(-14)=\frac{1800}{\pi},\quad \sum_{k=0}^{\infty}\frac{15k-256}{17^{2k}}\binom{2k}{k}p_k(18)=\frac{2312}{\pi}.$$

Conjecture 44. (i) *For any prime $p>3$, we have*

$$\sum_{k=0}^{p-1}\frac{20k-11}{(-576)^k}\binom{2k}{k}p_k(-32)\equiv p\left(5\left(\frac{-1}{p}\right)-16\left(\frac{2}{p}\right)\right)\pmod{p^2},$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{(-576)^k}p_k(-32)$$
$$\equiv\begin{cases}(\frac{-1}{p})(4x^2-2p)\pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{5}{p})=1\ \&\ p=x^2+10y^2\ (x,y\in\mathbb{Z}),\\ (\frac{-1}{p})(2p-8x^2)\pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{5}{p})=-1\ \&\ p=2x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{-10}{p})=-1.\end{cases}$$

(ii) *For any prime $p\neq 2,5$, we have*

$$\sum_{k=0}^{p-1}\frac{3k-2}{640^k}\binom{2k}{k}p_k(36)\equiv -2p\left(\frac{5}{p}\right)\pmod{p^2},$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{640^k}p_k(36)$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{5}{p})=1\ \&\ p=x^2+10y^2\ (x,y\in\mathbb{Z}),\\ 2p-8x^2\pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{5}{p})=-1\ \&\ p=2x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{-10}{p})=-1.\end{cases}$$

(iii) *For any* $n\in\mathbb{Z}^+$ *we have*

$$n\binom{2n-1}{n-1}\ \Big|\ \sum_{k=0}^{n-1}(20k-11)(-576)^{n-1-k}\binom{2k}{k}p_k(-32),$$
$$n\binom{2n}{n}\ \Big|\ \sum_{k=0}^{n-1}(3k-2)640^{n-1-k}\binom{2k}{k}p_k(36).$$

Remark 44. The two related conjectural series of the author [91] are

$$\sum_{k=0}^{\infty}\frac{20k-11}{(-576)^k}\binom{2k}{k}p_k(-32)=\frac{90}{\pi},\quad \sum_{k=0}^{\infty}\frac{3k-2}{640^k}\binom{2k}{k}p_k(36)=\frac{5\sqrt{10}}{\pi}.$$

Conjecture 45. (i) *For any prime* $p\neq 2,7$, *we have*

$$\sum_{k=0}^{p-1}\frac{20k-67}{(-3136)^k}\binom{2k}{k}p_k(-192)\equiv p\left(5\left(\frac{-1}{p}\right)-72\left(\frac{3}{p}\right)\right)\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{(-3136)^k}p_k(-192)$$
$$\equiv\begin{cases}(\frac{-1}{p})(4x^2-2p)\pmod{p^2} & \text{if } (\frac{-2}{p})=1\ \&\ p=x^2+2y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } p\equiv 5,7\pmod 8.\end{cases}$$

(ii) *For any prime* $p\neq 2,5$, *we have*

$$\sum_{k=0}^{p-1}\frac{7k-24}{3200^k}\binom{2k}{k}p_k(196)\equiv -24\left(\frac{6}{p}\right)\pmod{p^2},$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{3200^k}p_k(196)\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p=x^2+2y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } p\equiv 5,7\pmod 8.\end{cases}$$

(iii) *For any $n\in\mathbb{Z}^+$, we have*

$$n\binom{2n-1}{n-1}\ \Big|\ \sum_{k=0}^{n-1}(20k-67)(-3136)^{n-1-k}\binom{2k}{k}p_k(-192),$$
$$2n\binom{2n}{n}\ \Big|\ \sum_{k=0}^{n-1}(7k-24)3200^{n-1-k}\binom{2k}{k}p_k(196).$$

Remark 45. The two related conjectural series of the author [91] are

$$\sum_{k=0}^{\infty}\frac{20k-67}{(-3136)^k}\binom{2k}{k}p_k(-192)=\frac{490}{\pi},\quad \sum_{k=0}^{\infty}\frac{7k-24}{3200^k}\binom{2k}{k}p_k(196)=\frac{125\sqrt{2}}{\pi}.$$

Conjecture 46. (i) *For any prime $p>3$ with $p\neq 11$, we have*

$$\sum_{k=0}^{p-1}\frac{5k-32}{(-6336)^k}\binom{2k}{k}p_k(-392)\equiv\frac{p}{4}\left(5\left(\frac{-1}{p}\right)-133\left(\frac{11}{p}\right)\right)\pmod{p^2},$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{(-6336)^k}p_k(-392)$$
$$\equiv\begin{cases}(\frac{-1}{p})(4x^2-2p)\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{11})=1\ \&\ p=x^2+22y^2\ (x,y\in\mathbb{Z}),\\ (\frac{-1}{p})(2p-8x^2)\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{11})=-1\ \&\ p=2x^2+11y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{-22}{p})=-1.\end{cases}$$

(ii) *For any odd prime $p\neq 5$, we have*

$$\sum_{k=0}^{p-1}\frac{66k-427}{80^{2k}}\binom{2k}{k}p_k(396)\equiv-427p\pmod{p^2},$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{80^{2k}}p_k(396)$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{11})=1\ \&\ p=x^2+22y^2\ (x,y\in\mathbb{Z}),\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{11})=-1\ \&\ p=2x^2+11y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{-22}{p})=-1.\end{cases}$$

(iii) *For any $n\in\mathbb{Z}^+$, we have*

$$2n\binom{2n}{n}\ \Big|\ \sum_{k=0}^{n-1}(5k-32)(-6336)^{n-1-k}\binom{2k}{k}p_k(-392),$$

$$n\binom{2n-1}{n-1}\ \Big|\ \sum_{k=0}^{n-1}(66k-427)6400^{n-1-k}\binom{2k}{k}p_k(396).$$

Remark 46. The two related conjectural series of the author [91] are

$$\sum_{k=0}^{\infty}\frac{5k-32}{(-6336)^k}\binom{2k}{k}p_k(-392)=\frac{495}{2\pi},\quad \sum_{k=0}^{\infty}\frac{66k-427}{6400^k}\binom{2k}{k}p_k(396)=\frac{1000\sqrt{11}}{\pi}.$$

Conjecture 47. (i) *For any prime $p>3$, we have*

$$\sum_{k=0}^{p-1}\frac{34k-7}{(-2^{11}3^2)^k}\binom{2k}{k}p_k(-896)\equiv\frac{p}{2}\left(9\left(\frac{-2}{p}\right)-23\left(\frac{2}{p}\right)\right)\pmod{p^2},$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{(-2^{11}3^2)^k}p_k(-896)$$
$$\equiv\begin{cases}(\frac{2}{p})(4x^2-2p)\pmod{p^2} & \text{if } (\frac{p}{7})=1\ \&\ p=x^2+7y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{p}{7})=-1,\ \text{i.e., } p\equiv 3,5,6\pmod 7.\end{cases}$$

(ii) *Let p be an odd prime with $p\neq 17$. Then*

$$\sum_{k=0}^{p-1}\frac{24k-5}{136^{2k}}\binom{2k}{k}p_k(900)\equiv p\left(3\left(\frac{-1}{p}\right)-8\right)\pmod{p^2}.$$

Also,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{136^{2k}}p_k(900)$$
$$=\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{p}{7})=1\ \&\ p=x^2+7y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{p}{7})=-1, \text{i.e., } p\equiv 3,5,6\pmod 7.\end{cases}$$

(iii) *For any $n\in\mathbb{Z}^+$, we have*

$$n\binom{2n-1}{n-1}\ \Big|\ \sum_{k=0}^{n-1}(34k-7)(-18432)^{n-1-k}\binom{2k}{k}p_k(-896),$$

$$n\binom{2n-1}{n-1}\ \Big|\ \sum_{k=0}^{n-1}(24k-5)(136^2)^{n-1-k}\binom{2k}{k}p_k(900).$$

Remark 47. Two related conjectural series posed by the author [91] are

$$\sum_{k=0}^{\infty}\frac{34k-7}{(-18432)^k}\binom{2k}{k}p_k(-896)=\frac{54\sqrt2}{\pi},\quad \sum_{k=0}^{\infty}\frac{24k-5}{136^{2k}}\binom{2k}{k}p_k(900)=\frac{867}{16\pi}.$$

We mention that the author's conjectural identities in Remarks 41–47 remain open, but the following ones (discovered by the author in 2011 and published in [91]) have been proved (cf. [12, 47]):

$$\sum_{k=0}^{\infty}\frac{k-1}{72^k}\binom{2k}{k}p_k(4)=\frac{9}{\pi},\quad \sum_{k=0}^{\infty}\frac{k-2}{100^k}\binom{2k}{k}p_k(6)=\frac{50}{3\pi},$$

$$\sum_{k=0}^{\infty}\frac{k}{(-192)^k}\binom{2k}{k}p_k(-8)=\frac{3}{2\pi},\quad \sum_{k=0}^{\infty}\frac{6k-1}{256^k}\binom{2k}{k}p_k(12)=\frac{8\sqrt3}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{10k+1}{(-1530)^k}\binom{2k}{k}p_k(-32)=\frac{3\sqrt6}{\pi},\quad \sum_{k=0}^{\infty}\frac{12k+1}{1600^k}\binom{2k}{k}p_k(36)=\frac{75}{8\pi},$$

$$\sum_{k=0}^{\infty}\frac{24k+5}{3136^k}\binom{2k}{k}p_k(-60)=\frac{49\sqrt3}{8\pi},\quad \sum_{k=0}^{\infty}\frac{14k+3}{(-3072)^k}\binom{2k}{k}p_k(64)=\frac{6}{\pi}.$$

Conjecture 48. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{p_k(1)}{16^k}\equiv\left(\frac{p}{3}\right)\left(1-\frac{p}{4}q_p(3)\right)\pmod{p^2}.$$

Also,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{96^k}p_k(-2)\equiv\begin{cases}0\pmod p & \text{if } p\equiv7\pmod{12},\\ 0\pmod{p^2} & \text{if } p\equiv5\pmod 6.\end{cases}$$

If $(\frac{p}{7})=-1$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{14^k}p_k\left(\frac12\right)\equiv0\pmod{p^2}.$$

Remark 48. In view of [75, Lemma 2.1], it is easy to see that

$$\sum_{k=0}^{p-1}\frac{p_k(1)}{4^k}\equiv\left(\frac{p}{3}\right)p\pmod{p^2}$$

for any odd prime p. We have also shown that $n\mid\sum_{k=0}^{n-1}p_k(1)4^{n-1-k}$ for all $n\in\mathbb{Z}^+$. The series $\sum_{k=0}^{\infty}\binom{2k}{k}p_k(-2)/96^k$ convergence, but we are unable to guess its exact value. We also have conjectures on $\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{m^k}p_k(x)$ mod p^2 (with p an odd prime and m an integer not divisible by p) if (x,m) is among the following ordered pairs:

$$(-2,-36),\ (8,64),\ (8,128),\ (-12,64),\ (-12,-128),\ (16,192),\ (36,64),$$

$$(-36,-512),\ (40,576),\ (-96,-512),\ (-192,1024),\ (200,3136),\ (-252,64).$$

The author [84] introduced the polynomials

$$S_n(x) := \sum_{k=0}^{n} \binom{n}{k}^4 x^k \quad (n = 0, 1, 2, \ldots).$$

Note that $S_n(1) = \sum_{k=0}^{n} \binom{n}{k}^4$ with $n \in \mathbb{N}$ are called the Franel numbers of order 4. In 2005 Y. Yang found the interesting identity

$$\sum_{k=0}^{\infty} \frac{4k+1}{36^k} S_k(1) = \frac{18}{\sqrt{15}\,\pi}.$$

More such series for $1/\pi$ were given by S. Cooper [11].

Conjecture 49. *Let p be an odd prime.*

(i) (Sun [84]) *We have*

$$\sum_{k=0}^{p-1} S_k(1) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \ \&\ p = x^2 + 15y^2 \ (x, y \in \mathbb{Z}), \\ 12x^2 - 2p \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \ \&\ p = 3x^2 + 5y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } (\frac{p}{15}) = -1, \ \text{i.e., } p \equiv 7, 11, 13, 14 \pmod{15}. \end{cases}$$

Also,

$$\sum_{k=0}^{p-1} S_k(-2) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod 4 \ \&\ p = x^2 + 4y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

(ii) *Let $n \in \mathbb{Z}^+$. If $p > 3$, then*

$$\frac{1}{n^2(n+1)} \left(\sum_{k=0}^{pn-1} (3k+2) S_k(1) - p \left(\frac{p}{5}\right) \sum_{r=0}^{n-1} (3r+2) S_r(1) \right) \equiv 0 \pmod{p^2}. \tag{3.3}$$

If $p \equiv 1 \pmod 4$, then

$$\frac{1}{n^2} \left(\sum_{k=0}^{pn-1} (3k+2) S_k(-2) - p \sum_{r=0}^{n-1} (3r+2) S_r(-2) \right) \equiv 0 \pmod{p^2}. \tag{3.4}$$

Remark 49. The conjectural congruence (3.3) with $n = 1$ first appeared in [84, Conjecture 3.5]. The author [84, Conjecture 3.1] conjectured that for any odd prime p we have

$$\sum_{k=0}^{p-1} (3k+2) S_k(-2) \equiv \frac{p}{2} \left(1 + 3 \left(\frac{-1}{p} \right) \right) \pmod{p^2}.$$

Conjecture 50. (Sun [84]) *Let p be an odd prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}S_k(12)$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if } p\equiv 1\pmod{12}\ \&\ p=x^2+y^2\ (3\nmid x),\\ (\frac{xy}{3})4xy\pmod{p^2}&\text{if } p\equiv 5\pmod{12}\ \&\ p=x^2+y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2}&\text{if } p\equiv 3\pmod 4.\end{cases}$$

And

$$\sum_{k=0}^{p-1}(4k+3)S_k(12)\equiv p\left(1+2\left(\frac{3}{p}\right)\right)\pmod{p^2}.$$

Moreover,

$$\frac{1}{n}\sum_{k=0}^{n-1}(4k+3)S_k(12)\in\mathbb{Z}\qquad\text{for all } n=1,2,3,\ldots.$$

(ii) *We have*

$$\sum_{k=0}^{p-1}S_k(-20)$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if } (\frac{-1}{p})=(\frac{p}{5})=1\ \&\ p=x^2+y^2\ (x,y\in\mathbb{Z}\ \&\ 5\nmid x),\\ 4xy\pmod{p^2}&\text{if } (\frac{-1}{p})=-(\frac{p}{5})=1,\ p=x^2+y^2\ (x,y\in\mathbb{Z}\ \&\ 5\mid x-y),\\ 0\pmod{p^2}&\text{if } p\equiv 3\pmod 4.\end{cases}$$

And

$$\sum_{k=0}^{p-1}(6k+5)S_k(-20)\equiv p\left(\frac{-1}{p}\right)\left(2+3\left(\frac{-5}{p}\right)\right)\pmod{p^2}.$$

Moreover,

$$\frac{1}{n}\sum_{k=0}^{n-1}(6k+5)S_k(-20)\in\mathbb{Z}\qquad\text{for all } n=1,2,3,\ldots.$$

Remark 50. Note that for an odd prime $p=x^2+y^2$ with $x,y\in\mathbb{Z}$ and $x\equiv y\pmod 5$ we have the surprising conjectural congruence $\sum_{k=0}^{p-1}S_k(-20)\equiv 4xy\pmod{p^2}$.

