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The distribution of values of the Euler function

by

PAUL T. BATEMAN (Urbana, Ill.)

Dedicated to the memory of the late Professor Wacław Sierpiński

§ 1. Introduction. Let φ denote the arithmetic function of Euler, i.e., $\varphi(n)$ stands for the number of positive integers not exceeding n which are relatively prime to n. The distribution of the values taken by φ can be studied from many points of view. In this paper we take the following approach.

Let a_m denote the number of positive integers n with $\varphi(n) = m$. (This number is finite, since $\varphi(n)$ tends to infinity with n.) It is a familiar phenomenon that a_m is usually zero and on the other hand is occasionally very large. Accordingly, in order to get stable results it is reasonable to consider the summatory function

$$A(x) = \sum_{m\leqslant x} a_m.$$

(Thus $A(x)$ is the number of positive integers n with $\varphi(n) \leqslant x$.)

The following numerical data were obtained from Table II of [6]:

x	100	200	300	400	500	600	700	800	900	1000
$A(x)$	198	395	588	790	971	1174	1357	1569	1759	1941
$A(x)/x$	1.980	1.975	1.960	1.975	1.942	1.957	1.939	1.961	1.954	1.941

These data suggest that $A(x)/x$ has a finite limit α as $x \to +\infty$. In addition a simple heuristic argument suggests that this should be so with $\alpha = \zeta(2)\zeta(3)/\zeta(6) = 1.9435964\ldots$ Namely, since the arithmetic function $n/\varphi(n)$ is the multiplicative convolution of the two arithmetic functions 1 and

$$c(n) = |\mu(n)| \prod_{p|n} (p-1)^{-1},$$

we see that $n/\varphi(n)$ has a mean-value equal to

$$\sum_{d=1}^{\infty}\frac{c(d)}{d}=\prod_p\left(1+\frac{c(p)}{p}\right)=\prod_p\left(1+\frac{1}{p(p-1)}\right)=\frac{\zeta(2)\zeta(3)}{\zeta(6)}=\alpha.$$

Thus the number of positive integers n with $\varphi(n)\leqslant x$ should be near to the number of positive integers n with $n/\alpha\leqslant x$, which is $[\alpha x]$.

Two actual proofs of the relation

$$\lim_{x\to+\infty}\frac{A(x)}{x}=\alpha \tag{1.1}$$

are in the literature. The first proof, given by Erdös [5], is based on one of the earliest theorems in probabilistic number theory, namely, the theorem of Schoenberg [10] that $n/\varphi(n)$ possesses a distribution function. Erdös' proof as such proves only the existence of the limit in (1.1), but, as explained in the next section, it is comparatively trivial to evaluate the limit once its existence is known. The second proof was given by Dressler [3]. It is completely elementary and is based on density-theoretic ideas. As we shall also explain in the next section, (1.1) also follows immediately from the Wiener–Ikehara Theorem.

The main purpose of this paper is to illustrate the use of several techniques in analytic number theory by applying them to obtain estimates for the error term $A(x)-\alpha x$. The methods which we shall use are

(A) application of results of Nyman, Malliavin, and Diamond in Beurling's theory of generalized integers,

(B) the Nyman–Malliavin method of using the Plancherel formula,

(C) the classical method of contour integration.

The results obtained are as follows. If we use the prime-number theorem with the de la Vallée-Poussin form of the error term, Method A gives

$$A(x)=\alpha x+O\left(x\exp\{-c(\log x)^{1/3}\}\right) \tag{1.2}$$

for any positive constant c. The exponent 1/3 could be increased slightly by using sharper forms of the prime-number theorem. Method B gives

$$A(x)=\alpha x+O\left(x\exp\{-\tfrac{2}{3}(1-\varepsilon)(\log x\log\log x)^{1/2}\}\right) \tag{1.3}$$

for any fixed positive number ε. Method C gives the best result, namely

$$A(x)=\alpha x+O\left(x\exp\{-(1-\varepsilon)(\tfrac{1}{2}\log x\log\log x)^{1/2}\}\right) \tag{1.4}$$

for any fixed positive number ε. In particular the result

$$A(x)=\alpha x+O\left(x\exp\{-\tfrac{70}{99}(\log x\log\log x)^{1/2}\}\right)$$

follows from (1.4) but not from (1.3). Those readers interested only in (1.4) as an end in itself can omit the discussions of Methods A and B in § 3 and § 5.

The result (1.4) is undoubtedly not the best possible, but the optimal estimate for $A(x)-\alpha x$ may be somewhat elusive. The analytic function represented by $\sum a_m m^{-s}$ for $\operatorname{Re}s>1$ has no singularities in the half-plane $\operatorname{Re}s>0$ other than a simple pole at $s=1$, but the precise determination of its growth pattern in the strip $0<\operatorname{Re}s<1$ does not seem easy. It is probably true that $A(x)-\alpha x=O\left(x\exp\{-(\log x)^{1-\varepsilon}\}\right)$ for every positive ε.

Erdös [4] has proved that there is a positive number ϱ such that $a_m>m^{\varrho}$ for infinitely many m, so that in particular $A(x)-\alpha x\neq o(x^{\varrho})$. In addition he conjectured that if λ is any number less than one, then $a_m>m^{\lambda}$ for infinitely many m. Thus it would be reasonable to make the slightly weaker conjecture that $A(x)-\alpha x\neq o(x^{\lambda})$ for any $\lambda<1$. Erdös has suggested (in a private communication) that possibly $|A(x)-\alpha x|$ is infinitely often as big as $x\exp\{-c\log x/\log\log x\}$ for some positive c.

This paper originated in discussions with Harold G. Diamond and Robert E. Dressler. The author would like to express his thanks to Professors Diamond and Dressler for their helpful comments.

