CHAPTER IV

# NUMBER OF DIVISORS AND THEIR SUM

**§ 1. Number of divisors.** The number of the divisors of a given natural number $n$ is denoted by $d(n)$. In order to establish the table of the function $d(n)$ one may use the following method which is a modification of the sieve of Eratosthenes. In order to find the values $d(n)$ for $n \leqslant a$ we write down the natural numbers $1, 2, \ldots, n$ and we mark all of them. Next we mark those which are divisible by 2, then those which are divisible by 3, and so on. Finally we mark only the number $a$. The number of the divisors of a number $n$ is equal to the number of the marks on it (cf. Harris [1]). In particular, for $a = 20$ we have

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20.

Hence we find $d(1) = 1$, $d(2) = 2$, $d(3) = 2$, $d(4) = 3$, $d(5) = 2$, $d(6) = 4$, $d(7) = 2$, $d(8) = 4$, $d(9) = 3$, $d(10) = 4$, $d(11) = 2$, $d(12) = 6$, $d(13) = 2$, $d(14) = 4$, $d(15) = 4$, $d(16) = 5$, $d(17) = 2$, $d(18) = 6$, $d(19) = 2$, $d(20) = 6$.

Let $n$ be a natural number greater than 1 and let

$$n = q_1^{a_1} q_2^{a_2} \ldots q_k^{a_k} \tag{1}$$

be the factorization of $n$ into prime numbers. Suppose that $d$ is a divisor of $n$. Since every divisor of $d$ is a divisor of $n$, then in the factorization of the number $d$ into primes only the primes appearing in (1) can possibly appear and, moreover, the exponents of them cannot be greater than those on the corresponding primes in (1). Accordingly, every divisor $d$ of the number $n$ can be written in the form

$$d = q_1^{\lambda_1} q_2^{\lambda_2} \ldots q_k^{\lambda_k}, \tag{2}$$

where $\lambda_i$ $(i = 1, 2, \ldots, k)$ are integers satisfying the inequalities

$$0 \leqslant \lambda_i \leqslant a_i \quad \text{for} \quad i = 1, 2, \ldots, k. \tag{3}$$



On the other hand, it is obvious that every number that can be written in form (2), numbers $\lambda_i$ satisfying inequalities (3), is a natural divisor of the number $n$. This is because, in view of (3), $n/d = q_1^{a_1-\lambda_1} q_2^{a_2-\lambda_2} \ldots q_k^{a_k-\lambda_k}$ is an integer.

Finally, it is obvious that different systems of integers

$$\lambda_1, \lambda_2, \ldots, \lambda_k \tag{4}$$

satisfying (3) define different numbers (2). We have thus proved the following

THEOREM 1. *If $m$ is a natural number whose factorization into prime numbers is written as* (1), *then taking for the numbers in* (4) *all the different systems of $k$ integers which satisfy inequalities* (3) *we find that all the divisors of the number $n$ are given by* (2). *Moreover, to each system corresponds precisely one divisor.*

Consequently, the number of divisors of a natural number $n$ whose factorization into prime numbers is written as (1) is equal to the number of all the systems of integers (4) satisfying inequalities (3). It is a matter of simple computation to calculate the number of the systems. In fact, in order that an integer $\lambda_i$ should satisfy inequalities (3) it is necessary and sufficient that $\lambda_i$ should belong to the sequence

$$0, 1, 2, \ldots, a_i.$$

Thus for a given $i = 1, 2, \ldots, k$ the number $\lambda_i$ can take $a_i + 1$ different values. This proves

THEOREM 2. *The number $d(n)$ of the divisors of a natural number $n$ whose factorization into primes is written as* (1) *is given by*

$$d(n) = (a_1+1)(a_2+1)\ldots(a_k+1). \tag{5}$$

Let us calculate the number $d(60)$ for instance. We have $60 = 2^2 \cdot 3 \cdot 5$. Therefore, in view of (5), $d(60) = (2+1)(1+1)(1+1) = 12$. Similarly, since $100 = 2^2 \cdot 5^2$, we see that $d(100) = (2+1)(2+1) = 9$.

It follows from (5) that for every natural number $s > 1$ there exist infinitely many natural numbers which have precisely $s$ divisors. In fact, if $n = p^{s-1}$, where $p$ is a prime, then $d(n) = d(p^{s-1}) = s$.

Clearly, the equality $d(n) = 1$ implies $n = 1$. Formula (5) shows that $d(n) = 2$ whenever $k = 1$ and $a_1 = 1$, that is, whenever $n$ is a prime. Accordingly, the solutions of the equation $d(n) = 2$ are prime numbers. Consequently, for composite numbers $n$ we have $d(n) \geqslant 3$.

If follows from (5) that $d(n)$ is an odd number if and only if all the $a_i$'s $(i = 1, 2, \ldots, k)$ are even, that is, if and only if $n$ is the square of a natural number.