Conjecture 51. (Sun [84]) *Let p be an odd prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}S_k(36)$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{5})=1\ \&\ p=x^2+30y^2,\\ 12x^2-2p\pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1\ \&\ p=3x^2+10y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1\ \&\ p=2x^2+15y^2,\\ 2p-20x^2\pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1\ \&\ p=5x^2+6y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-30}{p})=-1,\end{cases}$$

where x and y are integers. And

$$\sum_{k=0}^{p-1}(8k+7)S_k(36)\equiv p\left(\frac{p}{15}\right)\left(3+4\left(\frac{-6}{p}\right)\right)\pmod{p^2}.$$

We also have

$$\frac{1}{n}\sum_{k=0}^{n-1}(8k+7)S_k(36)\in\mathbb{Z}\qquad \text{for all } n=1,2,3,\ldots.$$

(ii) *We have*

$$\sum_{k=0}^{p-1}S_k(196)$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{5})=(\frac{p}{7})=1\ \&\ p=x^2+70y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{p}{7})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1\ \&\ p=2x^2+35y^2,\\ 2p-20x^2\pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{7})=-1\ \&\ p=5x^2+14y^2,\\ 28x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{5})=(\frac{p}{7})=-1\ \&\ p=7x^2+10y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-70}{p})=-1,\end{cases}$$

where x and y are integers. And

$$\sum_{k=0}^{p-1}(120k+109)S_k(196)\equiv p\left(\frac{p}{7}\right)\left(49+60\left(\frac{-14}{p}\right)\right)\pmod{p^2}.$$

We also have

$$\frac{1}{n}\sum_{k=0}^{n-1}(120k+109)S_k(196)\in\mathbb{Z}\qquad \text{for all } n=1,2,3,\ldots.$$

(iii) *We have*

$$\sum_{k=0}^{p-1}S_k(-324)$$
$$\equiv\begin{cases}4x^2-2p\ (\mathrm{mod}\ p^2)&\text{if }(\frac{-1}{p})=(\frac{p}{5})=(\frac{p}{17})=1\ \&\ p=x^2+85y^2,\\ 2x^2-2p\ (\mathrm{mod}\ p^2)&\text{if }(\frac{p}{17})=1,\ (\frac{-1}{p})=(\frac{p}{5})=-1\ \&\ 2p=x^2+85y^2,\\ 2p-20x^2\ (\mathrm{mod}\ p^2)&\text{if }(\frac{-1}{p})=1,\ (\frac{p}{5})=(\frac{p}{17})=-1\ \&\ p=5x^2+17y^2,\\ 2p-10x^2\ (\mathrm{mod}\ p^2)&\text{if }(\frac{p}{5})=1,\ (\frac{-1}{p})=(\frac{p}{17})=-1\ \&\ 2p=5x^2+17y^2,\\ 0\ (\mathrm{mod}\ p^2)&\text{if }(\frac{-85}{p})=-1,\end{cases}$$

where x and y are integers. Provided $p>3$ we have

$$\sum_{k=0}^{p-1}(34k+31)S_k(-324)\equiv p\left(\frac{p}{5}\right)\left(17+14\left(\frac{-1}{p}\right)\right)\ (\mathrm{mod}\ p^2).$$

Moreover,

$$\frac{1}{n}\sum_{k=0}^{n-1}(34k+31)S_k(-324)\in\mathbb{Z}\qquad\text{for all } n=1,2,3,\ldots.$$

(iv) *We have*

$$\sum_{k=0}^{p-1}S_k(1296)$$
$$\equiv\begin{cases}4x^2-2p\ (\mathrm{mod}\ p^2)&\text{if }(\frac{-2}{p})=(\frac{p}{5})=(\frac{p}{13})=1\ \&\ p=x^2+130y^2,\\ 8x^2-2p\ (\mathrm{mod}\ p^2)&\text{if }(\frac{-2}{p})=1,\ (\frac{p}{5})=(\frac{p}{13})=-1\ \&\ p=2x^2+65y^2,\\ 2p-20x^2\ (\mathrm{mod}\ p^2)&\text{if }(\frac{p}{5})=1,\ (\frac{-2}{p})=(\frac{p}{13})=-1\ \&\ p=5x^2+26y^2,\\ 2p-40x^2\ (\mathrm{mod}\ p^2)&\text{if }(\frac{p}{13})=1,\ (\frac{-2}{p})=(\frac{p}{5})=-1\ \&\ p=10x^2+13y^2,\\ 0\ (\mathrm{mod}\ p^2)&\text{if }(\frac{-130}{p})=-1,\end{cases}$$

where x and y are integers. Provided $p>3$ we have

$$\sum_{k=0}^{p-1}(130k+121)S_k(1296)\equiv p\left(\frac{-2}{p}\right)\left(56+65\left(\frac{-26}{p}\right)\right)\ (\mathrm{mod}\ p^2).$$

Moreover,

$$\frac{1}{n}\sum_{k=0}^{n-1}(130k+121)S_k(1296)\in\mathbb{Z}\qquad\text{for all } n=1,2,3,\ldots.$$

(v) *We have*

$$\sum_{k=0}^{p-1} S_k(5776)$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{5})=(\frac{p}{19})=1\ \&\ p=x^2+190y^2,\\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{5})=(\frac{p}{19})=-1\ \&\ p=2x^2+95y^2,\\ 2p-20x^2 \pmod{p^2} & \text{if } (\frac{p}{19})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1\ \&\ p=5x^2+38y^2,\\ 2p-40x^2 \pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{19})=-1\ \&\ p=10x^2+19y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-190}{p})=-1,\end{cases}$$

where x *and* y *are integers. And*

$$\sum_{k=0}^{p-1}(816k+769)S_k(5776)\equiv p\left(\frac{p}{95}\right)\left(361+408\left(\frac{p}{19}\right)\right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n}\sum_{k=0}^{n-1}(816k+769)S_k(5776)\in\mathbb{Z}\qquad \text{for all } n=1,2,3,\ldots.$$

Remark 51. The reader may consult [84, Section 3] for more conjectures of this type.

4 Congruences Involving Some Special Numbers

We first present few conjectures on harmonic numbers.

Conjecture 52. (2016-12-20) *Let* $p>3$ *be a prime. Then*

$$p\sum_{k=1}^{p-1}\frac{3H_{k-1}^2+4H_{k-1}/k}{k^2\binom{2k}{k}}\equiv -3\frac{H_{p-1}}{p^2}-\frac{p^2}{5}B_{p-5} \pmod{p^3}$$

and

$$\sum_{k=1}^{p-1}\left(3H_k^2-4\frac{H_k}{k}\right)\frac{\binom{2k}{k}}{k}\equiv 6\frac{H_{p-1}}{p^2}+\frac{8}{5}p^2B_{p-5} \pmod{p^3},$$

Remark 52. The conjecture is related to the author's following conjectural identity

$$\sum_{k=1}^{\infty}\frac{3H_{k-1}^2+4H_{k-1}/k}{k^2\binom{2k}{k}}=\frac{\pi^4}{360}$$

discovered on Dec. 20, 2016.

Conjecture 53. (i) (Sun [92]) *For any prime $p>3$, we have*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}H_k^{(2)}\equiv\frac{2H_{p-1}}{3p^2}+\frac{76}{135}p^2B_{p-5}\pmod{p^3},$$
$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}H_k^{(2)}}{k2^k}\equiv-\frac{3}{16}\cdot\frac{H_{p-1}}{p^2}+\frac{479}{1280}p^2B_{p-5}\pmod{p^3},$$
$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}H_k^{(2)}}{k3^k}\equiv-\frac{8}{9}\cdot\frac{H_{p-1}}{p^2}+\frac{268}{1215}p^2B_{p-5}\pmod{p^3},$$
$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k4^k}H_k^{(2)}\equiv-\frac{3}{2}\cdot\frac{H_{p-1}}{p^2}+\frac{7}{80}p^2B_{p-5}\pmod{p^3}.$$

(ii) *For any prime $p>3$, we have*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\left(3H_{k-1}+\frac{1}{k}\right)\equiv-2\frac{H_{p-1}}{p}+\frac{18}{5}p^3B_{p-5}\pmod{p^4},$$
$$\sum_{k=1}^{p-1}\frac{3H_k-1/k}{k^2\binom{2k}{k}}\equiv-2\frac{H_{p-1}}{p^2}-\frac{2}{5}p^2B_{p-5}\pmod{p^3},$$
$$p\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}}\left(5H_k^{(3)}+\frac{1}{k^3}\right)\equiv 2B_{p-5}\pmod{p}.$$

Remark 53. `Mathematica 9` yields that

$$\sum_{k=1}^{\infty}\frac{2^kH_{k-1}^{(2)}}{k^2\binom{2k}{k}}=\frac{\pi^4}{384},\ \sum_{k=1}^{\infty}\frac{3^kH_{k-1}^{(2)}}{k^2\binom{2k}{k}}=\frac{2\pi^4}{243},$$
$$\sum_{k=1}^{\infty}\frac{\binom{2k}{k}}{k4^k}H_k^{(2)}=\frac{3}{2}\zeta(3),\ \sum_{k=1}^{\infty}\frac{4^kH_{k-1}^{(2)}}{k^2\binom{2k}{k}}=\frac{\pi^4}{24}.$$

Part (ii) of the conjecture is related to the author's observation

$$\sum_{k=1}^{\infty}\frac{3H_k-1/k}{k^2\binom{2k}{k}}=\zeta(3)$$

(cf. Sun [93, Remark 3.1]) and his conjectural identity (cf. [93, (4.5)])

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_k^{(3)}+1/(5k^3)}{k^3\binom{2k}{k}}=\frac{2}{5}\zeta(3)^2.$$

In 2016, Mao and Sun [44] determined $\sum_{k=1}^{p-1}\binom{2k}{k}H_k/k$ and $\sum_{k=1}^{p-1}\binom{2k}{k}H_{2k}/k$ modulo any prime $p>3$.

Conjecture 54. (2010-03-02) *Let p be an odd prime.*

(i) *If $p\equiv 1 \pmod 4$ then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{(-8)^k}(H_{2k}-H_k)\equiv\frac{1}{2}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k}(H_{2k}-H_k)$$
$$\equiv\frac{1}{3}\left(\frac{2}{p}\right)\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{(-512)^k}(H_{2k}-H_k) \pmod{p^2};$$

when $p\equiv 3 \pmod 4$ we have

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{(-8)^k}(H_{2k}-H_k)\equiv-\frac{7}{2}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k}(H_{2k}-H_k) \pmod{p^2},$$
$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k}(H_{2k}-H_k)\equiv-\left(\frac{2}{p}\right)\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{(-512)^k}(H_{2k}-H_k) \pmod{p^2}.$$

(ii) *If $p\equiv 1 \pmod 3$ then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{16^k}(H_{2k}-H_k)\equiv\frac{1}{2}\left(\frac{-1}{p}\right)\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{256^k}(H_{2k}-H_k) \pmod{p^2}.$$

If $p\equiv 2 \pmod 3$ then

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{256^k}(H_{2k}-H_k)\equiv 0 \pmod{p^2}.$$

(iii) *If $p>3$ and $p\equiv 3,5,6 \pmod 7$, then*

$$\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^3(H_{2k}-H_k)\equiv 0 \pmod{p^2}.$$

Remark 54. In 2009, M. Jameson and K. Ono tried to prove the author's conjecture on $\sum_{k=0}^{p-1}\binom{2k}{k}^3$ modulo p^2 with p an odd prime. As a by-product, they realized that $\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^3(H_{2k}-H_k)\equiv 0 \pmod p$ for any prime $p>3$ but they did not have a proof of this observation. When $p>3$ is a prime with $p\equiv 3 \pmod 4$, the author [79] showed that

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k}H_{2k}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k}H_k\equiv 0 \pmod p.$$

Recall that the Apéry numbers are those integers

$$A_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2\ (n\in\mathbb{N})$$

which play a central role in Apéry's proof of the irrationality of $\zeta(3)=\sum_{n=1}^\infty 1/n^3$. Another kind of Apéry numbers are give by

$$\beta_n=\sum_{k=0}^n\binom nk^2\binom{n+k}k\ (n\in\mathbb N).$$

On August 14, 2013 the author conjectured that $\det[A_{i+j}]_{0\leqslant i,j\leqslant n}$ and $\det[\beta_{i+j}]_{0\leqslant i,j\leqslant n}$ are always positive, which remains open up to now.

We define the Apéry polynomials by

$$A_n(x)=\sum_{k=0}^n\binom nk^2\binom{n+k}k^2x^k\quad(n=0,1,2,\ldots).$$

Note that $A_n(1)=A_n$.