§ 2. Further preliminary remarks. Analytical approaches to our problem naturally involve the Dirichlet series with coefficients $a_1, a_2, \ldots$ This can be rewritten

$$\sum_{m=1}^{\infty}a_m m^{-s}=\sum_{n=1}^{\infty}\varphi(n)^{-s}=\prod_p\{1+\varphi(p)^{-s}+\varphi(p^2)^{-s}+\varphi(p^3)^{-s}+\ldots\} \tag{2.1}$$
$$=\prod_p\{1+(p-1)^{-s}+(p-1)^{-s}p^{-s}+(p-1)^{-s}p^{-2s}+\ldots\},$$

where the products are over all the prime numbers and absolute convergence prevails if $\operatorname{Re}s>1$. If in the last expression $(p-1)^{-s}$ were replaced at each occurrence by p^{-s}, we would get the ordinary Riemann zeta-function

$$\zeta(s)=\prod_p(1-p^{-s})^{-1}.$$

Accordingly it is reasonable to factor out $\zeta(s)$ as a first approximation to our generating function. Thus

$$\sum_{m=1}^{\infty}a_m m^{-s}=\zeta(s)f(s)\qquad(\operatorname{Re}s>1), \tag{2.2}$$

where

$$f(s)=\prod_p\{1+(p-1)^{-s}-p^{-s}\}. \tag{2.3}$$

The Dirichlet series for f has abscissa of absolute convergence one and so our generating function cannot be obtained from the zeta-function by a comparatively trivial modification, i.e., by multiplication by a Dirichlet series with small abscissa of absolute convergence. Nevertheless the product representation (2.3) converges uniformly on compact subsets of the half-plane $\operatorname{Re} s > 0$, since

$$(2.4)\qquad |(p-1)^{-s}-p^{-s}| = \Big|s\int_{p-1}^{p} v^{-s-1}dv\Big| \leqslant |s|(p-1)^{-\operatorname{Re}s-1}.$$

Thus f is an analytic function in the right half-plane.

Let us recall the following abelian theorem, given as Supplement II of [7].

DIRICHLET–DEDEKIND THEOREM. *If $c_1, c_2, \ldots$ is a sequence of complex numbers such that*

$$\lim_{x\to+\infty} x^{-1}\sum_{m\leqslant x} c_m = \gamma$$

for some complex number γ, then $\sum c_m m^{-s}$ converges for real $s > 1$ and

$$\lim_{s\to1+}(s-1)\sum_{m=1}^{\infty} c_m m^{-s} = \gamma,$$

where the latter limit is taken over real values of s greater than 1.

Since

$$\lim_{s\to1+}(s-1)\sum_{m=1}^{\infty} a_m m^{-s} = \lim_{s\to1+}(s-1)\zeta(s)f(s) = f(1)$$
$$= \prod_p \{1+(p-1)^{-1}-p^{-1}\} = \frac{\zeta(2)\zeta(3)}{\zeta(6)},$$

the Dirichlet–Dedekind Theorem tells us that once we know the existence of the limit of $A(x)/x$, e.g., from Erdös' proof, we can immediately conclude that the value of the limit must be $a = \zeta(2)\zeta(3)/\zeta(6)$.

Now let us also recall the following powerful tauberian theorem ([11], p. 127), which is a corrected form of the false converse of the Dirichlet–Dedekind Theorem.

WIENER–IKEHARA THEOREM. *Suppose $c_1, c_2, \ldots$ is a sequence of non-negative real numbers such that $\sum c_m m^{-s}$ converges in the open half-plane $\operatorname{Re} s > 1$. If there exists a constant γ and a continuous function h on the closed half-plane $\operatorname{Re} s \geqslant 1$ such that*

$$\sum_{m=1}^{\infty} c_m m^{-s} - \gamma(s-1)^{-1} = h(s)$$

on the open half-plane $\operatorname{Re} s > 1$, then

$$\lim_{x\to+\infty} x^{-1}\sum_{m\leqslant x} c_m = \gamma.$$

Now for $\operatorname{Re} s > 1$ we have

$$\sum_{m=1}^{\infty} a_m m^{-s} - a(s-1)^{-1} = f(s)\{\zeta(s)-(s-1)^{-1}\} + \{f(s)-f(1)\}\{s-1\}^{-1},$$

where both terms on the right are analytic functions in the half-plane $\operatorname{Re} s > 0$, except for removable singularities at $s = 1$. Thus the Wiener–Ikehara Theorem gives (1.1) immediately.

§ 3. Method A. Method A is based on Beurling's theory of generalized integers, specifically on the inverse theorems to the prime-number theorem obtained successively by Nyman [9], Malliavin [8], and Diamond [2].

Suppose we have a sequence of positive numbers $\pi_1, \pi_2, \ldots$ such that

$$1 < \pi_1 \leqslant \pi_2 \leqslant \pi_3 \leqslant \ldots, \qquad \pi_j \to +\infty.$$

The product

$$\prod_{j=1}^{\infty}(1-\pi_j^{-s})^{-1} = \prod_{j=1}^{\infty}(1+\pi_j^{-s}+\pi_j^{-2s}+\ldots)$$

may be formally expanded into a general Dirichlet series

$$\prod_{i=1}^{\infty} \beta_i \nu_i^{-s},$$

where $\nu_1 = 1, \nu_2, \nu_3, \ldots$ is an increasing sequence of positive numbers containing the distinct elements of the multiplicative semigroup generated by $\pi_1, \pi_2, \ldots$, and where $\beta_1 = 1, \beta_2, \beta_3, \ldots$ are non-negative integers.