EXERCISES. 1. Prove that for natural numbers $n$ we have $d(n) \leqslant 2\sqrt{n}$.

The proof follows from the fact that of two complementary divisors of a natural number $n$ one is always not greater than $\sqrt{n}$.

2. Find all the natural numbers which have precisely 10 divisors.

Solution. If $d(n) = 10$, then, in view of (5), we have $(a_1+1)(a_2+1)\dots(a_k+1) = 10$. We may, of course, assume that $a_1 \leqslant a_2 \leqslant \dots \leqslant a_k$. Since there are two ways of presenting 10 as the product of natural numbers $> 1$ written in the order of their magnitude, namely $10 = 2\cdot 5$ and $10 = 10$, then either $k = 2$, $a_1 = 1$, $a_2 = 4$, or $k = 1$, $a = 9$. It follows that the natural numbers which have precisely 10 divisors are either the numbers $p\cdot q^4$, where $p, q \neq p$ are arbitrary primes, or the numbers $p^9$, where $p$ is an arbitrary prime.

3. Find the least natural number $n$ for which $d(n) = 10$.

Solution. In view of exercise 2 and the fact that of the numbers $2^9$, $2\cdot 3^4$, and $3\cdot 2^4$ the latter is least, it follows that the least natural number $n$ for which $d(n) = 10$ is the number $n = 3\cdot 2^4 = 48$.

Remark. In general it is easy to prove that for given prime numbers $p, q$ with $q > p$ the least natural number that has precisely $pq$ divisors is the number $2^{q-1}\cdot 3^{p-1}$.

4. Prove that, if $n$ is a natural number $> 1$, then in the infinite sequence

$$n,\ d(n),\ d(d(n)),\ ddd(n),\ \dots$$

all the terms starting from a certain place onwards are equal to 2. Prove that the place can be arbitrarily given.

The proof follows immediately from the remark that if $n$ is a natural number greater than 2, then $d(n) < n$, and from the equality $d(2) = 2$. To prove the second part of the exercise we use the equality $d(2^{n-1}) = n$.

5. Prove that for any natural number $m$ the set of the natural numbers $n$ such that the number of the divisors of $n$ is divisible by $m$ contains an infinite arithmetical progression.

Proof. We note that the numbers $2^m t + 2^{m-1}$ $(t = 0, 1, 2, \dots)$ form an infinite arithmetical progression and belong to the set defined above for the number $m$. In fact, the exponent of the number 2 in the factorization of the number $n = 2^m t + 2^{m-1}$ is $m-1$. Hence, by (1), we see that $m \mid d(n)$.

Remark. As an immediate consequence of the theorem proved above we obtain that for any natural number $m$ the set of natural numbers $n$ such that $m \mid d(n)$ has positive lower density. This means that there exists a positive number $a$ with the property that the number $S_m(x)$ of the natural numbers $n \leqslant x$ for which $m \mid d(n)$ is greater than $ax$ for all $x$ large enough. E. Cohen [1] proved that for any natural number $m$ the limit $\lim\limits_{x\to\infty} \dfrac{S_m(x)}{x}$ exists and is positive.

In the year 1940 the tables of the function $d(n)$ for $n \leqslant 10000$ were published, cf. Glaisher [1]. As we check in the tables, the equalities $d(n) = d(n+1) = d(n+2) = d(n+3) = 8$ hold for $n = 3655$, 4503, 5943, 6853, 7256, 8393, 9367.

As found by J. Mycielski, for $n = 40311$ we have

$$d(n) = d(n+1) = d(n+2) = d(n+3) = d(n+4).$$



The proof follows immediately from the factorizations into primes of the numbers $40311 = 3^3\cdot 1493$, $40312 = 2^3\cdot 5039$, $40313 = 7\cdot 13\cdot 443$, $40314 = 2\cdot 3\cdot 6719$, $40315 = 5\cdot 11\cdot 733$. A similar situation occurs for $n = 99655$.

A conjecture has been formulated that there are infinitely many natural numbers $n$ for which $d(n) = d(n+1)$ (cf. Erdös and Mirsky [1]). We have $d(2) = d(3)$, $d(14) = d(15)$, $d(33) = d(34) = d(35) = 4$, $d(242) = d(243) = d(244) = d(245) = 6$.

We do not know whether there exists an infinite sequence of increasing natural numbers $n_k$ $(k = 1, 2, \dots)$ such that $\lim\limits_{k\to\infty} d(n_k+1)/d(n_k) = 1$. Neither do we know whether the numbers $d(n+1)/d(n)$ are dense in the set of the positive real numbers. However, P. Erdös has proved that they are dense in a non-trivial interval. (Cf. Erdös [18], footnote (¹).)