Conjecture 55. (Sun [78])

(i) *For any odd prime p, we have*

$$\sum_{k=0}^{p-1}A_k\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if } p=x^2+2y^2\ (x,y\in\mathbb Z),\\0\pmod{p^2}&\text{if } p\equiv5,7\pmod 8.\end{cases}$$

(ii) *Let $p>3$ be a prime. If $p\equiv1,3\pmod 8$, then*

$$\sum_{k=0}^{p-1}A_k\equiv\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{256^k}\pmod{p^3}.$$

If $p\equiv1\pmod 3$, then

$$\sum_{k=0}^{p-1}(-1)^kA_k\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}k^3}{16^k}\pmod{p^3}.$$

Remark 55. Let p be an odd prime. The author [78] showed the congruence in part (i) modulo p. Sun [78] also proved that

$$\sum_{k=0}^{p-1}A_k(x)\equiv\left(\frac xp\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{(256x)^k}\pmod p$$

and

$$\sum_{k=0}^{p-1}(-1)^kA_k(x)\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}k^3}{16^k}x^k\pmod{p^2}$$

for any p-adic integer $x\not\equiv0\pmod p$.

Conjecture 56. *Let p be an odd prime and let $n\in\mathbb Z^+$. Then*

$$\frac1{n^3}\left(\sum_{k=0}^{pn-1}(2k+1)(-1)^kA_k-\left(\frac p3\right)p\sum_{r=0}^{n-1}(2r+1)(-1)^rA_r\right)\equiv0\pmod{p^3}.\qquad(4.1)$$

If $p>3$, then

$$\frac{1}{n^4}\bigg(\sum_{k=0}^{pn-1}(2k+1)A_k-p\sum_{r=0}^{n-1}(2r+1)A_r\bigg)\equiv 0 \pmod{p^4}, \tag{4.2}$$

$$\frac{1}{n^6}\bigg(\sum_{k=0}^{pn-1}(2k+1)^3A_k-p\sum_{r=0}^{n-1}(2r+1)^3A_r\bigg)\equiv 0 \pmod{p^6}. \tag{4.3}$$

Remark 56. The author [78] proved that $\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)A_k(x)\in\mathbb{Z}[x]$ for all $n\in\mathbb{Z}^+$ and that $\sum_{k=0}^{p-1}(2k+1)A_k\equiv p+\frac{7}{6}p^4B_{p-3}\pmod{p^5}$ for any prime $p>3$. Guo and Zeng [25] confirmed the author's conjecture that $\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)(-1)^kA_k(x)\in\mathbb{Z}[x]$ for all $n\in\mathbb{Z}^+$. Motivated by the author's work in [78], Guo and Zeng [26] proved that $n^3\mid\sum_{k=0}^{n-1}(2k+1)^3A_k$ for all $n\in\mathbb{Z}^+$ and $\sum_{k=0}^{p-1}(2k+1)^3A_k\equiv p^3\pmod{p^6}$ for any prime $p>3$. The author [94, (2.19)] proved that for any prime $p>3$ we have

$$\sum_{k=0}^{p-1}(2k+1)^3(-1)^kA_k\equiv-\frac{p}{3}\left(\frac{p}{3}\right)\pmod{p^3}.$$

The usual Franel numbers are given by

$$f_n:=\sum_{k=0}^{n}\binom{n}{k}^3\quad(n=0,1,2,\ldots).$$

There are some series for $1/\pi$ involving Franel numbers or Franel polynomials, see, e.g., [7] and [62]. B.-X. Zhu and Sun [112] proved that $6^{-n}\det[f_{i+j}]_{0\leqslant i,j\leqslant n}$ is a positive odd integer for every $n\in\mathbb{N}$, which was first conjectured by the author in 2013. For $r=4,5,\ldots$, the Franel numbers of order r are given by

$$f_n^{(r)}:=\sum_{k=0}^{n}\binom{n}{k}^r\quad(n=0,1,2,\ldots).$$

On August 14, 2013, the author conjectured that the Hankel-type determinant $\det[f_{i+j}^{(r)}]_{0\leqslant i,j\leqslant n}$ is positive for any integers $n\geqslant 0$ and $r\geqslant 4$.

Conjecture 57. (i) *Let p be a prime and let $n\in\mathbb{Z}^+$. Then*

$$\frac{1}{n^2}\bigg(\sum_{k=0}^{pn-1}(-1)^kf_k-\left(\frac{p}{3}\right)\sum_{r=0}^{n-1}(-1)^rf_r\bigg)\equiv 0 \pmod{p^2}, \tag{4.4}$$

$$\frac{1}{n^3}\bigg(\sum_{k=0}^{pn-1}(3k+2)(-1)^kf_k-p^2\sum_{r=0}^{n-1}(3r+2)(-1)^rf_r\bigg)\equiv 0 \pmod{p^3}. \tag{4.5}$$

If $p>2$ then

$$\frac{1}{n^2}\bigg(\sum_{k=0}^{pn-1}\frac{f_k}{8^k}-\left(\frac{p}{3}\right)\sum_{r=0}^{n-1}\frac{f_r}{8^r}\bigg)\equiv 0 \pmod{p^2}. \tag{4.6}$$

(ii) (2014-07-07) *For any prime $p>3$, we have*

$$\sum_{k=0}^{p-1}(-1)^k f_k\equiv\left(\frac p3\right)+\frac23p^2B_{p-2}\left(\frac13\right)\pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{f_k}{8^k}\equiv\left(\frac p3\right)-\frac{p^2}{12}B_{p-2}\left(\frac13\right)\pmod{p^3},$$

$$\sum_{k=0}^{p-1}(-1)^k f_kH_k^{(2)}\equiv-\sum_{k=0}^{p-1}\frac{f_k}{8^k}H_k^{(2)}\equiv\frac12B_{p-2}\left(\frac13\right)\pmod{p}.$$

(iii) (2019) *For each odd prime p, we have*

$$\sum_{k=1}^{p-1}\frac{f_{k-1}}{k8^{k-1}}\equiv-p^2B_{p-3}\pmod{p^3}\tag{4.7}$$

and

$$\sum_{k=1}^{p-1}\frac{f_k}{k8^k}\equiv3q_p(2)-\frac32p\,q_p(2)^2+p^2q_p(2)^3\pmod{p^3}.\tag{4.8}$$

Remark 57. In the case $n=1$, (4.4) was first established by the author [85, (1.5)]. Sun [85] also proved that

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k}f_k\equiv0\pmod{p^2}\quad\text{and}\quad\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}f_k\equiv0\pmod{p}$$

for each prime $p>3$. The author's conjecture (cf. [86, Conjecture 1.3]) that

$$\sum_{k=0}^{p-1}(3k+2)(-1)^kf_k\equiv2p^2(2^p-1)^2\pmod{p^5}$$

for any prime $p>3$, was confirmed by Guo [21]. (4.6) with $n=1$ was conjectured by the author in [85, Remark 1.1]. Both (4.7) and (4.8) hold modulo p by [85, Remark 1.1].

For $n\in\mathbb{N}$ we define

$$g_n:=\sum_{k=0}^n\binom nk^2\binom{2k}k\quad\text{and}\quad h_n:=\sum_{k=0}^n\binom nk^2C_k.\tag{4.9}$$

It is known that $g_n=\sum_{k=0}^n\binom nk f_k$ for all $n\in\mathbb{N}$ (cf. [2]).

Conjecture 58. *Let p be an odd prime p and let $n\in\mathbb{Z}^+$. If $\max\{p,n\}>3$, then*

$$\frac1{(pn)^2}\sum_{k=n}^{pn-1}g_k\equiv\frac58\left(\frac p3\right)B_{p-2}\left(\frac13\right)g_{n-1}\pmod{p}$$

and

$$\frac{1}{(pn)^2}\sum_{k=n}^{pn-1} h_k \equiv \frac{3}{4}\left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) g_{n-1} \pmod{p}.$$

When $p > 3$, we have

$$\sum_{k=1}^{p-1}\frac{g_{k-1}}{k} \equiv -\left(\frac{p}{3}\right) q_p(9) \pmod{p^2} \tag{4.10}$$

and

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{g_k}{9^k} - \left(\frac{p}{3}\right)\sum_{r=0}^{n-1}\frac{g_r}{9^r}\right) \equiv -\frac{5}{8}B_{p-2}\left(\frac{1}{3}\right)\frac{g_n}{9^n} \pmod{p}.$$

Remark 58. Mao and Sun [44, Theorem 1.2] proved the first congruence and the second one in this conjecture in the case $n = 1$, which extend the author's previous results (cf. [94, Theorem 1.1(i)] and [77, Corollary 1.5]). The congruence (4.10) and that the left-hand side of the last congruence with $n = 1$ is p-adic integral, were conjectured by the author [94, Remark 1.1].

Conjecture 59. (Sun [86, Conjecture 1.2]) *Let $p > 3$ be a prime. When $p \equiv 1 \pmod{3}$ and $p = x^2 + 3y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod{3}$, we have*

$$\sum_{k=0}^{p-1}\frac{g_k}{3^k} \equiv \sum_{k=0}^{p-1}\frac{g_k}{(-3)^k} \equiv 2x - \frac{p}{2x} \pmod{p^2}$$

and

$$x \equiv \sum_{k=0}^{p-1}(k+1)\frac{g_k}{3^k} \equiv \sum_{k=0}^{p-1}(k+1)\frac{g_k}{(-3)^k} \pmod{p^2}.$$

If $p \equiv 2 \pmod{3}$, then

$$2\sum_{k=0}^{p-1}\frac{g_k}{3^k} \equiv -\sum_{k=0}^{p-1}\frac{g_k}{(-3)^k} \equiv \frac{3p}{\binom{(p+1)/2}{(p+1)/6}} \pmod{p^2}.$$

Remark 59. The author [86] determined $\sum_{k=0}^{p-1} g_k/(\pm 3)^k$ modulo a prime $p > 3$.

The author [94] introduced the polynomials

$$g_n(x) := \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}x^k \quad (n = 0, 1, 2, \ldots),$$

and proved that

$$\sum_{k=1}^{p-1}\frac{g_k(-1)}{k} \equiv 0 \pmod{p^2} \quad\text{and}\quad \sum_{k=1}^{p-1}\frac{g_k(-1)}{k^2} \equiv 0 \pmod{p}$$

for any prime $p>5$. Guo, Mao and Pan [23] confirmed the author's conjecture that

$$\frac{1}{n}\sum_{k=0}^{n-1}(4k+3)g_k(x)\in\mathbb{Z}[x]\quad\text{for all } n\in\mathbb{Z}^+.$$

Conjecture 60. (i) *Let p be an odd prime and let $n\in\mathbb{Z}^+$. Then*

$$\frac{1}{n^2}\bigg(\sum_{k=0}^{pn-1}g_k(-1)-\left(\frac{-1}{p}\right)\sum_{r=0}^{n-1}g_r(-1)\bigg)\equiv 0 \pmod{p^2},$$

$$\frac{1}{n^2}\bigg(\sum_{k=0}^{pn-1}g_k(-3)-\left(\frac{p}{3}\right)\sum_{r=0}^{n-1}g_r(-3)\bigg)\equiv 0 \pmod{p^2}.$$

(ii) *For any $n\in\mathbb{Z}^+$, the number*

$$\frac{1}{n\binom{2n-1}{n-1}}\sum_{k=0}^{n-1}(16k+5)324^{n-1-k}\binom{2k}{k}g_k(-20)$$

is an odd integer.

Remark 60. The two congruences in part (i) with $n=1$ were proved in [94, Theorem 1.1(i)]. Part (ii) is related to the author's conjectural series (cf. [84, Conjecture 7.9])

$$\sum_{k=0}^{\infty}\frac{16k+5}{324^k}\binom{2k}{k}g_k(-20)=\frac{189}{25\pi}.$$

For $n\in\mathbb{N}$ we define

$$F(n):=\sum_{k=0}^{n}\binom{n}{k}^3(-8)^k\quad\text{and}\quad G(n):=\sum_{k=0}^{n}\binom{n}{k}^2(6k+1)C_k.$$

Conjecture 61. *Let p be an odd prime.*

(i) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{n^2}\bigg(\sum_{k=0}^{pn-1}(-1)^kF(k)-\left(\frac{p}{3}\right)\sum_{r=0}^{n-1}(-1)^rF(r)\bigg)\equiv 0 \pmod{p^2},\tag{4.11}$$

and

$$\frac{1}{n^3}\sum_{k=n}^{pn-1}G(k)\equiv 0 \pmod{p^3}\quad\textit{and}\quad \frac{1}{p^{3n}}\sum_{k=p^{n-1}}^{p^n-1}G(k)\equiv-\frac{4}{3}B_{p-3} \pmod{p}\tag{4.12}$$

provided $p>3$.

(ii) (2014-07-17) *We have*

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k}F(k)\equiv -6q_p(2) \pmod{p}.$$

When $p>3$, we also have

$$\sum_{k=0}^{p-1}(-1)^kF(k)\equiv\left(\frac{p}{3}\right)-\frac{p^2}{12}B_{p-2}\left(\frac{1}{3}\right)\pmod{p^3}$$

$$\sum_{k=0}^{p-1}(-1)^kF(k)H_k^{(2)}\equiv B_{p-2}\left(\frac{1}{3}\right)\pmod{p},$$

$$\sum_{k=0}^{p-1}G_kH_k^{(2)}\equiv\frac{5}{3}pB_{p-3}\pmod{p^2}.$$

Remark 61. (4.11) and (4.12) with $n=1$, were first stated in [85, Remark 1.1] and [94, Conjecture 4.3] respectively.

Define

$$\bar{P}_n:=\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}\quad(n=0,1,2,\ldots).$$

The author [91, Remark 4.3] observed that $2^n\bar{P}_n$ coincides with the Catalan-Larcombe-French number given by (2.32). See [91] for some congruences and series for $1/\pi$ related to $\bar{P}_n$.

Conjecture 62. *Let p be an odd prime and let $n\in\mathbb{Z}^+$. If $p>3$ or $3\nmid n$, then*

$$\frac{1}{(pn)^2}\sum_{k=n}^{pn-1}\frac{\bar{P}_k}{4^k}\equiv\left(\frac{-1}{p}\right)2E_{p-3}\frac{\bar{P}_{n-1}}{4^{n-1}}\pmod{p}$$

and

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{\bar{P}_k}{8^k}-\left(\frac{-1}{p}\right)\sum_{r=0}^{n-1}\frac{\bar{P}_r}{8^r}\right)\equiv-2E_{p-3}\frac{\bar{P}_n}{8^n}\pmod{p}.$$

Remark 62. This conjecture with $n=1$ was stated in [87, Remark 3.13] and proved by Mao [43].