Beurling [1] showed that if

$$\sum_{\nu_i\leqslant x} \beta_i$$

behaves sufficiently like a constant times x, then we have analogues of the prime-number theorem for the counting function of the sequence $\pi_1, \pi_2, \ldots$, i.e., for

$$\sum_{\pi_j\leqslant x} 1.$$

In the opposite direction if this counting function behaves sufficiently like $\int_2^x (\log u)^{-1}du$, we can expect $\sum_{\nu_i\leqslant x}\beta_i$ to behave very much like a constant times x. The known results are as follows.

Nyman–Malliavin–Diamond Theorem. *In the above notation suppose*

(3.1) $$\sum_{\pi_j\leqslant x} 1 = \int_2^x (\log u)^{-1}du + O\big(x\exp\{-b(\log x)^a\}\big),$$

where $b>0$ *and* $0<a<1$. *Then*

(3.2) $$\sum_{\nu_i\leqslant x}\beta_i = Bx + O(x\{\log x\}^{-M}) \quad \textit{for every } M>0 \quad \text{(Nyman)},$$

(3.3) $$\sum_{\nu_i\leqslant x}\beta_i = Bx + O\big(x\exp\{-c(\log x)^{a/(a+2)}\}\big) \ \textit{for some } c>0$$

(Malliavin),

(3.4) $$\sum_{\nu_i\leqslant x}\beta_i = Bx + O\big(x\exp\{-c(\log x)^{a/(a+1)}\}\big) \ \textit{for every } c>0$$

(Diamond).

Here B *is a positive number given, for example, by*

$$B = \lim_{s\to1+}(s-1)\prod_{j=1}^{\infty}(1-\pi_j^{-s})^{-1} = \lim_{s\to1+}\Big\{\prod_p(1-p^{-s})\prod_{j=1}^{\infty}(1-\pi_j^{-s})^{-1}\Big\}.$$

In order to apply this theorem to our problem we return to the expansion (2.1). In § 2 we noted that a rough approximation to the generating function is obtained by replacing $(p-1)^{-s}$ throughout the last line of (2.1) by p^{-s}. An even better approximation is obtained if we turn things around and replace p^{-s} throughout the last line of (2.1) by $(p-1)^{-s}$ (except when $p=2$). In fact we have

$$\sum_{m=1}^{\infty}a_m m^{-s} = \prod_{p>2}\{1+(p-1)^{-s}+(p-1)^{-2s}+(p-1)^{-3s}+\dots\}\sum_{k=1}^{\infty}d_k k^{-s}$$
$$= \sum_{l=1}^{\infty}b_l l^{-s}\cdot\sum_{k=1}^{\infty}d_k k^{-s},$$

where $\sum d_k k^{-s}$ has abscissa of absolute convergence not exceeding $\frac12$. The precise values of $d_1, d_2, \dots$ are not required, but they can be determined recursively from the product representation

$$\sum d_k k^{-s} = (2+2^{-s}+2^{-2s}+\dots)\prod_{p>2}\Big(1-\{(p-1)^{-2s}-(p-1)^{-s}p^{-s}\}\sum_{j=0}^{\infty}p^{-js}\Big).$$

(Actually, since

$$\sum_{4\nmid k}d_k k^{-s} = 2^{-s}+2\sum_{p\equiv3\,(\mathrm{mod}\,4)}\sum_{j=0}^{\infty}(p-1)^{-s}p^{-(j+1)s},$$

it is easy to see that the abscissa of absolute convergence of $\sum d_k k^{-s}$ is exactly $\frac12$.)

We apply the above theorem with $\pi_j = p_{j+1}-1$, $\nu_i = i$, and $\beta_i = b_i$, where p_j is the jth prime-number. For this purpose we recall that

$$\sum_{p>2,\,p-1\leqslant x} 1 = \int_2^x(\log u)^{-1}du + O\big(x\exp\{-b(\log x)^{1/2}\}\big)$$

from de la Vallée Poussin's version of the prime-number theorem, b being a certain positive constant. Combining this with Diamond's result (3.4), we obtain

(3.5) $$B(x) = \sum_{l\leqslant x}b_l = \beta x + O\big(x\exp\{-c(\log x)^{1/3}\}\big)$$

for every positive c, where

$$\beta = \lim_{s\to1+}\prod_p\{1-p^{-s}\}\prod_{p>2}\{1-(p-1)^{-s}\}^{-1}$$
$$= (1-\tfrac12)\prod_{p>2}\{1-p^{-1}\}\{1-(p-1)^{-1}\}^{-1}.$$

Since $\sum d_k k^{-s}$ has abscissa of absolute convergence (at most) $\frac12$, an elementary argument then gives

$$A(x) = \sum_{k=1}^{[x]}d_k B(x/k) = x\beta\sum_{k=1}^{\infty}d_k k^{-1} + O\big(x\exp\{-c(\log x)^{1/3}\}\big)$$

for every positive c, where

$$\beta\sum_{k=1}^{\infty}d_k k^{-1} = 3\beta\prod_{p>2}\{1-(p-1)^{-3}\} = \prod_p\Big\{1+\frac{1}{p(p-1)}\Big\} = \frac{\zeta(2)\zeta(3)}{\zeta(6)} = \alpha.$$

Thus (1.2) is proved. (In view of the remarks in § 2 centering around the Dirichlet–Dedekind Theorem, the last calculation is not actually necessary, but it is a useful check on the work.)

It is easy to verify that $\big(\sum d_k k^{-s}\big)^{-1}$ has a Dirichlet series whose abscissa of absolute convergence is less than one. Thus the improvements on (1.2) obtained later on in this paper will imply corresponding improvements on the estimate (3.5) for $B(x)-\beta x$.