For $n \leqslant 10000$ we have $d(n) \leqslant 64$ and the maximum value $d(n) = 64$ is taken only for the numbers $n = 7560$ and 9240.

A. Schinzel [2] has proved that for all natural numbers $h$ and $m$ there exists a natural number $n > h$ such that

$$d(n)/d(n\pm i) > m \quad \text{for} \quad i = 1, 2, \dots, h.$$

**§ 2. Sums $d(1)+d(2)+\dots+d(n)$.** For real numbers $x \geqslant 1$ we denote by $T(x)$ the sum

$$T(x) = \sum_{k=1}^{[x]} d(k) = d(1)+d(2)+\dots+d([x]). \tag{6}$$

In order to find this sum we prove first that for a given natural number $k$ the number $d(k)$ is the number of the solutions of the equation

$$mn = k \tag{7}$$

in natural numbers $m$ and $n$.

In fact, if a natural number $n$ is a divisor of a number $k$, then $m = k/n$ is a natural number and the pair $m, n$ is a solution of equation (7) in natural numbers. Conversely, if a pair of natural numbers $m, n$ satisfies equation (7), then $n$ is a divisor of the number $k$. Accordingly, to each natural divisor of the number $k$ corresponds precisely one solution of equation (7) and *vice versa*. It follows that the number $d(k)$ is equal to the number of the solutions of equation (7) in natural numbers, and this is what was to be proved.

Consequently, in view of (6), $T(x)$ can be regarded as the number of solutions of the inequality $mn \leqslant [x]$ in natural numbers $m, n$, this being clearly equivalent to the inequality

$$mn \leqslant x. \tag{8}$$

All the solutions of the last inequality in natural numbers $m, n$ we divide into classes simply by saying that a solution $m, n$ belongs to the $n$th class. If $k_n$ denotes the number of the solutions belonging to the $n$th class, then, clearly,

(9) $$T(x) = k_1+k_2+k_3+\ldots$$

We now calculate the number of the solutions of the $n$th class. For a given $n$ the number $m$ can take only the natural values satisfying inequality (8), i.e. the inequality

$$m \leqslant \frac{x}{n}.$$

This means that $m$ can be any of the numbers $1, 2, \ldots, \left[\frac{x}{n}\right]$, which are $\left[\frac{x}{n}\right]$ in number. Therefore $k_n = \left[\frac{x}{n}\right]$, which, by (9), gives

(10) $$T(x) = \left[\frac{x}{1}\right]+\left[\frac{x}{2}\right]+\left[\frac{x}{3}\right]+\ldots$$

The right-hand side of this equality is not an infinite series, in fact: only the first $[x]$ terms of it are different from zero. Thus formula (10) can be rewritten in the form

(11) $$T(x) = \sum_{k=1}^{[x]}\left[\frac{x}{k}\right].$$

The calculation of the number $T(x)$ from (11), though much more convenient than by finding the consecutive values of the function $d(k)$, is somewhat tedious for larger values of $x$. For instance, in order to find $T(100)$ by the use of (11) one has to add a hundred numbers. For this reason it seems worth-while to find a more convenient formula for $T(x)$.

In order to do this we divide the class of all the solutions of inequality (8) in natural numbers into two classes including in the first class the solutions for which $n \leqslant \sqrt{x}$ and in the second the remaining solutions, i.e. those for which $n > \sqrt{x}$. We calculate the number of the solutions in each of these two classes.

If $n$ takes natural values $\leqslant \sqrt{x}$ and if $m, n$ is a solution of inequality (8) in natural numbers, i.e. if $m$ is a natural number such that $m \leqslant x/n$, then $m, n$ belongs to the first class. Then for every natural number $n \leqslant \sqrt{x}$ the number of the solutions in the first class is $\left[\frac{x}{n}\right]$. Since $n$ takes the values $1, 2, \ldots, [\sqrt{x}]$, the number of the solutions in the first class is



$$\sum_{n=1}^{[\sqrt{x}]}\left[\frac{x}{n}\right].$$

We now calculate the number of the solutions belonging to the second class. That is we find how many of the pairs of natural numbers $m, n$ satisfy the inequalities

$$mn \leqslant x \quad \text{and} \quad n > \sqrt{x},$$

i.e. the inequalities

(12) $$\sqrt{x} < n \leqslant \frac{x}{m}.$$

If we had $m > \sqrt{x}$, then $x/m < \sqrt{x}$ and inequalities (12) would not be satisfied for any $n$. Accordingly, let $m$ denote a fixed natural number $\leqslant \sqrt{x}$. In order to find the number of possible values of $n$ for which inequalities (12) are satisfied, it is sufficient to subtract from the number of all the natural numbers $n \leqslant \frac{x}{m}$ (i.e. from the number $\left[\frac{x}{n}\right]$) the number of the natural numbers $n$ which do not satisfy the inequality $\sqrt{x} < n$, i.e. the number of the $n$'s which satisfy the inequality $n \leqslant \sqrt{x}$ (clearly, they are $[\sqrt{x}]$ in number). Hence $\left[\frac{x}{m}\right]-[\sqrt{x}]$ is the number of all the pairs $m, n$ which for $m \leqslant \sqrt{x}$ satisfy inequalities (12). But since $m$ can take only the values $1, 2, \ldots, [\sqrt{x}]$, the number of the pairs of natural numbers $m, n$ satisfying inequalities (12), i.e. the number of the solutions belonging to the second class, is