Those integers

$$D_n=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\ (n\in\mathbb{N})$$

are called central Delannoy numbers; they arise naturally in many enumeration problems in combinatorics. For $n\in\mathbb{N}$ we define

$$D_n(x)=\sum_{k=0}^{n}\binom{n}{k}^2x^k(x+1)^{n-k}=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k.$$

Note that $D_n(1)$ is the central Delannoy number D_n. Actually $D_n((x-1)/2)$ coincides with the Legendre polynomial $P_n(x)$ of degree n.

Conjecture 63. *Let p be a prime, and let $n\in\mathbb{Z}^+$. Then*

$$\frac{1}{n^2}\bigg(\sum_{k=0}^{pn-1}(2k+1)D_k(x)-p\sum_{r=0}^{n-1}(2r+1)D_r(x)\bigg)\equiv 0 \pmod{p^2}. \tag{4.13}$$

for any p-adic integer $x\not\equiv 0 \pmod p$. For each p-adic integer x, we have

$$\frac{1}{n^3}\bigg(\sum_{k=0}^{pn-1}(2k+1)D_k(x)^2-p^2\sum_{r=0}^{n-1}(2r+1)D_r(x)^2\bigg)\equiv 0 \pmod{p^3}. \tag{4.14}$$

Also, for any p-adic integer $x\not\equiv 0,-1 \pmod p$, we have

$$\frac{1}{n^2}\bigg(\sum_{k=0}^{pn-1}(2k+1)D_k(x)^3-p\left(\frac{-4x-3}{p}\right)\sum_{r=0}^{n-1}(2r+1)D_r(x)^3\bigg)\equiv 0 \pmod{p^2} \tag{4.15}$$

and

$$\frac{1}{n^2}\bigg(\sum_{k=0}^{pn-1}(2k+1)D_k(x)^4-p\sum_{r=0}^{n-1}(2r+1)D_r(x)^4\bigg)\equiv 0 \pmod{p^2}. \tag{4.16}$$

Remark 63. The congruences (4.13) and (4.14) with $n=1$ were obtained by the author [89, Theorem 1.5(ii) and Theorem 1.8(ii)]. The congruences (4.15) and (4.16) with $n=1$ were first conjectured by the author [89] and later confirmed by Guo [22].

Conjecture 64. (i) (Sun [78]) *For any $n\in\mathbb{Z}^+$, the numbers*

$$s(n)=\frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)(-1)^kA_k\left(\frac14\right)$$

and

$$t(n)=\frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)(-1)^kD_k\left(-\frac14\right)^3$$

are rational numbers with denominators $2^{2\nu_2(n!)}$ and $2^{3(n-1+\nu_2(n!))-\nu_2(n)}$ respectively.

(ii) (Sun [78]) *Let p be a prime. For any $n\in\mathbb{Z}^+$ and p-adic integer x, we have*

$$\nu_p\bigg(\frac1n\sum_{k=0}^{n-1}(2k+1)(-1)^kA_k(x)\bigg)\geqslant\min\{\nu_p(n),\nu_p(4x-1)\}$$

and

$$\nu_p\bigg(\frac1n\sum_{k=0}^{n-1}(2k+1)(-1)^kD_k(x)^3\bigg)\geqslant\min\{\nu_p(n),\nu_p(4x+1)\}.$$

(iii) (Sun [94]) *Let n be any positive integer. Then*

$$\nu_3\bigg(\sum_{k=0}^{n-1}(2k+1)(-1)^kA_k\bigg)=3\nu_3(n)\leqslant\nu_3\bigg(\sum_{k=0}^{n-1}(2k+1)^3(-1)^kA_k\bigg).$$

If n is a positive multiple of 3, then

$$\nu_3\bigg(\sum_{k=0}^{n-1}(2k+1)^3(-1)^kA_k\bigg)=3\nu_3(n)+2.$$

Remark 64. Though this conjecture appeared in the author's papers for several years, it seems that nobody has studied it seriously.

Conjecture 65. (Sun [74]) *Let $p>3$ be a prime. Then*

$$\begin{aligned}&\sum_{k=0}^{p-1}D_k(-3)^3=\sum_{k=0}^{p-1}(-1)^kD_k(2)^3\\&\equiv\sum_{k=0}^{p-1}(-1)^kD_k\left(-\frac{1}{4}\right)^3\equiv\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}(-1)^kD_k\left(\frac{1}{8}\right)^3\\&\equiv\begin{cases}(\frac{-1}{p})(4x^2-2p)\pmod{p^2} & \text{if } p\equiv 1\pmod 3\ \&\ p=x^2+3y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } p\equiv 2\pmod 3.\end{cases}\end{aligned}$$

Also,

$$\begin{aligned}&\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}(-1)^kD_k\left(\frac{1}{2}\right)^3\\&\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,7\pmod{24}\text{ and } p=x^2+6y^2\ (x,y\in\mathbb{Z}),\\ 8x^2-2p\pmod{p^2} & \text{if } p\equiv 5,11\pmod{24}\text{ and } p=2x^2+3y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{-6}{p})=-1.\end{cases}\end{aligned}$$

And

$$\begin{aligned}&\sum_{k=0}^{p-1}D_k(3)^3=\sum_{k=0}^{p-1}(-1)^kD_k(-4)^3\equiv\left(\frac{-5}{p}\right)\sum_{k=0}^{p-1}(-1)^kD_k\left(-\frac{1}{16}\right)^3\\&\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,4\pmod{15}\text{ and } p=x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 12x^2-2p\pmod{p^2} & \text{if } p\equiv 2,8\pmod{15}\text{ and } p=3x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{p}{15})=-1.\end{cases}\end{aligned}$$

Remark 65. It is known that $(-1)^nD_n(x)=D_n(-x-1)$ (cf. [74, Remark 1.2]).

The central trinomial coefficients are given by

$$T_n:=[x^n](1+x+x^2)^n=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}\quad(n\in\mathbb{N}).$$

Conjecture 66. *For any prime $p>3$ and $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{pn\binom{2n}{n}}\left(\sum_{k=0}^{pn-1}\frac{\binom{2k}{k}}{12^k}T_k-\left(\frac{p}{3}\right)\sum_{r=0}^{n-1}\frac{\binom{2r}{r}}{12^r}T_r\right)\equiv\left(\frac{p}{3}\right)\frac{q_p(3)}{8}\cdot\frac{T_{n-1}}{12^{n-1}} \pmod{p}. \tag{4.17}$$

Remark 66. The conjecture with $n=1$ was first stated by the author in [90, Conjecture 2.1] and recently confirmed by C. Wang and the author.

Conjecture 67. *Let $n\in\mathbb{Z}^+$. Then*

$$\frac{1}{2n\binom{2n}{n}}\sum_{k=0}^{n-1}(-1)^{n-1-k}(105k+44)\binom{2k}{k}^2T_k\in\mathbb{Z}^+.$$

Also, for any prime $p\equiv 1 \pmod 3$, we have

$$\begin{aligned}&\frac{\sum_{k=0}^{pn-1}(105k+44)(-1)^k\binom{2k}{k}^2T_k-p\sum_{r=0}^{n-1}(105r+44)(-1)^r\binom{2r}{r}^2T_r}{(pn)^2\binom{2n}{n}^2}\\&\quad\equiv(-1)^n6q_p(3)T_{n-1} \pmod{p}.\end{aligned} \tag{4.18}$$

Remark 67. Let $p>3$ be a prime. The author [84, Conjecture 1.3] conjectured that

$$\sum_{k=0}^{p-1}(105k+44)(-1)^k\binom{2k}{k}^2T_k\equiv p\left(20+24\left(\frac{p}{3}\right)(2-3^{p-1})\right) \pmod{p^3}.$$

[84, Conjecture 1.3] also contains the determination of $\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}^2T_k \bmod p^2$ via binary quadratic forms.

Those numbers

$$M_n=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}C_k\quad (n=0,1,2,\ldots)$$

are called Motzkin numbers. They play important roles in enumerative combinatorics.

Conjecture 68. (i) (Sun [89]) *For any prime $p>3$, we have*

$$\begin{aligned}\sum_{k=0}^{p-1}M_k^2&\equiv(2-6p)\left(\frac{p}{3}\right) \pmod{p^2},\\ \sum_{k=0}^{p-1}kM_k^2&\equiv(9p-1)\left(\frac{p}{3}\right) \pmod{p^2},\\ \sum_{k=0}^{p-1}T_kM_k&\equiv\frac{4}{3}\left(\frac{p}{3}\right)+\frac{p}{6}\left(1-9\left(\frac{p}{3}\right)\right) \pmod{p^2}.\end{aligned}$$

(ii) (2017-11-14) *For any* $n\in\mathbb{Z}^+$ *we have*

$$\sum_{k=0}^{n-1}(8k+9)W_k^2\equiv n \pmod{2n},\ \ where\ W_k:=\sum_{j=0}^{\lfloor k/2\rfloor}\binom{k}{2j}\frac{\binom{2j}{j}}{2j-1}.$$

Also, for any odd prime p *we have*

$$\frac{1}{p}\sum_{k=0}^{p-1}(8k+9)W_k^2\equiv 24+10\left(\frac{-1}{p}\right)-9\left(\frac{p}{3}\right)-18\left(\frac{3}{p}\right) \pmod{p^2}.$$

Remark 68. The author [89] proved that $\sum_{k=0}^{p-1}T_k^2\equiv(\frac{-1}{p}) \pmod{p}$ for any odd prime p. For any prime $p>3$ the author [97] showed that

$$\sum_{k=0}^{p-1}(2k+1)M_k^2\equiv 12p\left(\frac{p}{3}\right) \pmod{p^2}$$

and hence the first and the second congruences in Conjecture 68 are equivalent.

For $b,c\in\mathbb{Z}$ and $n\in\mathbb{N}$, as in [89] we define the generalized central trinomial coefficient

$$T_n(b,c):=[x^n](x^2+bx+c)^n=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}b^{n-2k}c^k.$$

Conjecture 69. (i) *Let* p *be an odd prime and let* $n\in\mathbb{Z}^+$ *be odd. If* $p>3$, *then*

$$\frac{1}{n^2\binom{n-1}{(n-1)/2}}\left(\sum_{k=0}^{(pn-1)/2}\frac{\binom{2k}{k}}{16^k}T_{2k}(4,1)-\sum_{r=0}^{(n-1)/2}\frac{\binom{2r}{r}}{16^r}T_{2r}(4,1)\right)\equiv 0 \pmod{p^2}.$$

We also have

$$\frac{1}{n^2\binom{n-1}{(n-1)/2}}\left(\sum_{k=0}^{(pn-1)/2}\frac{\binom{2k}{k}}{16^k}T_{2k}(8,9)-\left(\frac{3}{p}\right)\sum_{r=0}^{(n-1)/2}\frac{\binom{2r}{r}}{16^r}T_{2r}(8,9)\right)\equiv 0 \pmod{p^2}.$$

(ii) (Sun [90]) *Let* $p>3$ *be a prime. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{C_k}{16^k}T_{2k}(4,1)\equiv\frac{4}{3}\left(\left(\frac{3}{p}\right)-p\left(\frac{-1}{p}\right)\right) \pmod{p^2},$$

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{4^k}T_{2k}(3,4)\equiv\left(\frac{-1}{p}\right)\frac{7-3^p}{4} \pmod{p^2}.$$

(iii) *Let* $p>3$ *be a prime. If* $p\equiv 1 \pmod{3}$ *and* $p=x^2+3y^2$ *with* $x,y\in\mathbb{Z}$ *and* $x\equiv 1 \pmod{3}$, *then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{16^k}T_{2k}(2,3)\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{16^k}T_{2k}(4,-3)\equiv\left(\frac{-1}{p}\right)\left(2x-\frac{p}{2x}\right) \pmod{p^2}.$$

When $p\equiv 2\pmod 3$, we have

$$-2\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{16^k}T_{2k}(2,3)\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{16^k}T_{2k}(4,-3)\equiv\left(\frac{-1}{p}\right)\frac{3p}{\binom{(p+1)/2}{(p+1)/6}}\pmod{p^2}.$$

Remark 69. Part (i) with $n=1$, and the first assertion in part (iii), also appeared in Sun [90].

Conjecture 70. (Sun [89]) *Let p be an odd prime. We have*

$$\sum_{k=0}^{p-1}\frac{T_k(2,2)^2}{4^k}-\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k}\equiv\begin{cases}0\pmod{p^3}&\text{if } p\equiv 1\pmod 4,\\0\pmod{p^2}&\text{if } p\equiv 3\pmod 4.\end{cases}$$

If $p>3$, then

$$\sum_{k=0}^{p-1}\frac{T_k(4,1)^2}{4^k}\equiv\sum_{k=0}^{p-1}\frac{T_k(4,1)^2}{36^k}\equiv\left(\frac{-1}{p}\right)\pmod{p^2}.$$

Remark 70. The author [89] proved that for any prime $p>3$ we have

$$\sum_{k=0}^{p-1}\frac{T_k(6,-3)^2}{48^k}\equiv\left(\frac{-1}{p}\right)+\frac{p^2}{3}E_{p-3}\pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{T_k(2,-1)^2}{8^k}\equiv\left(\frac{-2}{p}\right)\pmod{p^2},\ \sum_{k=0}^{p-1}\frac{T_k(2,-3)^2}{16^k}\equiv\left(\frac{p}{3}\right)\pmod{p^2}.$$

Conjecture 71. (Sun [89, Conjecture 5.7]) *Let $p>3$ be a prime. Then*

$$\begin{aligned}&\left(\frac{3}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(2,3)^3}{8^k}\equiv\sum_{k=0}^{p-1}\frac{T_k(2,3)^3}{(-64)^k}\\&\equiv\sum_{k=0}^{p-1}\frac{T_k(2,9)^3}{(-64)^k}\equiv\left(\frac{3}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(2,9)^3}{512^k}\\&\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if } p\equiv 1,7\pmod{24}\text{ and } p=x^2+6y^2,\\2p-8x^2\pmod{p^2}&\text{if } p\equiv 5,11\pmod{24}\text{ and } p=2x^2+3y^2,\\0\pmod{p^2}&\text{if } (\frac{-6}{p})=-1,\end{cases}\end{aligned}$$

where $x,y\in\mathbb{Z}$. And

$$\begin{aligned}\sum_{k=0}^{p-1}(3k+2)\frac{T_k(2,3)^3}{8^k}&\equiv p\left(3\left(\frac{3}{p}\right)-1\right)\pmod{p^2},\\\sum_{k=0}^{p-1}(3k+1)\frac{T_k(2,3)^3}{(-64)^k}&\equiv p\left(\frac{-2}{p}\right)\pmod{p^3}.\end{aligned}$$

Also,

$$\sum_{k=0}^{n-1}(3k+2)T_k(2,3)^3 8^{n-1-k} \equiv 0 \pmod{2n}$$

and

$$\sum_{k=0}^{n-1}(3k+1)T_k(2,3)^3(-64)^{n-1-k} \equiv 0 \pmod{n}$$

for all $n \in \mathbb{Z}^+$.