Use of the sharpest known error term in the prime-number theorem would give a slightly higher exponent on $\log x$ in (3.5) and (1.2), but not results nearly as good as (1.3) or (1.4). In fact the results obtained by Methods B and C are roughly what we get from Diamond's result (3.4) if we simply assume the Riemann hypothesis. This is to be expected, since (3.4) is a general assertion based only on (3.1). In applying Methods B and C we are able to take account of the specific information that here we are dealing with a sequence of generalized primes obtained from the ordinary odd primes by moving each one a bounded distance.

§ 4. Estimation of the generating function. In order to apply either Method B or Method C we need a "reasonable" estimate for our generating function (2.2) in as wide a region as possible to the left of the line $\mathrm{Re}\, s = 1$. Here a "reasonable" estimate means an estimate in terms of $\mathrm{Im}\, s$ which is of lower order of magnitude than any positive power of $|\mathrm{Im}\, s|$ when $|\mathrm{Im}\, s|$ is large. Since the behavior of the zeta function is familiar, our main task will be to estimate the function $f(s)$ defined by (2.3). For brevity we use the customary notation $\sigma = \mathrm{Re}\, s$, $t = \mathrm{Im}\, s$. We shall see that if $\theta < 1$, it is possible to get a "reasonable" estimate for $f(s)$ provided $\sigma \geqslant 1-\theta(\log\log|t|)/(\log|t|)$ when $|t|$ is large. However if $\theta \geqslant 1$, the estimate obtained by our method would not be satisfactory.

LEMMA 4.1. *If* $|t| \geqslant 8$ *and* $\sigma \geqslant \sigma_0(t)$, *where* $\sigma_0(t)$ *is some function of* t *such that* $\frac{1}{3} \leqslant \sigma_0(t) \leqslant 1$, *then*

$$|f(s)| \leqslant \exp\{50\,|t|^{1-\sigma_0(t)}\log\log|t|\}.$$

Proof. We may assume $\sigma \leqslant 2$. The estimate (2.4) which we used to establish the analyticity of $f(s)$ for $\mathrm{Re}\, s > 0$ is weaker than the trivial inequality

$$|(p-1)^{-s}-p^{-s}| \leqslant 2(p-1)^{-\sigma} \tag{4.1}$$

when p is small relative to $|t|$. We use (2.4) when $p > |t|$ and (4.1) when $p \leqslant |t|$. This gives

$$\log|f(s)| \leqslant \sum_{p\leqslant|t|}\log\{1+2(p-1)^{-\sigma}\}+\sum_{p>|t|}\log\{1+(2+|t|)(p-1)^{-\sigma-1}\}$$
$$\leqslant \sum_{p\leqslant|t|}2(p-1)^{-\sigma_0}+\sum_{p>|t|}\tfrac{5}{4}|t|(p-1)^{-\sigma_0-1} \leqslant 4\sum_{p\leqslant|t|}p^{-\sigma_0}+5|t|\sum_{p>|t|}p^{-\sigma_0-1}.$$

Now, since

$$\vartheta(x) = \sum_{p\leqslant x}\log p \leqslant (x-\tfrac{3}{2})\log 4 \qquad (x \geqslant \tfrac{3}{2})$$

and $(u^{\sigma_0}\log u)^{-1}$ decreases as u increases, we have by two integrations by parts

$$4\sum_{p\leqslant|t|}p^{-\sigma_0} = 4\int_{3/2}^{|t|}(u^{\sigma_0}\log u)^{-1}d\vartheta(u) \leqslant 4\int_{3/2}^{|t|}(u^{\sigma_0}\log u)^{-1}(\log 4)\,du$$
$$\leqslant 7\,|t|^{1-\sigma_0}\int_{3/2}^{|t|}(u\log u)^{-1}du \leqslant 20\,|t|^{1-\sigma_0}\log\log|t|.$$

On the other hand

$$5|t|\sum_{p>|t|}p^{-\sigma_0-1} \leqslant 5|t|\int_{|t|-1}^{\infty}u^{-\sigma_0-1}du = \frac{5|t|}{\sigma_0(|t|-1)^{\sigma_0}} \leqslant 20\,|t|^{1-\sigma_0}$$
$$\leqslant 30\,|t|^{1-\sigma_0}\log\log|t|.$$

Thus the assertion of the lemma follows.

LEMMA 4.2. *If* $|t| \geqslant 8$ *and* $\sigma \geqslant \sigma_0(t)$, *where* $\sigma_0(t)$ *is some function of* t *such that* $1/3 \leqslant \sigma_0(t) \leqslant 1$, *then*

$$|\zeta(s)| \leqslant 4\,|t|^{1-\sigma_0(t)}\log|t|.$$

Proof. From the familiar formula

$$\zeta(s) = \sum_{n=1}^{N}\frac{1}{n^s}+\frac{1}{(s-1)N^{s-1}}+s\int_N^{\infty}\frac{[x]-x}{x^{s+1}}dx \qquad (\mathrm{Re}\, s > 0,\ s \neq 1),$$

we have

$$|\zeta(s)| \leqslant \sum_{n=1}^{N}\frac{1}{n^\sigma}+\frac{1}{|t|N^{\sigma-1}}+\frac{\sigma+|t|}{\sigma N^\sigma} \qquad (\sigma > 0,\ t \neq 0).$$

Now the right-hand side of the preceding inequality is a decreasing function of σ. Thus if $\sigma \geqslant \sigma_0(t)$, we may replace σ by σ_0 and obtain

$$|\zeta(s)| \leqslant N^{1-\sigma_0}\left\{\sum_{n=1}^{N}\frac{1}{n}+\frac{1}{|t|}+\frac{1}{N}+\frac{|t|}{\sigma_0 N}\right\}$$
$$\leqslant N^{1-\sigma_0}\left\{\log N+1+\frac{1}{|t|}+\frac{1}{N}+\frac{3|t|}{N}\right\}.$$

Taking $N = [|t|]$ and using the assumption that $|t| \geqslant 8$, we get

$$|\zeta(s)| \leqslant |t|^{1-\sigma_0}\{\log|t|+5\} \leqslant 4\,|t|^{1-\sigma_0}\log|t|.$$

Lemma 4.2 gives a "reasonable" estimate for $\zeta(s)$ as long as $\sigma_0(t) \to 1$ as $|t| \to +\infty$. However, in order to get a "reasonable" estimate for $f(s)$ from Lemma 4.1 we must place a more stringent restriction on $\sigma_0(t)$, namely that

$$\log\log|t|-\log\log\log|t|-\{1-\sigma_0(t)\}\log|t| \to +\infty$$

as $|t| \to +\infty$. The choice

$$\sigma_0(t) = 1-\theta\,\frac{\log\log|t|}{\log|t|},$$

where θ is a fixed positive number less than one, satisfies this restriction. At the same time it would be easy to check that a more contrived choice, such as

$$\sigma_0(t) = 1-\frac{\log\log|t|-2\log\log\log|t|}{\log|t|},$$

would not give us better results than (1.3) and (1.4) in the end.

LEMMA 4.3. *Suppose* $0 < \theta < 1$ *and* $\varepsilon > 0$. *If* $|t| \geqslant 8$ *and* $\sigma \geqslant 1 - \theta(\log\log|t|)/(\log|t|)$, *then*

$$|\zeta(s)f(s) - as/(s-1)| \leqslant M(\theta, \varepsilon)|t|^{\varepsilon/2},$$

where $M(\theta, \varepsilon)$ *is a positive number depending on* θ *and* ε.

Proof. Since $1-(\log\log|t|)/(\log|t|) \geqslant 1-e^{-1} > 1/3$ if $|t| > 1$, we may apply the two preceding lemmas. On the set specified we have

$$|f(s)| < \exp\{50(\log|t|)^{\theta}\log\log|t|\}$$

by Lemma 4.1 and

$$|\zeta(s)| < 4(\log|t|)^2$$

by Lemma 4.2. Thus our result follows.

§ 5. Method B. By writing $\sum a_m m^{-s}$ as a Stieltjes integral and integrating by parts we obtain

$$\zeta(s)f(s) = \sum_{m=1}^{\infty}\frac{a_m}{m^s} = s\int_1^{\infty}\frac{A(x)}{x^{s+1}}dx \quad (\mathrm{Re}\, s > 1)$$

and hence

$$(5.1)\qquad g(s) = \frac{\zeta(s)f(s)}{s} - \frac{a}{s-1} = \int_1^{\infty}\frac{A(x)-ax}{x^{s+1}}dx \quad (\mathrm{Re}\, s > 1).$$

Instead of inverting the transform (5.1) and then moving the contour of integration (as in the next section), the method of Nyman [9] and Malliavin [8] involves differentiating (5.1) a large number of times and then invoking the Plancherel formula. Although this method may be a little more cumbersome than the more familiar method of contour integration and although it seems to give results which are poorer by a factor $2\sqrt{2}/3 = 0.9428\ldots$ in the exponent, it is worth study because of the tremendous advantage that it can be applied (as Nyman and Malliavin did) in cases where analytic continuation to the left of the line $\mathrm{Re}\, s = 1$ is impossible.

By differentiation of (5.1) we obtain

$$g^{(k)}(\sigma+it) = \int_1^{\infty}\frac{A(x)-ax}{x^{\sigma}}(-\log x)^k x^{-it}\frac{dx}{x} \quad (\sigma > 1).$$

The Plancherel formula then gives

$$(5.2)\qquad \int_{-\infty}^{\infty}|g^{(k)}(\sigma+it)|^2 dt = \frac{1}{2\pi}\int_1^{\infty}\frac{\{A(y)-ay\}^2}{y^{2\sigma}}(\log y)^{2k}\frac{dy}{y} \quad (\sigma > 1).$$

Our first task will be to estimate the left-hand side of (5.2) as a function of k uniformly in σ for $\sigma > 1$. Given a number ε with $0 < \varepsilon < 1/10$, we choose the number θ in Lemma 4.3 so that $\theta = 1-\varepsilon/2$. If $|t|$ is sufficiently large, the disc

$$\{z\colon\ |z-(\sigma+it)| \leqslant (1-\varepsilon)(\log\log|t|)/(\log|t|)\}$$

is contained in the set

$$\{z\colon\ |\mathrm{Im}\, z| \geqslant 8,\ \mathrm{Re}\, z \geqslant 1-\theta(\log\log|\mathrm{Im}\, z|)/(\log|\mathrm{Im}\, z|)\}.$$

In view of Lemma 4.3 we therefore have

$$|g(z)| \leqslant A_\varepsilon|t|^{-1+\varepsilon/2}$$

on the above disc provided $|t| > T_\varepsilon$, where A_ε and T_ε are positive numbers depending only on ε. Thus by Cauchy's inequality for the coefficients of a power series

$$(5.3)\qquad |g^{(k)}(\sigma+it)| \leqslant k!\left\{\frac{\log|t|}{(1-\varepsilon)\log\log|t|}\right\}^k A_\varepsilon|t|^{-1+\varepsilon/2} \quad (\sigma > 1,\ |t| > T_\varepsilon).$$

On the other hand, since g is regular in the right half-plane, there is a constant B_ε such that $|g(z)| \leqslant B_\varepsilon$ on the set

$$\{z\colon\ \mathrm{Re}\, z \geqslant 1/2,\ |\mathrm{Im}\, z| \leqslant T_\varepsilon+1/2\}.$$