Remark 71. See Sun [90] for more such conjectures.

Conjecture 72. *Let* p *be an odd prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k(7,12)^2}{4^k} \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p\equiv 1 \pmod{12}\ \&\ p=x^2+9y^2, \\ 4xy \pmod{p^2} & \text{if } p\equiv 5 \pmod{12}\ \&\ p=x^2+y^2\ (3\mid x-y), \\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod{4}, \end{cases}$$

where $x,y\in\mathbb{Z}$. *If* $p\neq 3$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k(7,12)^2}{4^k} \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_{2k}(3,3)^2}{36^k} \quad \left(\mathrm{mod}\ p^{(5+(\frac{-1}{p}))/2}\right).$$

(ii) *We have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_{2k}(9,20)^2}{4^k} \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,9 \pmod{20}\ \&\ p=x^2+25y^2, \\ 4xy \pmod{p^2} & \text{if } p\equiv 13,17 \pmod{20}\ \&\ p=x^2+y^2\ (5\mid x-y), \\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod{4}, \end{cases}$$

where $x,y\in\mathbb{Z}$. *If* $p\neq 11$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_{2k}(9,20)^2}{4^k} \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k(19,-20)^2}{22^{2k}} \pmod{p^2}.$$

Remark 72. Note that $T_k(7,12)=D_k(3)$ and $T_k(9,20)=D_k(4)$ for all $k\in\mathbb{N}$. The conjecture essentially appeared as Conjectures 4.24 and 4.25 of Sun [90].

Conjecture 73. *Let $p>3$ be a prime.*

(i) (Sun [84, Conjecture 7.13]) *We have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k(3,-3)^2}{(-108)^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{7})=1\ \&\ p=x^2+21y^2,\\ 12x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{7})=-1,\ (\frac{p}{3})=1\ \&\ p=3x^2+7y^2,\\ 2x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=-1,\ (\frac{p}{7})=1\ \&\ 2p=x^2+21y^2,\\ 6x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=1,\ (\frac{p}{3})=(\frac{p}{7})=-1\ \&\ 2p=3x^2+7y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-21}{p})=-1,\end{cases}$$

where $x,y\in\mathbb{Z}$. Also,

$$\sum_{k=0}^{p-1}\frac{56k+19}{(-108)^k}\binom{2k}{k}T_k(3,-3)^2\equiv\frac{p}{2}\left(21\left(\frac{p}{7}\right)+17\right)\pmod{p^2}.$$

(ii) (2011-06-18) *We have*

$$\sum_{n=0}^{p-1}\binom{2n}{n}\sum_{k=0}^{n}\frac{\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}}{64^k}$$
$$\equiv\begin{cases}x^2-2p\pmod{p^2} & \text{if } (\frac{p}{3})=(\frac{p}{17})=1\ \&\ 4p=x^2+51y^2,\\ 3x^2-2p\pmod{p^2} & \text{if } (\frac{p}{3})=(\frac{p}{17})=-1\ \&\ 4p=3x^2+17y^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{51})=-1,\end{cases}$$

where $x,y\in\mathbb{Z}$. Also,

$$\sum_{n=0}^{p-1}(17n+9)\binom{2n}{n}\sum_{k=0}^{n}\frac{\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}}{64^k}\equiv\frac{p}{3}\left(34\left(\frac{p}{17}\right)-7\right)\pmod{p^2}.$$

Remark 73. There are many similar congruences and related series for $1/\pi$ (cf. [84, 90]). The author's some conjectural series for $1/\pi$ involving central trinomial coefficients (cf. [76, 90]) were confirmed by Chan, Wan and Zudilin [8], Wan and Zudilin [108], and Zudilin [115]. Motivated by Sun [90, Conjecture 4.17], in 2011 the author believed that

$$c:=\pi\sum_{k=0}^{\infty}\frac{15k+2}{(-3456)^k}\binom{2k}{k}\binom{3k}{k}T_{3k}(2,-1)$$

is an algebraic number; on the author's request, Prof. H. H. Chan got in 2015 that

$$c=\frac{1}{2}\sqrt{72+54\sqrt[3]{4}+12\sqrt[3]{2}}.$$

Conjecture 74. *Let p be an odd prime*

(i) (Sun [90, Conjecture 4.20]) *If $p\neq 3,7,11,17,31$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k(73,576)^2}{434^{2k}}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{17})=1,\ p=x^2+102y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{p}{17})=1,(\frac{2}{p})=(\frac{p}{3})=-1,\ p=2x^2+51y^2,\\ 12x^2-2p\pmod{p^2} & \text{if } (\frac{p}{3})=1,(\frac{2}{p})=(\frac{p}{17})=-1,\ p=3x^2+34y^2,\\ 24x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=1,(\frac{p}{3})=(\frac{p}{17})=-1,\ p=6x^2+17y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-102}{p})=-1,\end{cases}$$

where x and y are integers.

(ii) *When $p\neq 7,31$, we have*

$$\sum_{k=0}^{p-1}\frac{2800512k+435257}{434^{2k}}\binom{2k}{k}T_k(73,576)^2$$
$$\equiv p\left(466752\left(\frac{-6}{p}\right)-31495\right)\pmod{p^2}.$$

For any $n\in\mathbb{Z}^+$, the number

$$\frac{1}{n\binom{2n-1}{n-1}}\sum_{k=0}^{n-1}(2800512k+435257)434^{2(n-1-k)}\binom{2k}{k}T_k(73,576)^2$$

is an odd integer.

Remark 74. This corresponds to the author's conjectural series (cf. [90, VII7])

$$\sum_{k=0}^{p-1}\frac{2800512k+435257}{434^{2k}}\binom{2k}{k}T_k(73,576)^2=\frac{10406669}{2\sqrt{6}\,\pi}.$$

Sun's another similar conjectural identity (cf. [90, VII2])

$$\sum_{k=0}^{\infty}\frac{24k+5}{28^{2k}}\binom{2k}{k}T_k(4,9)^2=\frac{49}{9\pi}(\sqrt{3}+\sqrt{6})$$

was motivated by [90, Conjecture 4.18], for this one we also conjecture that

$$n\binom{2n-1}{n-1}\,\Big|\,\sum_{k=0}^{n-1}(24k+5)28^{2(n-1-k)}\binom{2k}{k}T_k(4,9)^2.$$

Conjecture 75. (2011-10-01) (i) *Let $p>5$ be a prime. Then*

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(22,21^2)^3}{(-80)^{3k}}\equiv\left(\frac{5}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(22,21^2)^3}{16^{3k}}$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,4 \pmod{15}\ \&\ p=x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 2p-12x^2 \pmod{p^2} & \text{if } p\equiv 2,8 \pmod{15}\ \&\ p=3x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{p}{15})=-1,\ i.e.,\ p\equiv 7,11,13,14 \pmod{15}.\end{cases} \tag{4.19}$$

Also,

$$\sum_{k=0}^{p-1}\frac{126k+31}{(-80)^{3k}}T_k(22,21^2)^3\equiv 31p\left(\frac{-5}{p}\right) \pmod{p^2}.$$

(ii) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{(-1)^{n-1}}{n}\sum_{k=0}^{n-1}(126k+31)(-80)^{3(n-1-k)}T_k(22,21^2)^3\in\mathbb{Z}^+,$$
$$\frac{1}{n}\sum_{k=0}^{n-1}(66k+17)(2^{11}3^3)^{n-1-k}T_k(10,11^2)^3\in\mathbb{Z}^+,$$
$$\frac{(-1)^{n-1}}{n}\sum_{k=0}^{n-1}(3990k+1147)(-288)^{3(n-1-k)}T_k(62,95^2)^3\in\mathbb{Z}^+.$$

Remark 75. This conjecture is related to the author's conjectural formula

$$\sum_{k=0}^{\infty}\frac{126k+31}{(-80)^{3k}}T_k(22,21^2)^3=\frac{880\sqrt{5}}{21\pi}$$

(cf. [90, (VI2)]). The author [90] promised to offer 300 US dollars as the prize for the person (not joint authors) who can provide first rigorous proofs of this formula and the two other identities (cf. [90, (VI1) and (VI3)])

$$\sum_{k=0}^{\infty}\frac{66k+17}{(2^{11}3^3)^k}T_k(10,11^2)^3=\frac{540\sqrt{2}}{11\pi},$$
$$\sum_{k=0}^{\infty}\frac{3990k+1147}{(-288)^{3k}}T_k(62,95^2)^3=\frac{432}{95\pi}(195\sqrt{14}+94\sqrt{2}).$$

The Domb numbers in combinatorics are given by

$$\mathrm{Domb}(n):=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}\quad(n\in\mathbb{N}).$$

Conjecture 76. (Sun [84, Conjecture 5.1]) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\mathrm{Domb}(k)\equiv\sum_{k=0}^{p-1}\frac{\mathrm{Domb}(k)}{64^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,4\pmod{15}\ \&\ p=x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 2p-12x^2\pmod{p^2} & \text{if } p\equiv 2,8\pmod{15}\ \&\ p=3x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{p}{15})=-1,\ i.e.,\ p\equiv 7,11,13,14\pmod{15}.\end{cases}$$

Remark 76. Such conjectures can be easily checked via a computer.

Conjecture 77. (i) *For any $n\in\mathbb{Z}^+$, the numbers*

$$\frac{1}{4n}\sum_{k=0}^{n-1}(5k+4)\mathrm{Domb}(k),\ \frac{1}{2n}\sum_{k=0}^{n-1}(2k+1)\mathrm{Domb}(k)(-2)^{n-1-k},$$
$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)\mathrm{Domb}(k)8^{n-1-k},\ \frac{1}{n}\sum_{k=0}^{n-1}(2k+1)\mathrm{Domb}(k)(-8)^{n-1-k}$$
$$\frac{(-1)^{n-1}}{n}\sum_{k=0}^{n-1}(2k+1)\mathrm{Domb}(k)(-32)^{n-1-k},\ \frac{1}{n}\sum_{k=0}^{n-1}(5k+1)\mathrm{Domb}(k)64^{n-1-k}$$

are all positive integers.

(ii) *For any prime $p>3$, we have*

$$\sum_{k=0}^{p-1}(5k+4)\mathrm{Domb}(k)\equiv 4p\left(\frac{p}{3}\right)+\frac{14}{3}p^3B_{p-2}\left(\frac{1}{3}\right)\pmod{p^4},$$
$$\sum_{k=0}^{p-1}\frac{3k+2}{(-2)^k}\mathrm{Domb}(k)\equiv 2p\left(\frac{-1}{p}\right)+6p^3E_{p-3}\pmod{p^4},$$
$$\sum_{k=0}^{p-1}\frac{2k+1}{(-8)^k}\mathrm{Domb}(k)\equiv p\left(\frac{p}{3}\right)+\frac{5}{12}p^3B_{p-2}\left(\frac{1}{3}\right)\pmod{p^4},$$
$$\sum_{k=0}^{p-1}\frac{3k+1}{(-32)^k}\mathrm{Domb}(k)\equiv p\left(\frac{-1}{p}\right)+p^3E_{p-3}\pmod{p^4},$$
$$\sum_{k=0}^{p-1}\frac{5k+1}{64^k}\mathrm{Domb}(k)\equiv p\left(\frac{p}{3}\right)-\frac{p^3}{3}B_{p-2}\left(\frac{1}{3}\right)\pmod{p^4},$$
$$\sum_{k=0}^{p-1}\frac{2k+1}{8^k}\mathrm{Domb}(k)\equiv p+\frac{35}{24}p^4B_{p-3}\pmod{p^5}.$$

(iii) *Let p be a prime and let $n\in\mathbb{Z}^+$. Then*

$$\frac{1}{n^3}\bigg(\sum_{k=0}^{pn-1}(5k+4)\mathrm{Domb}(k)-\Big(\frac{p}{3}\Big)p\sum_{r=0}^{n-1}(5r+4)\mathrm{Domb}(r)\bigg)\equiv 0 \pmod{p^3}.$$

If $p>2$, then

$$\frac{1}{n^3}\bigg(\sum_{k=0}^{pn-1}(3k+2)\frac{\mathrm{Domb}(k)}{(-2)^k}-\Big(\frac{-1}{p}\Big)p\sum_{r=0}^{n-1}(3r+2)\frac{\mathrm{Domb}(r)}{(-2)^r}\bigg)\equiv 0 \pmod{p^3},$$

$$\frac{1}{n^3}\bigg(\sum_{k=0}^{pn-1}(2k+1)\frac{\mathrm{Domb}(k)}{(-8)^k}-\Big(\frac{p}{3}\Big)p\sum_{r=0}^{n-1}(2r+1)\frac{\mathrm{Domb}(r)}{(-8)^r}\bigg)\equiv 0 \pmod{p^3},$$

$$\frac{1}{n^3}\bigg(\sum_{k=0}^{pn-1}(3k+1)\frac{\mathrm{Domb}(k)}{(-32)^k}-\Big(\frac{-1}{p}\Big)p\sum_{r=0}^{n-1}(3r+1)\frac{\mathrm{Domb}(r)}{(-32)^r}\bigg)\equiv 0 \pmod{p^3},$$

$$\frac{1}{n^3}\bigg(\sum_{k=0}^{pn-1}(5k+1)\frac{\mathrm{Domb}(k)}{64^k}-\Big(\frac{p}{3}\Big)p\sum_{r=0}^{n-1}(5r+1)\frac{\mathrm{Domb}(r)}{64^r}\bigg)\equiv 0 \pmod{p^3}.$$

When $p>3$, we have

$$\frac{1}{n^4}\bigg(\sum_{k=0}^{pn-1}(2k+1)\frac{\mathrm{Domb}(k)}{8^k}-p\sum_{r=0}^{n-1}(2r+1)\frac{\mathrm{Domb}(r)}{8^r}\bigg)\equiv 0 \pmod{p^4}.$$

Remark 77. Note that

$$\sum_{k=0}^{\infty}\frac{5k+1}{64^k}\mathrm{Domb}(k)=\frac{8}{\sqrt{3}\,\pi}\quad\text{and}\quad\sum_{k=0}^{\infty}\frac{3k+1}{(-32)^k}\mathrm{Domb}(k)=\frac{2}{\pi}$$

by H.H. Chan, S.H. Chan and Z.-Liu [6], and M.D. Rogers [61]. The supercongruence $\sum_{k=0}^{p-1}(5k+1)\mathrm{Domb}(k)/64^k\equiv(\frac{p}{3})p \pmod{p^3}$ for primes $p>3$ was first pointed out by Zudilin [114, (34)]. Other congruences in part (ii) modulo p^3, as well as $4n\mid\sum_{k=0}^{n-1}(5k+4)\mathrm{Domb}(k)$ for all $n\in\mathbb{Z}^+$, were posed by the author [84, Conjectures 5.1-5.3].