Thus by Cauchy's inequality again

$$(5.4)\qquad |g^{(k)}(\sigma+it)| \leqslant k!\,2^k B_\varepsilon \leqslant B_\varepsilon(2k)^k \quad (\sigma > 1,\ |t| \leqslant T_\varepsilon).$$

From (5.4) we get

$$(5.5)\qquad \int_{-T_\varepsilon}^{T_\varepsilon}|g^{(k)}(\sigma+it)|^2 dt \leqslant 2T_\varepsilon B_\varepsilon^2(2k)^{2k} \quad (\sigma > 1).$$

From (5.3) we obtain

$$\int_{-\infty}^{-T_\varepsilon}|g^{(k)}(\sigma+it)|^2 dt + \int_{T_\varepsilon}^{+\infty}|g^{(k)}(\sigma+it)|^2 dt$$

$$\leqslant 2A_\varepsilon^2(k!)^2\int_{T_\varepsilon}^{\infty}t^{-1+\varepsilon}\left\{\frac{\log t}{(1-\varepsilon)\log\log t}\right\}^{2k}\frac{dt}{t}$$

$$\leqslant 2A_\varepsilon^2(k!)^2\frac{(1+\varepsilon)^{2k+1}}{(1-\varepsilon)^{4k+1}}\int_{T_\varepsilon^{(1-\varepsilon)/(1+\varepsilon)}}^{\infty}u^{-1-\varepsilon}\left(\frac{\log u}{\log\log u}\right)^{2k}\frac{du}{u},$$

where we have made the substitution $t = u^{(1+\varepsilon)/(1-\varepsilon)}$. Now for large k the maximum of $(\log u/\log\log u)^{2k}u^{-1}$ occurs when $\log u$ is somewhere between $2k\{1-2/(\log 2k)\}$ and $2k$. Hence for large k we have

$$\left(\frac{\log u}{\log\log u}\right)^{2k}\frac{1}{u} \leqslant \left(\frac{2k}{\log 2k}\right)^{2k}\exp\{-2k+4k/(\log 2k)\} \leqslant (1+\varepsilon)^k\left(\frac{2k}{e\log 2k}\right)^{2k}.$$

Also $(k!)^2 < (1+\varepsilon)^k k^{2k}e^{-2k}$ for large k. Thus for large k we have

$$(5.6)\qquad \int_{-\infty}^{-T_\varepsilon}|g^{(k)}(\sigma+it)|^2\,dt+\int_{T_\varepsilon}^{+\infty}|g^{(k)}(\sigma+it)|^2\,dt < 2A_\varepsilon^2\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{4k+1}\left(\frac{2k^2}{e^2\log 2k}\right)^{2k}\int_1^\infty u^{-1-\varepsilon}\,du.$$

Combining (5.5) and (5.6), we obtain

$$(5.7)\qquad \int_{-\infty}^{\infty}|g^{(k)}(\sigma+it)|^2\,dt < C_\varepsilon\left\{\frac{(1+6\varepsilon)(2k)^2}{2e^2\log 2k}\right\}^{2k}\qquad (\sigma>1)$$

for all positive integers k, since the constant C_ε can be adjusted to take care of any finite number of values of k.

Combining (5.2) and (5.7), we have

$$(5.8)\qquad \int_1^\infty\frac{\{A(y)-ay\}^2}{y^{2\sigma}}(\log y)^{2k}\frac{dy}{y} \leqslant 2\pi C_\varepsilon\left\{\frac{(1+6\varepsilon)(2k)^2}{2e^2\log 2k}\right\}^{2k}$$

for any $\sigma > 1$. It is clear that (5.8) implies some sort of bound on $|A(x)-ax|$. The specific deduction is based on the following simple lemma.

LEMMA 5.1. *Suppose F is a non-decreasing function on $[1, +\infty)$ such that $F(y) = O(y^{1+\delta})$ for every positive δ and suppose G is a real-valued function on $[1, +\infty)$ such that $|G(y+u)-G(y)| \leqslant Ku$ for $y \geqslant 1$ and $u \geqslant 0$. Then, if $x > 2$, we have*

$$\int_1^\infty\frac{\{F(y)-G(y)\}^2}{y^{2\sigma}}(\log y)^{2k}\frac{dy}{y} \geqslant \frac{1}{8}\left(\frac{2}{3}\right)^{2\sigma+1}\frac{\{F(x)-G(x)\}^2}{x^{2\sigma}}\left(\log\frac{x}{2}\right)^k\min\left\{\frac{|F(x)-G(x)|}{Kx},1\right\}$$

for any $\sigma > 1$.

Proof. We distinguish three cases.

(i) If $F(x)-G(x) = 0$, there is nothing to prove.

(ii) If $F(x)-G(x) > 0$, we replace the integral of the lemma by an integral over the interval

$$\left[x,\ x+\min\left\{\frac{F(x)-G(x)}{2K},\frac{x}{2}\right\}\right].$$

In that interval we have

$$F(y)-G(y) \geqslant F(x)-G(y) \geqslant F(x)-G(x)-K(y-x) \geqslant \{F(x)-G(x)\}/2,$$

and so

$$\int_1^\infty\frac{\{F(y)-G(y)\}^2}{y^{2\sigma}}(\log y)^{2k}\frac{dy}{y} \geqslant \left\{\frac{F(x)-G(x)}{2}\right\}^2\frac{1}{(3x/2)^{2\sigma+1}}(\log x)^{2k}\min\left\{\frac{F(x)-G(x)}{2K},\frac{x}{2}\right\}.$$

(iii) If $F(x)-G(x) < 0$, we replace the integral of the lemma by an integral over the interval

$$\left[x-\min\left\{\frac{|F(x)-G(x)|}{2K},\frac{x}{2}\right\},\ x\right].$$

In that interval we have

$$F(y)-G(y) \leqslant F(x)-G(y) \leqslant F(x)-G(x)+K(x-y) \leqslant \{F(x)-G(x)\}/2,$$

and so

$$\int_1^\infty\frac{\{F(y)-G(y)\}^2}{y^{2\sigma}}(\log y)^{2k}\frac{dy}{y} \geqslant \left\{\frac{F(x)-G(x)}{2}\right\}^2\frac{1}{x^{2\sigma+1}}\left(\log\frac{x}{2}\right)^{2k}\min\left\{\frac{|F(x)-G(x)|}{2K},\frac{x}{2}\right\}.$$

In either case we get the inequality of the Lemma.

We apply Lemma 5.1 with $F(x) = A(x)$, $G(x) = ax$, and $K = 2$. Combining the result obtained with (5.8) and then letting σ approach 1, we get

$$\frac{1}{27}\left\{\frac{A(x)-ax}{x}\right\}^2\min\left\{\frac{|A(x)-ax|}{2x},1\right\} \leqslant 2\pi C_\varepsilon\left\{\frac{(1+6\varepsilon)(2k)^2}{2e^2\log 2k}\right\}^{2k}\left(\log\frac{x}{2}\right)^{-2k}$$

for any positive integer k. By holding k fixed and letting x go to infinity we see that $\{A(x)-ax\}/x \to 0$. Thus for large x we have

$$(5.9)\qquad \left|\frac{A(x)-ax}{x}\right|^3 \leqslant 108\pi C_\varepsilon\left\{\frac{(1+6\varepsilon)(2k)^2}{2e^2\log 2k}\right\}^{2k}\left(\log\frac{x}{2}\right)^{-2k}.$$

Following Malliavin, we choose k as a function of x in such a way as to make the right-hand side of (5.9) as small as possible. Specifically, we choose

$$k = \left[\frac{1}{2}\left(\log\frac{x}{2}\log\log x\right)^{1/2}\right]$$

and obtain

$$\left|\frac{A(x)-ax}{x}\right|^3 \leqslant 108\pi C_\varepsilon\left(\frac{1+6\varepsilon}{e^2}\right)^{2k} \leqslant D_\varepsilon^3\exp\{-2(1-3\varepsilon)(\log x\log\log x)^{1/2}\}$$

or

$$|A(x)-ax| \leqslant D_\varepsilon x\exp\{-(\tfrac{2}{3}-2\varepsilon)(\log x\log\log x)^{1/2}\},$$

where D_ε is a constant depending on ε. Since ε is arbitrary, this gives (1.3).

In Nyman's use of the above method he did not derive an explicit dependence of his estimates on k. This approach would have saved us a lot of work, but would merely enable us to get the weaker estimate

$$\left|\frac{A(x)-ax}{x}\right|^3 = O\left(\left\{\log\frac{x}{2}\right\}^{-2k}\right)$$

for any positive integer k, which of course also follows from Nyman's result (3.2).

§ 6. Method C. Method C also begins with formula (5.1), but instead of using the Plancherel theorem one uses an inversion formula. Although it would be possible to use the straightforward inversion formula

$$\frac{A(x+0)+A(x-0)}{2}-ax = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{2-iT}^{2+iT} x^s g(s)\,ds,$$

it is more convenient to use the smoothed inversion formula

$$\int_0^x \frac{A(u)-au}{u}\,du = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{x^s g(s)}{s}\,ds, \tag{6.1}$$

in which the integral on the right is absolutely convergent. (Note that one factor $1/s$ has already been absorbed in the definition of $g(s)$ in (5.1).)

Given a number ε with $0<\varepsilon<1/10$, we choose $\theta = 1-\varepsilon$. In view of Lemma 4.3 we may move the path of integration on the right-hand side of (6.1) from the line $\sigma = 2$ to the contour

$$\sigma = 1-\theta\frac{\log\log\max(|t|,100)}{\log\max(|t|,100)}.$$

By Lemma 4.3 we have $|g(s)| = O(|t|^{-1+\varepsilon/2})$ on the part of the path of integration with $|t|\geqslant 100$. Thus

$$\begin{aligned}\int_0^x \frac{A(u)}{u}\,du-ax &= O\left(x\int_{100}^{\infty}\exp\left\{-(1-\varepsilon)\log x\frac{\log\log t}{\log t}\right\}t^{-2+\varepsilon/2}dt\right)+O(x^{3/4})\\ &= O\left(x\int_{100}^{\infty}\exp\left\{-(1-\varepsilon)\left(\log x\frac{\log\log t}{\log t}+\log t\right)\right\}t^{-1-\varepsilon/2}dt\right)+O(x^{3/4}).\end{aligned} \tag{6.2}$$

Now if $x\geqslant\exp 10^4$, say, the minimum of

$$\log x\frac{\log\log t}{\log t}+\log t$$

on $[100,+\infty)$ is taken when t has a value $t_0 = t_0(x)$ such that

$$\frac{(\log t_0)^2}{\log\log t_0-1} = \log x.$$

Since

$$2\log\log t_0-\log\log\log t_0+O(1) = \log\log x$$

and accordingly

$$\log\log\log t_0+O(1) = \log\log\log x,$$

we have by addition

$$2\log\log t_0 = \log\log x+\log\log\log x+O(1).$$

Thus

$$\begin{aligned}\log t_0 &= \{\log x\}^{1/2}\{\log\log t_0-1\}^{1/2}\\ &= \{\log x\}^{1/2}\{\tfrac12\log\log x+\tfrac12\log\log\log x+O(1)\}^{1/2}.\end{aligned}$$