Conjecture 78. *Let p be an odd prime. For any $n\in\mathbb{Z}^+$, we have*

$$\frac{(-256)^{n-1}}{(pn)^3\binom{2n}{n}^2}D_{p,n}\equiv\frac{q_p(2)}{2}\mathrm{Domb}(n-1) \pmod{p},$$

where

$$D_{p,n}:=\sum_{k=0}^{pn-1}\frac{40k^2+26k+5}{(-256)^k}\binom{2k}{k}^2\mathrm{Domb}(k)-p^2\sum_{r=0}^{n-1}\frac{40r^2+26r+5}{(-256)^r}\binom{2r}{r}^2\mathrm{Domb}(r).$$

Moreover,

$$\sum_{k=0}^{p-1}\frac{40k^2+26k+5}{(-256)^k}\binom{2k}{k}^2\mathrm{Domb}(k)\equiv 5p^2+2p^3q_p(2)-3p^4q_p(2)^2 \pmod{p^5},\qquad(4.20)$$

and

$$\frac{(-1)^{n-1}}{n^2\binom{2n}{n}^2}\sum_{k=0}^{n-1}(40k^2+26k+5)\binom{2k}{k}^2\mathrm{Domb}(k)(-256)^{n-1-k}\in\mathbb{Z}^+$$

for every $n=2,3,\ldots$.

Remark 78. This corresponds to the author's conjectural series

$$\sum_{k=0}^{\infty}\frac{40k^2+26k+5}{(-256)^k}\binom{2k}{k}^2\mathrm{Domb}(k)=\frac{24}{\pi^2}$$

stated in [75, Conjecture 1.4]. The congruence (4.20) modulo p^3 was conjectured by Sun [75, Conjecture 5.15].

Conjecture 79. (i) *For any odd prime* p, *we have*

$$\sum_{k=1}^{p-1}\frac{\mathrm{Domb}(k)}{k}\equiv\left(\frac{p}{3}\right)\frac{2}{5}p\,B_{p-2}\left(\frac{1}{3}\right)\pmod{p^2}.$$

(ii) (Sun [75]) *For any prime* $p>3$, *we have*

$$\sum_{n=0}^{p-1}\frac{3n^2+n}{16^n}\mathrm{Domb}(n)\equiv-4p^4q_p(2)+6p^5q_p(2)^2\pmod{p^6}.$$

(iii) (2013-08-20) *For any prime* p, *we have*

$$\det[\mathrm{Domb}(i+j)]_{0\leqslant i,j\leqslant p-1}\equiv\begin{cases}(\frac{-1}{p})(4x^2-2p)\pmod{p^2} & \text{if } p=x^2+3y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } p\equiv 2\pmod 3.\end{cases}\tag{4.21}$$

Remark 79. Part (i) was found by the author in 2019. The congruence in part (ii) modulo p^5 was proved by Y.-P. Mu and the author [53]. For the Catalan-Larcombe-French numbers given by (2.32), the author has proved that

$$\det[P_{i+j}]_{0\leqslant i,j\leqslant p-1}\equiv\left(\frac{-1}{p}\right)\pmod{p^2}$$

for any odd prime p.

Conjecture 80. (i) (2013-08-22) *For* $n=0,1,2,\ldots$ *let*

$$H(n)=\det\left[h_{i+j}\right]_{0\leqslant i,j\leqslant n}.$$

Then $H(n)$ *is always positive and odd, and not congruent to* 7 *modulo* 8. *For any prime* $p\equiv 1\pmod 3$ *with* $p=x^2+3y^2$ ($x,y\in\mathbb{Z}$ *and* $x\equiv 1\pmod 3$), *we have*

$$H(p-1)\equiv\left(\frac{-1}{p}\right)\left(2x-\frac{p}{2x}\right)\pmod{p^2}.$$

For any odd prime $p\equiv 2\pmod{3}$, *we have*

$$H(p-1)\equiv-\left(\frac{-1}{p}\right)\frac{3p}{\binom{(p+1)/2}{(p+1)/6}}\pmod{p^2}.$$

(ii) (2013-08-24) *For* $n\in\mathbb{N}$ *let* $a_n=\sum_{k=0}^n\binom{n}{k}^2\binom{2k}{k+1}$. *Then*

$$\det[a_{i+j}]_{0\leqslant i,j\leqslant p-1}\equiv 0\pmod{p^2}\quad\textit{for any prime } p>3.$$

Remark 80. Recall that $h_n=\sum_{k=0}^n\binom{n}{k}^2C_k$ for all $n\in\mathbb{N}$. On August 17, 2013, the author also conjectured that for any $m,n\in\mathbb{Z}^+$ we have

$$(-1)^n\det[H_{i+j}^{(m)}]_{0\leqslant i,j\leqslant n}>0\ \text{ and }\ \det[B_{i+j}^2]_{0\leqslant i,j\leqslant n}<0<\det[E_{i+j}^2]_{0\leqslant i,j\leqslant n}.$$

Conjecture 81. (2013-08-23) *For* $n=0,1,2,\ldots$ *let*

$$w_n:=\sum_{k=0}^{\lfloor n/3\rfloor}(-1)^k3^{n-3k}\binom{n}{3k}\binom{2k}{k}\binom{3k}{k}\ \textit{and}\ W(n)=\det\left[w_{i+j}\right]_{0\leqslant i,j\leqslant n}.$$

(i) *When* $n\equiv 0,2\pmod{3}$, *the number* $(-1)^{\lfloor (n+1)/3\rfloor}W(n)/6^n$ *is always a positive odd integer.*

(ii) *For any prime* $p\equiv 1\pmod{3}$, *if we write* $4p=x^2+27y^2$ *with* $x,y\in\mathbb{Z}$ *and* $x\equiv 1\pmod{3}$, *then*

$$W(p-1)\equiv\left(\frac{-1}{p}\right)\left(\frac{p}{x}-x\right)\pmod{p^2}.$$

Remark 81. The sequence $(w_n)_{n\geqslant 0}$ was first introduced by D. Zagier [109]. On August 23, 2013, the author observed that $W(3n+1)=0$ for all $n=0,1,2,\ldots$, which was later confirmed by C. Krattenthaler in a private message.

Conjecture 82. (2016-11-13). *For* $n=0,1,2,\ldots$ *define*

$$a_n:=\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{2k}\binom{n-k}{k},$$
$$b_n:=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}^2\binom{n-k}{k},$$
$$c_n:=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}^2\binom{n-k}{k}.$$

Let n *be any positive integer. Then*

$$\frac{a_{pn}-a_n}{(pn)^2}\in\mathbb{Z}_p\quad\textit{for each prime } p>3.$$

Also, for any prime $p > 5$ we have

$$\frac{b_{pn} - b_n}{(pn)^3} \in \mathbb{Z}_p \quad \text{and} \quad \frac{c_{pn} - c_n}{(pn)^3} \in \mathbb{Z}_p.$$

Remark 82. One may consult Osburn, B. Sahu and A. Straub [55] for some known supercongruences of similar types. For any prime $p > 5$ and $n \in \mathbb{Z}^+$, we are able to show that

$$\frac{a_{pn} - a_n}{p^2 n} \in \mathbb{Z}_p, \quad \frac{b_{pn} - b_n}{p^2 n} \in \mathbb{Z}_p \quad \text{and} \quad \frac{c_{pn} - c_n}{p^2 n} \in \mathbb{Z}_p.$$

For $n = 0, 1, 2, \ldots$, Sun [95] defined

$$s_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k}\binom{x+k}{k} = \sum_{k=0}^{n} \binom{n}{k}(-1)^k \binom{x}{k}\binom{-1-x}{k}.$$

Note that $s_n(-1/2)$ coincides with

$$\tilde{J}_2(n) := \sum_{k=0}^{n} \binom{n}{k}(-1)^k \binom{-1/2}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}(-1)^k \frac{\binom{2k}{k}^2}{16^k}$$

defined by K. Kimoto and M. Wakayama [37, (3.4)]. Long, Osburn and Swisher [42] proved that

$$\sum_{k=0}^{p-1} s_k\left(-\frac{1}{2}\right)^2 \equiv \left(\frac{-1}{p}\right) \pmod{p^3}$$

for any odd prime p, which was conjectured by Kimoto and Wakayama [37]. Sun [95] conjectured further that

$$\sum_{k=0}^{p-1} s_k\left(-\frac{1}{2}\right)^2 \equiv \left(\frac{-1}{p}\right)(1 - 7p^3 B_{p-3}) \pmod{p^4}$$

for any odd prime p, which was later confirmed by J.-C. Liu [40].

Conjecture 83. (i) (Sun [95, Conjecture 6.10]) *For any prime $p > 3$ and p-adic integer $x \neq -1/2$, we have the congruence*

$$\sum_{k=0}^{p-1} s_k(x)^2 \equiv (-1)^{\langle x\rangle_p} \frac{p + 2(x - \langle x\rangle_p)}{2x+1} \pmod{p^3}.$$

(ii) (Sun [95, Conjecture 6.11]) *For any prime $p>3$, we have*

$$\sum_{k=0}^{p-1}s_k\left(-\frac{1}{3}\right)^2\equiv p-\frac{14}{3}\left(\frac{p}{3}\right)p^3B_{p-2}\left(\frac{1}{3}\right)\pmod{p^4},$$
$$\sum_{k=0}^{p-1}s_k\left(-\frac{1}{4}\right)^2\equiv\left(\frac{2}{p}\right)p-26\left(\frac{-2}{p}\right)p^3E_{p-3}\pmod{p^4}$$
$$\sum_{k=0}^{p-1}s_k\left(-\frac{1}{6}\right)^2\equiv\left(\frac{3}{p}\right)p-\frac{155}{12}\left(\frac{-1}{p}\right)p^3B_{p-2}\left(\frac{1}{3}\right)\pmod{p^4}.$$

(iii) *Let p be an odd prime and let $n\in\mathbb{Z}^+$. Then*

$$\frac{1}{n^3}\left(\sum_{k=0}^{pn-1}s_k\left(-\frac{1}{2}\right)^2-\left(\frac{-1}{p}\right)\sum_{r=0}^{n-1}s_r\left(-\frac{1}{2}\right)^2\right)\equiv 0\pmod{p^3}$$

and

$$\frac{1}{n^3}\left(\sum_{k=0}^{pn-1}s_k\left(-\frac{1}{4}\right)^2-\left(\frac{2}{p}\right)p\sum_{r=0}^{n-1}s_r\left(-\frac{1}{4}\right)^2\right)\equiv 0\pmod{p^3}.$$

If $p>3$, then

$$\frac{1}{n^3}\left(\sum_{k=0}^{pn-1}s_k\left(-\frac{1}{3}\right)^2-p\sum_{r=0}^{n-1}s_r\left(-\frac{1}{3}\right)^2\right)\equiv 0\pmod{p^3}$$

and

$$\frac{1}{n^3}\left(\sum_{k=0}^{pn-1}s_k\left(-\frac{1}{6}\right)^2-\left(\frac{3}{p}\right)p\sum_{r=0}^{n-1}s_r\left(-\frac{1}{6}\right)^2\right)\equiv 0\pmod{p^3}.$$

Remark 83. Sun [95] proved the congruence in part (i) modulo p^2.

Sun [95] introduced two new kinds of polynomials

$$d_n(x):=\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}2^k\ \text{and}\ t_n(x):=\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}\binom{x+k}{k}2^k\quad(n\in\mathbb{N}).$$

Conjecture 84. (i) *Let p be an odd prime and let $n\in\mathbb{Z}^+$. Then*

$$\frac{1}{n^2}\bigg(\sum_{k=0}^{pn-1}d_k\left(-\frac12\right)^2-\left(\frac{-1}{p}\right)\sum_{r=0}^{n-1}(-1)^r d_r\left(-\frac12\right)^2\bigg)\equiv 0 \pmod{p^2},$$

$$\frac{1}{n^2}\bigg(\sum_{k=0}^{pn-1}d_k\left(-\frac12\right)^2-\left(\frac{-1}{p}\right)\sum_{r=0}^{n-1}d_r\left(-\frac12\right)^2\bigg)\equiv 0 \pmod{p^2},$$

$$\frac{1}{n^2}\bigg(\sum_{k=0}^{pn-1}t_k\left(-\frac12\right)^2-\left(\frac{-1}{p}\right)\sum_{r=0}^{n-1}t_r\left(-\frac12\right)^2\bigg)\equiv 0 \pmod{p^2},$$

$$\frac{1}{n^2}\bigg(\sum_{k=0}^{pn-1}(-1)^k d_k\left(-\frac14\right)^2-\left(\frac{-2}{p}\right)\sum_{r=0}^{n-1}(-1)^r d_r\left(-\frac14\right)^2\bigg)\equiv 0 \pmod{p^2},$$

$$\frac{1}{n^2}\bigg(\sum_{k=0}^{pn-1}d_k\left(-\frac14\right)^2-\left(\frac{2}{p}\right)p\sum_{r=0}^{n-1}d_r\left(-\frac14\right)^2\bigg)\equiv 0 \pmod{p^2},$$

$$\frac{1}{n^2}\bigg(\sum_{k=0}^{pn-1}t_k\left(-\frac14\right)^2-\left(\frac{2}{p}\right)p\sum_{r=0}^{n-1}t_r\left(-\frac14\right)^2\bigg)\equiv 0 \pmod{p^2}.$$

If $p>3$, then

$$\frac{1}{n^2}\bigg(\sum_{k=0}^{pn-1}(-1)^k d_k\left(-\frac13\right)^2-\left(\frac{p}{3}\right)\sum_{r=0}^{n-1}(-1)^r d_r\left(-\frac13\right)^2\bigg)\equiv 0 \pmod{p^2},$$

$$\frac{1}{n^2}\bigg(\sum_{k=0}^{pn-1}d_k\left(-\frac13\right)^2-p\sum_{r=0}^{n-1}d_r\left(-\frac13\right)^2\bigg)\equiv 0 \pmod{p^2},$$

$$\frac{1}{n^2}\bigg(\sum_{k=0}^{pn-1}t_k\left(-\frac13\right)^2-p\sum_{r=0}^{n-1}t_r\left(-\frac13\right)^2\bigg)\equiv 0 \pmod{p^2},$$

and

$$\frac{1}{n^2}\bigg(\sum_{k=0}^{pn-1}(-1)^k d_k\left(-\frac16\right)^2-\left(\frac{-1}{p}\right)\sum_{r=0}^{n-1}(-1)^r d_r\left(-\frac16\right)^2\bigg)\equiv 0 \pmod{p^2},$$

$$\frac{1}{n^2}\bigg(\sum_{k=0}^{pn-1}d_k\left(-\frac16\right)^2-\left(\frac{3}{p}\right)p\sum_{r=0}^{n-1}d_r\left(-\frac16\right)^2\bigg)\equiv 0 \pmod{p^2},$$

$$\frac{1}{n^2}\bigg(\sum_{k=0}^{pn-1}t_k\left(-\frac16\right)^2-\left(\frac{3}{p}\right)p\sum_{r=0}^{n-1}t_r\left(-\frac16\right)^2\bigg)\equiv 0 \pmod{p^2}.$$

(ii) (Sun [95]) *For any odd prime p and p-adic integer x, we have*

$$\sum_{k=0}^{p-1}t_k(x)^2\equiv\begin{cases}(\frac{-1}{p}) \pmod{p^2} & \text{if } 2x\equiv -1 \pmod{p},\\ (-1)^{\langle x\rangle_p}\frac{p+2x-2\langle x\rangle_p}{2x+1} \pmod{p^2} & \text{otherwise.}\end{cases}$$

Also, for any $n \in \mathbb{Z}^+$ *and* $x \in \mathbb{Z}$, *the number*

$$\frac{1}{n}\sum_{k=0}^{n-1}(8k+5)t_k(x)^2$$

is always an integer congruent to 1 *modulo* 4.

Remark 84. Sun [95] determined $\sum_{k=0}^{p-1}(\pm 1)^k d_k(x)^2$ and $\sum_{k=0}^{p-1}(2k+1)d_k(x)^2$ modulo p^2 for any odd prime p.

5 Congruences Involving Lucas Sequences

Recall that the Fibonacci numbers $F_0, F_1, F_2, \ldots$ and the Lucas numbers $L_0, L_1, L_2, \ldots$ are given by

$$F_0 = 0,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1}\ (n = 1, 2, 3, \ldots),$$

and

$$L_0 = 2,\ L_1 = 1,\ L_{n+1} = L_n + L_{n-1}\ (n = 1, 2, 3, \ldots)$$

respectively. Actually, $F_n = u_n(1,-1)$ and $L_n = v_n(1,-1)$ for all $n \in \mathbb{N}$.

Conjecture 85. (i) (Sun [90]) *Let* $p > 5$ *be a prime. If* $p \equiv 1, 4 \pmod{15}$ *and* $p = x^2 + 15y^2$ $(x, y \in \mathbb{Z})$ *with* $x \equiv 1 \pmod 3$, *then*

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{3k}{k}}{27^k}F_k \equiv& \frac{2}{15}\left(\frac{p}{x} - 2x\right) \pmod{p^2},\\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}L_k \equiv& 4x - \frac{p}{x} \pmod{p^2}\end{aligned}$$

and

$$\sum_{k=0}^{p-1}(3k+2)\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}L_k \equiv 4x \pmod{p^2}.$$

If $p \equiv 2, 8 \pmod{15}$ *and* $p = 3x^2 + 5y^2$ $(x, y \in \mathbb{Z})$ *with* $y \equiv 1 \pmod 3$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}F_k \equiv \frac{p}{5y} - 4y \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{3k}{k}}{27^k}F_k \equiv \sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{3k}{k}}{27^k}L_k \equiv \frac{4}{3}y \pmod{p^2}.$$

(ii) (2011-09-29) *Let $p>3$ be a prime. If $p\equiv 1,7\pmod{24}$ and $p=x^2+6y^2$ $(x,y\in\mathbb{Z})$ with $x\equiv 1\pmod{3}$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{108^k}ku_k(4,2)\equiv\frac{1}{6}\left(2x-\frac{p}{x}\right)\pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{108^k}v_k(4,2)\equiv 4x-\frac{p}{x}\pmod{p^2},$$
$$\sum_{k=0}^{p-1}(3k-1)\frac{\binom{2k}{k}\binom{3k}{k}}{108^k}v_k(4,2)\equiv -2x\pmod{p^2}.$$

If $p\equiv 5,11\pmod{24}$ and $p=2x^2+3y^2$ $(x,y\in\mathbb{Z})$ with $x\equiv 1\pmod{3}$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{108^k}u_k(4,2)\equiv 2x-\frac{p}{4x}\pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{108^k}ku_k(4,2)\equiv\frac{x}{3}\pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{108^k}kv_k(4,2)\equiv\frac{4}{3}x\pmod{p^2}.$$

Remark 85. By Sun [83, Theorem 1.6], we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}F_k\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{108^k}u_k(4,2)\equiv 0\pmod{p^2}$$

for any prime $p\equiv 1\pmod{3}$, and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}L_k\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{108^k}v_k(4,2)\equiv 0\pmod{p^2}$$

for each odd prime $p\equiv 2\pmod{3}$. For more such conjectures, one may consult Sun [80] and [90, Conjectures 4.1-4.2].

Conjecture 86. (2012-11-03) *Let $p\neq 2,5$ be a prime. If $(\frac{-1}{p})=(\frac{5}{p})=1$ (i.e., $p\equiv 1,9 \pmod{20}$) and $p=x^2+5y^2$ with $x,y\in\mathbb{Z}$, then*

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}F_{6k}\equiv&0 \pmod{p^3},\\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}L_{6k}\equiv&(-1)^y(8x^2-4p) \pmod{p^2},\\ \sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{64^k}F_{6k}\equiv&\frac{(-1)^y}{10}(3p-4x^2) \pmod{p^2}.\end{aligned}$$

If $(\frac{-5}{p})=-1$ (i.e., $p\equiv 11,13,17,19 \pmod{20}$), then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}F_{6k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}L_{6k}\equiv 0 \pmod{p^2},\ \text{and}\ \sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{64^k}F_{6k}\equiv 0 \pmod{p}.$$

Conjecture 87. (2012-11-03) *Let $p\neq 2,5$ be a prime. If $(\frac{-2}{p})=(\frac{5}{p})=1$ (i.e., $p\equiv 1,9,11,19 \pmod{40}$) and $p=x^2+10y^2$ with $x,y\in\mathbb{Z}$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}L_{12k}\equiv\left(\frac{-1}{p}\right)(8x^2-4p) \pmod{p^2};$$

if $p\equiv 1,9 \pmod{40}$ then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}F_{12k}\equiv 0 \pmod{p^3}.$$

If $(\frac{-2}{p})=(\frac{5}{p})=-1$ (i.e., $p\equiv 7,13,23,37 \pmod{40}$) and $p=2x^2+5y^2$ with $x,y\in\mathbb{Z}$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}F_{12k}\equiv 16\left(\frac{-1}{p}\right)(4x^2-p) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}L_{12k}\equiv 36\left(\frac{-1}{p}\right)(p-4x^2) \pmod{p^2}.$$

If $(\frac{-10}{p})=-1$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}F_{12k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}L_{12k}\equiv 0 \pmod{p^2}.$$

Conjecture 88. (2012-11-03) *Let $p \neq 2,5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}F_{24k} \equiv \begin{cases} 0 \pmod{p^3} & \text{if } p \equiv 1,9 \pmod{20}, \\ 0 \pmod{p^2} & \text{if } p \equiv 3,7,11,19 \pmod{20}, \\ 288(p-2x^2) \pmod{p^2} & \text{if } p = x^2+4y^2 \equiv 13,17 \pmod{20}, \end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{64^k}F_{24k} \equiv \begin{cases} (-1)^y(3p-4x^2)/6 \pmod{p^2} & \text{if } p = x^2+25y^2 \equiv 1,9 \pmod{20}, \\ 110x^2/3 \pmod{p} & \text{if } p = x^2+4y^2 \ \&\ (\frac{p}{5}) = -1, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

where x and y are integers. Also,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}L_{24k} \equiv \begin{cases} (81-80(\frac{p}{5}))(8x^2-4p) \pmod{p^2} & \text{if } p = x^2+4y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{64^k}L_{24k}$$
$$\equiv \begin{cases} (-1)^y(3p-4x^2)/2 \pmod{p^2} & \text{if } p = x^2+25y^2 \equiv 1,9 \pmod{20}, \\ -82x^2 \pmod{p} & \text{if } p = x^2+4y^2 \ \&\ (\frac{p}{5}) = -1, \\ 0 \pmod{p} & \text{if } p > 3 \ \&\ p \equiv 3 \pmod{4}, \end{cases}$$

where x and y are integers.

The Pell sequence $(P_n)_{n\geqslant 0}$ and its companion $(Q_n)_{n\geqslant 0}$ are given by

$$P_0 = 0,\ P_1 = 1,\ \text{and } P_{n+1} = 2P_n + P_{n-1}\ (n = 1,2,3,\ldots),$$

and

$$Q_0 = 2,\ Q_1 = 2,\ \text{and } Q_{n+1} = 2Q_n + Q_{n-1}\ (n = 1,2,3,\ldots).$$

In other words, $P_n = u_n(2,-1)$ and $Q_n = v_n(2,-1)$ for all $n \in \mathbb{N}$.

Conjecture 89. (2012-11-02) *Let p be an odd prime. When $p\equiv 1,3 \pmod 8$ and $p=x^2+2y^2$ with $x,y\in\mathbb{Z}$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}Q_{3k}\equiv\left(2-\left(\frac{-1}{p}\right)\right)(8x^2-4p) \pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}P_{3k}\equiv\begin{cases}0 \pmod{p^3} & \text{if } p\equiv 1 \pmod 8,\\ 4p-8x^2 \pmod{p^2} & \text{if } p\equiv 3 \pmod 8,\end{cases}$$

$$14\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{(-8)^k}P_{3k}\equiv\begin{cases}3p-4x^2 \pmod{p^2} & \text{if } p\equiv 1 \pmod 8,\\ 20x^2+21p \pmod{p^2} & \text{if } p\equiv 3 \pmod 8.\end{cases}$$

If $p\equiv 1 \pmod 8$, then

$$\sum_{k=0}^{p-1}(7k+2)\frac{\binom{2k}{k}^3}{(-8)^k}Q_{3k}\equiv 4p \pmod{p^3};$$

if $p\equiv 3 \pmod 8$, then

$$\sum_{k=0}^{p-1}(21k+4)\frac{\binom{2k}{k}^3}{(-8)^k}Q_{3k}\equiv -132p \pmod{p^3}$$

and

$$\sum_{k=0}^{p-1}(28k+5)\frac{\binom{2k}{k}^3}{(-8)^k}P_{3k}\equiv 62p \pmod{p^3}.$$

If $p\equiv 5,7 \pmod 8$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}P_{3k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}Q_{3k}\equiv 0 \pmod{p^2},$$

$$14\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{(-8)^k}P_{3k}\equiv -p\left(16+15\left(\frac{-1}{p}\right)\right) \pmod{p^2},$$

and

$$\sum_{k=0}^{p-1}(21k+4)\frac{\binom{2k}{k}^3}{(-8)^k}Q_{3k}\equiv 12p\left(5+6\left(\frac{-1}{p}\right)\right) \pmod{p^2}.$$

Conjecture 90. (2013-03-12) *Let p be an odd prime. If $(\frac{-6}{p})=-1$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}P_{4k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}Q_{4k}\equiv 0 \pmod{p^2}.$$

If $p\equiv 1,7 \pmod{24}$ and $p=x^2+6y^2$ $(x,y\in\mathbb{Z})$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}P_{4k}\equiv 0 \pmod{p^3}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}Q_{4k}\equiv(-1)^y(8x^2-4p)\pmod{p^2}.$$

When $p\equiv 5,11\pmod{24}$ *and* $p=2x^2+3y^2$ $(x,y\in\mathbb{Z})$, *we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}P_{4k}\equiv 4\left(\frac{-1}{p}\right)(p-4x^2)\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}Q_{4k}\equiv 12\left(\frac{-1}{p}\right)(4x^2-p)\pmod{p^2}.$$

Conjecture 91. (2013-03-11) *Let* p *be an odd prime. If* $(\frac{-22}{p})=-1$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}P_{12k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}Q_{12k}\equiv 0\pmod{p^2}.$$

If $(\frac{2}{p})=(\frac{p}{11})=1$ *and* $p=x^2+22y^2$ $(x,y\in\mathbb{Z})$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}P_{12k}\equiv 0\pmod{p^3}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}Q_{12k}\equiv(-1)^y(8x^2-4p)\pmod{p^2}.$$

When $(\frac{2}{p})=(\frac{p}{11})=-1$ *and* $p=2x^2+11y^2$ $(x,y\in\mathbb{Z})$, *we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}P_{12k}\equiv 140\left(\frac{-1}{p}\right)(p-4x^2)\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}Q_{12k}\equiv 396\left(\frac{-1}{p}\right)(4x^2-p)\pmod{p^2}.$$