Hence for large x we have

$$\inf_{t\geqslant 100}\left(\log x\frac{\log\log t}{\log t}+\log t\right) = \log x\frac{\log\log t_0}{\log t_0}+\log t_0 \geqslant (2\log x\log\log x)^{1/2}.$$

Using this in (6.2) we obtain

$$\int_0^x\frac{A(u)}{u}\,du = ax+O\left(x\exp\{-(1-\varepsilon)(2\log x\log\log x)^{1/2}\}\right). \tag{6.3}$$

To derive (1.4) we put

$$\delta(x) = \exp\{-(1-\varepsilon)(2^{-1}\log x\log\log x)^{1/2}\}$$

and note that (6.3) gives

$$A(x)\log\{1+\delta(x)\} \leqslant \int_0^{x\{1+\delta(x)\}} A(u)u^{-1}du-\int_0^x A(u)u^{-1}du = ax\delta(x)+O\left(x\delta(x)^2\right)$$

and

$$\begin{aligned}A(x)\log\{1-\delta(x)\}^{-1} &\geqslant \int_0^x A(u)u^{-1}du-\int_0^{x\{1-\delta(x)\}} A(u)u^{-1}du\\ &= ax\delta(x)+O\left(x\delta(x)^2\right).\end{aligned}$$

Combining these two inequalities gives

$$A(x) = ax+O\left(x\delta(x)\right),$$

which is (1.4).

§ 7. Corresponding results for the sum of divisors function. If $\sigma(n)$ stands for the sum of the divisors of the positive integer n, let a_m^* denote the number of positive integers n such that $\sigma(n) = m$ and let

$$A^*(x) = \sum_{m\leqslant x} a_m^* = \sum_{\sigma(n)\leqslant x} 1.$$

Erdös [5] has remarked that the method which he used to prove the existence of the limit of $A(x)/x$ would also work for $A^*(x)/x$, since $\sigma(n)/n$ is known to have a distribution function. In a forthcoming paper in the Journal of Number Theory, Dressler gives an elementary proof that

$$\lim_{x\to\infty} \frac{A^*(x)}{x} = \prod_p \left\{\left(1-\frac{1}{p}\right)\left(1+\frac{1}{p+1}+\frac{1}{p^2+p+1}+\frac{1}{p^3+p^2+p+1}+\ldots\right)\right\} = \alpha^*.$$

However there is necessarily a considerable difference in detail between this elementary proof and the corresponding proof for $A(x)$ given by Dressler in [3].

All the analytical methods discussed in this paper, including the use of the Wiener–Ikehara Theorem, would work without substantial change if φ were replaced throughout by σ. In place of (2.1) our generating function would be

$$\sum_{m=1}^{\infty} a_m^* m^{-s} = \prod_p \{1+(p+1)^{-s}+(p^2+p+1)^{-s}+(p^3+p^2+p+1)^{-s}+\ldots\}.$$

Accordingly, in order to apply Methods B and C, we would write

$$\sum_{m=1}^{\infty} a_m^* m^{-s} = \sum_{l=1}^{\infty} b_l^* l^{-s}\cdot \sum_{k=1}^{\infty} d_k^* k^{-s},$$

where $\sum d_k^* k^{-s}$ has abscissa of absolute convergence not exceeding 1/2 and where the Nyman–Malliavin–Diamond Theorem is applicable to

$$\sum_{l=1}^{\infty} b_l^* l^{-s} = \prod_p \{1-(p+1)^{-s}\}^{-1} = \prod_p \{1+(p+1)^{-s}+(p+1)^{-2s}+\ldots\}.$$

(In fact

$$\sum_{k=1}^{\infty} d_k^* k^{-s} = \prod_p \Big\{1+\sum_{j=2}^{\infty}\big(\{p^j+\ldots+p+1\}^{-s}-\{(p+1)(p^{j-1}+\ldots+p+1)\}^{-s}\big)\Big\}$$

for $\mathrm{Re}\, s > \frac{1}{2}$.) Thus Method A could be applied to give the analogue of (1.2) for $A^*(x)-\alpha^* x$.

In order to apply Methods B and C to $A^*(x)$ we would replace (2.2) and (2.3) by

$$\sum_{m=1}^{\infty} a_m^* m^{-s} = \zeta(s) f^*(s),$$

where

$$f^*(s) = \prod_p \Big\{1+\sum_{j=1}^{\infty}\big(\{p^j+\ldots+p+1\}^{-s}-\{p^j+\ldots+p\}^{-s}\big)\Big\}.$$

In place of (2.4) and (4.1) we would have (for $\mathrm{Re}\, s > 0$)

$$\sum_{j=1}^{\infty} |\{p^j+\ldots+p+1\}^{-s}-\{p^j+\ldots+p\}^{-s}| \leqslant |s| \sum_{j=1}^{\infty} (p^j+\ldots+p)^{-\mathrm{Re}\, s-1} \leqslant |s|(p^{\mathrm{Re}\, s+1}-1)^{-1}$$

and

$$\sum_{j=1}^{\infty} |\{p^j+\ldots+p+1\}^{-s}-\{p^j+\ldots+p\}^{-s}| \leqslant 2 \sum_{j=1}^{\infty} (p^j+\ldots+p)^{-\mathrm{Re}\, s} \leqslant 2(p^{\mathrm{Re}\, s}-1)^{-1}.$$

As a result, estimates similar to those in Lemmas 4.1 and 4.3 would be readily obtainable for $f^*(s)$. Thus Methods B and C would give the analogues of (1.3) and (1.4) for $A^*(x)-\alpha^* x$. In particular

$$A^*(x) = \alpha^* x + O\big(x \exp\{-(1-\varepsilon)(\tfrac{1}{2}\log x \log\log x)^{1/2}\}\big)$$

for any fixed positive number ε.

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