Conjecture 92. (2011-11-03) *Let* p *be an odd prime. If* $(\frac{-13}{p})=-1$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}u_{6k}(3,-1)\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}v_{6k}(3,-1)\equiv 0\pmod{p^2}.$$

If $(\frac{-1}{p})=(\frac{p}{13})=1$ *and* $p=x^2+13y^2$ $(x,y\in\mathbb{Z})$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}u_{6k}(3,-1)\equiv 0\pmod{p^3}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}v_{6k}(3,-1)\equiv(-1)^y(8x^2-4p)\pmod{p^2}.$$

Conjecture 93. (2012-11-03) *Let p be an odd prime. If $(\frac{-58}{p})=-1$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}u_{12k}(5,-1)\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}v_{12k}(5,-1)\equiv 0 \pmod{p^2}.$$

If $(\frac{-2}{p})=(\frac{29}{p})=1$ and $p=x^2+58y^2$ $(x,y\in\mathbb{Z})$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}u_{12k}(5,-1)\equiv 0\pmod{p^3}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}v_{12k}(5,-1)\equiv\left(\frac{-1}{p}\right)(8x^2-4p)\pmod{p^2}.$$

If $(\frac{-2}{p})=(\frac{29}{p})=-1$ and $p=2x^2+29y^2$ $(x,y\in\mathbb{Z})$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}u_{12k}(5,-1)\equiv 7280\left(\frac{-1}{p}\right)(4x^2-p)\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}v_{12k}(5,-1)\equiv 39204\left(\frac{-1}{p}\right)(p-4x^2)\pmod{p^2}.$$

Conjecture 94. (2012-11-03) *Let p be an odd prime. If $(\frac{-37}{p})=-1$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}u_{6k}(12,-1)\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}v_{6k}(12,-1)\equiv 0 \pmod{p^2}.$$

If $(\frac{-1}{p})=(\frac{37}{p})=1$ and $p=x^2+37y^2$ $(x,y\in\mathbb{Z})$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}u_{6k}(12,-1)\equiv 0\pmod{p^3}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}v_{6k}(12,-1)\equiv(-1)^y(8x^2-4p)\pmod{p^2}.$$

Conjecture 95. (2013-03-12) *Let p be an odd prime. If $p\equiv 5,7\pmod{8}$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}u_{4k}(10,1)\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}v_{4k}(10,1)\equiv 0 \pmod{p^2}.$$

If $p\equiv 1,\ 19 \pmod{24}$ *and* $p=x^2+2y^2$ $(x,y\in\mathbb{Z})$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}u_{4k}(10,1)\equiv 0 \pmod{p^3}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}v_{4k}(10,1)\equiv (-1)^y(8x^2-4p) \pmod{p^2}.$$

If $p\equiv 11,\ 17 \pmod{24}$ *and* $p=x^2+2y^2$ $(x,y\in\mathbb{Z})$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}u_{4k}(10,1)\equiv 20\left(\frac{-1}{p}\right)(p-2x^2) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}v_{4k}(10,1)\equiv 196\left(\frac{-1}{p}\right)(2x^2-p) \pmod{p^2}.$$

Conjecture 96. (2013-03-12) *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-2^{12})^k}u_{4k}(5,8)\equiv 0 \pmod{p^2}.$$

When $(\frac{p}{7})=1$ (*i.e.*, $p\equiv 1,2,4 \pmod{7}$), *we even have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-2^{12})^k}u_{4k}(5,8)\equiv 0 \pmod{p^3}.$$

Also,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-4096)^k}v_{4k}(5,8)$$
$$\equiv\begin{cases}8x^2-4p \pmod{p^2} & \text{if } (\frac{p}{7})=1\ \&\ p=x^2+7y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{p}{7})=-1.\end{cases}$$

If $(\frac{p}{7})=-1$ *and* $p>3$, *then*

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{(-4096)^k}u_{4k}(5,8)\equiv\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{(-4096)^k}v_{4k}(5,8)\equiv 0 \pmod{p}.$$

If $(\frac{p}{7})=1$ *and* $p=x^2+7y^2$ $(x,y\in\mathbb{Z})$, *then*

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{(-4096)^k}u_{4k}(5,8)\equiv\frac{3p-4x^2}{42} \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{(-4096)^k}v_{4k}(5,8)\equiv\frac{3}{2}p-2x^2 \pmod{p^2}.$$

Conjecture 97. (2013-03-13) *Let $p>3$ be a prime.*

(i) *Assume that $(\frac{p}{7})=-1$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3(-1)^k u_{3k}(16,1)\equiv\sum_{k=0}^{p-1}\binom{2k}{k}^3(-1)^k v_{3k}(16,1)\equiv 0 \pmod{p^2},$$

and also

$$\sum_{k=0}^{p-1}k\binom{2k}{k}^3(-1)^k u_{3k}(16,1)\equiv\sum_{k=0}^{p-1}k\binom{2k}{k}^3(-1)^k v_{3k}(16,1)\equiv 0 \pmod{p}$$

provided $p\neq 19$.

(ii) *Suppose $(\frac{p}{7})=1$ and write $p=x^2+7y^2$ with $x,y\in\mathbb{Z}$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3(-1)^k u_{3k}(16,1)$$
$$\equiv\begin{cases}0 \pmod{p^3} & \text{if } p\equiv 1 \pmod{4},\\ (-1)^y 32(p-2x^2) \pmod{p^2} & \text{if } p\equiv 3 \pmod{4},\end{cases}$$

and

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3(-1)^k v_{3k}(16,1)\equiv\left(64\left(\frac{-1}{p}\right)-63\right)(8x^2-4p) \pmod{p^2}.$$

Also,

$$\sum_{k=0}^{p-1}k\binom{2k}{k}^3(-1)^k u_{3k}(16,1)\equiv\begin{cases}\frac{8}{399}(3p-4x^2) \pmod{p^2} & \text{if } p\equiv 1 \pmod{4},\\ -\frac{8}{3591}(3492x^2+4535p) \pmod{p^2} & \text{if } p\equiv 3 \pmod{4},\end{cases}$$

and

$$\sum_{k=0}^{p-1}k\binom{2k}{k}^3(-1)^k v_{3k}(16,1)\equiv\begin{cases}\frac{32}{57}(3p-4x^2) \pmod{p^2} & \text{if } p\equiv 1 \pmod{4},\\ \frac{32}{171}(660x^2+857p) \pmod{p^2} & \text{of } p\equiv 3 \pmod{4}.\end{cases}$$

Conjecture 98. *Let p be an odd prime.*

(i) (2013-03-14) *If $p>7$ and $p\equiv 3 \pmod{4}$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-72)^k}u_k(24,-3)\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-72)^k}v_k(24,-3)\equiv 0 \pmod{p^2}.$$

If $p\equiv 1 \pmod{12}$ and $p=x^2+9y^2$ with $x,y\in\mathbb{Z}$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-72)^k}u_k(24,-3)\equiv 0 \pmod{p^3}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-72)^k}v_k(24,-3)\equiv 8x^2-4p \pmod{p^2}.$$

If $p\equiv 5 \pmod{12}$ *and* $p=x^2+y^2$ *with* $x,y\in\mathbb{Z}$*, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-72)^k}u_k(24,-3)\equiv \frac{8}{7}\left(\frac{xy}{3}\right)xy \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-72)^k}v_k(24,-3)\equiv -32\left(\frac{xy}{3}\right)xy \pmod{p^2}.$$

(ii) (2013-03-18) *We have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{819200^k}u_k(720,-5)$$
$$\equiv\begin{cases}0 \pmod{p^2} & \text{if } (\frac{p}{35})=-1 \ \&\ p\neq 23,\\ 0 \pmod{p^3} & \text{if } (\frac{-1}{p})=(\frac{p}{5})=(\frac{p}{7})=1,\end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{819200^k}v_k(720,-5)$$
$$\equiv\begin{cases}0 \pmod{p^2} & \text{if } (\frac{p}{35})=-1,\\ \pm(2x^2-4p) \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{5})=(\frac{p}{7})=1 \ \&\ 4p=x^2+35y^2,\\ 0 \pmod{p^3} & \text{if } (\frac{p}{5})=(\frac{p}{7})=1 \ \&\ p\equiv 3 \pmod 4,\end{cases}$$

where x *and* y *are integers.*

Conjecture 99. (2011-10-02) *Let* $p>3$ *be a prime.*

(i) *If* $(\frac{p}{15})=-1$*, then*

$$\sum_{k=0}^{p-1}A_ku_k(7,1)\equiv\sum_{k=0}^{p-1}A_kv_k(7,1)\equiv 0 \pmod{p^2},$$
$$\sum_{k=0}^{p-1}kA_ku_k(7,1)\equiv\frac{p}{90}\left(25\left(\frac{p}{3}\right)+27\right) \pmod{p^2},$$
$$\sum_{k=0}^{p-1}kA_kv_k(7,1)\equiv-\frac{p}{2}\left(5\left(\frac{p}{3}\right)+3\right) \pmod{p^2}.$$

When $p\equiv 1,4 \pmod{15}$ *and* $p=x^2+15y^2$ $(x,y\in\mathbb{Z})$, *we have*

$$\sum_{k=0}^{p-1}A_ku_k(7,1)\equiv 0 \pmod{p^3},$$

$$\sum_{k=0}^{p-1}kA_ku_k(7,1)\equiv \frac{3p-4x^2}{45} \pmod{p^2},$$

$$\sum_{k=0}^{p-1}A_kv_k(7,1)\equiv 8x^2-2p \pmod{p^2},$$

$$\sum_{k=0}^{p-1}(2k+1)A_kv_k(7,1)\equiv 2p \pmod{p^3}.$$

If $p\equiv 2,8 \pmod{15}$ *and* $p=3x^2+5y^2$ $(x,y\in\mathbb{Z})$, *then*

$$\sum_{k=0}^{p-1}A_ku_k(7,1)\equiv 2p-12x^2 \pmod{p^2},$$

$$\sum_{k=0}^{p-1}(45k+19)A_ku_k(7,1)\equiv 26p \pmod{p^3},$$

$$\sum_{k=0}^{p-1}A_kv_k(7,1)\equiv 84x^2-14p \pmod{p^2},$$

$$\sum_{k=0}^{p-1}(7k+3)A_kv_k(7,1)\equiv -28p \pmod{p^3}.$$

(ii) *If* $p\equiv 3 \pmod{4}$, *then*

$$\sum_{k=0}^{p-1}(-1)^kA_ku_k(14,1)\equiv \sum_{k=0}^{p-1}(-1)^kA_kv_k(14,1)\equiv 0 \pmod{p^2},$$

$$\sum_{k=0}^{p-1}(-1)^kkA_ku_k(14,1)\equiv -\frac{p}{48}\left(15\left(\frac{p}{3}\right)+16\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1}(-1)^kkA_kv_k(14,1)\equiv p\left(5\left(\frac{p}{3}\right)+4\right) \pmod{p^2}.$$

When $p\equiv 1 \pmod{12}$ *and* $p=x^2+9y^2$ $(x,y\in\mathbb{Z})$, *we have*

$$\sum_{k=0}^{p-1}(-1)^kA_ku_k(14,1)\equiv 0 \pmod{p^3},$$

$$\sum_{k=0}^{p-1}(-1)^kkA_ku_k(14,1)\equiv \frac{3p-4x^2}{48} \pmod{p^2},$$

$$\sum_{k=0}^{p-1}(-1)^k A_k v_k(14,1) \equiv 8x^2-4p \pmod{p^2},$$
$$\sum_{k=0}^{p-1}(-1)^k (2k+1) A_k v_k(14,1) \equiv 2p \pmod{p^3}.$$

If $p \equiv 5 \pmod{12}$ *and* $p = x^2+y^2$ $(x, y \in \mathbb{Z})$, *then*

$$\sum_{k=0}^{p-1}(-1)^k A_k u_k(14,1) \equiv -4xy\left(\frac{xy}{3}\right) \pmod{p^2},$$
$$\sum_{k=0}^{p-1}(-1)^k (48k+17) A_k u_k(14,1) \equiv 31p \pmod{p^3},$$
$$\sum_{k=0}^{p-1}(-1)^k A_k v_k(14,1) \equiv 56xy\left(\frac{xy}{3}\right) \pmod{p^2},$$
$$\sum_{k=0}^{p-1}(-1)^k (14k+5) A_k v_k(14,1) \equiv -126p \pmod{p^3}.$$

Conjecture 100. (2011-09-30) *Let* $p > 3$ *be a prime. If* $(\frac{-6}{p}) = -1$, *then*

$$\sum_{k=0}^{p-1} D_k^3 u_k(6,1) \equiv \sum_{k=0}^{p-1} D_k^3 v_k(6,1) \equiv 0 \pmod{p^2}.$$

If $p \equiv 1, 7 \pmod{24}$ *and* $p = x^2+6y^2$ $(x, y \in \mathbb{Z})$, *then*

$$\sum_{k=0}^{p-1} D_k^3 u_k(6,1) \equiv 0 \pmod{p^2},$$
$$\sum_{k=0}^{p-1} k D_k^3 u_k(6,1) \equiv -\frac{11}{96}x^2 \pmod{p^2},$$
$$\sum_{k=0}^{p-1} D_k^3 v_k(6,1) \equiv 8x^2-4p \pmod{p^2},$$
$$\sum_{k=0}^{p-1} (2k+1) D_k^3 v_k(6,1) \equiv -\frac{p}{4} \pmod{p^2}.$$

If $p \equiv 5, 11 \pmod{24}$ *and* $p = 2x^2 + 3y^2$ $(x, y \in \mathbb{Z})$, *then*

$$\sum_{k=0}^{p-1} D_k^3 u_k(6,1) \equiv 8x^2 - 2p \pmod{p^2},$$

$$\sum_{k=0}^{p-1} (128k+53) D_k^3 u_k(6,1) \equiv 30p \pmod{p^3},$$

$$\sum_{k=0}^{p-1} D_k^3 v_k(6,1) \equiv 12p - 48x^2 \pmod{p^2},$$

$$\sum_{k=0}^{p-1} (144k+61) D_k^3 v_k(6,1) \equiv -186p \pmod{p^2}.$$

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