$$\sum_{m=1}^{[\sqrt{x}]}\left(\left[\frac{x}{m}\right]-[\sqrt{x}]\right) = \sum_{m=1}^{[\sqrt{x}]}\left[\frac{x}{m}\right]-\sum_{m=1}^{[\sqrt{x}]}[\sqrt{x}].$$

The second of the sums on the right-hand side of the last equality is equal to the number $[\sqrt{x}]^2$ because it is the sum of $[\sqrt{x}]$ summands, each being equal to $[\sqrt{x}]$. Consequently, the number of the solutions in the second class is equal to

$$\sum_{m=1}^{[\sqrt{x}]}\left[\frac{x}{m}\right]-[\sqrt{x}]^2.$$

Thus, combining this with the number of the solutions belonging to the first class previously obtained, we get

$$\sum_{n=1}^{[\sqrt{x}]}\left[\frac{x}{n}\right]+\sum_{m=1}^{[\sqrt{x}]}\left[\frac{x}{m}\right]-[\sqrt{x}]^2$$

as the number of all the solutions of inequalities (8) in natural numbers $m, n$, i.e. the value of the function $T(x)$. We have $\sum_{m=1}^{[\sqrt{x}]}\left[\frac{x}{m}\right]=\sum_{n=1}^{[\sqrt{x}]}\left[\frac{x}{n}\right]$ because both the sums are abbreviated forms of the sum $\left[\frac{x}{1}\right]+\left[\frac{x}{2}\right]+\ldots+\left[\frac{x}{\sqrt{x}}\right]$; therefore we may write

$$T(x)=2\sum_{n=1}^{[\sqrt{x}]}\left[\frac{x}{n}\right]-[\sqrt{x}]^2. \tag{13}$$

This formula has been found by Lejeune Dirichlet. Applying it we calculate $T(100)$ as follows:

$$T(100)=2\sum_{n=1}^{10}\left[\frac{100}{n}\right]-10^2=2(100+50+33+25+20+$$

$$+16+14+12+11+10)-100=2\cdot 291-100=482.$$

Similarly, by an easy calculation, we find

$$T(200)=1098,\quad T(500)=3190,\quad T(1000)=7069.$$

Slightly longer calculations lead us to the values

$$T(5000)=43376,\quad T(10000)=93668.$$

From formula (11) an approximate formula for the average value of the function $d(n)$ is easily obtainable. If on the right-hand side of formula (11) we replace $\left[\frac{x}{k}\right]$ simply by $\frac{x}{k}$, then the error in each of the summands is less than 1, and consequently in the whole sum it is less than the number of the summands, i.e. less than $[x]\leqslant x$. Therefore, as an approximate value of $T(x)$ we have $\sum_{n=1}^{[x]}\frac{x}{n}$, the error of the approximation being less than $x$. For natural values of $x=k$ we then have

$$\frac{d(1)+d(2)+\ldots+d(k)}{k}\approx\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{k}, \tag{14}$$



the error of the approximation being less than 1. Since the right-hand side of (14) increases to infinity with $k$, the ratio of the left-hand side to the right-hand side of (14) tends to 1 when $k$ tends to infinity.

As is known from Analysis, for the approximate value of the sum $\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{k}$ we may take the number $\log k$, the error being less than for $k>1$. Consequently, $\log k$ is an approximate value of the left-hand side of (14).

As one proves in Analysis, the difference $\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{k}-\log k$ tends to a finite limit called *Euler's constant* $C=0.57721566\ldots$ (we do not know whether it is an irrational number). On the basis of this and formula (13) the expression $x\log x+(2C-1)x$ has been found, which approximates $T(x)$ with an error less than a finite multiple of $\sqrt{x}$. G. Voronoï proved that the error is not greater than a finite multiple of $\sqrt[3]{x}\log x$. Later other authors found a more precise evaluation of this error (cf. Yin Wen-Lin [2]).

**§ 3. Numbers $d(n)$ as coefficients of expansions.** The function $d(n)$ appears in Analysis as the coefficient of expansion in infinite series. For instance, consider the series (convergent for $|x|<1$)

$$\sum_{k=1}^{\infty}\frac{x^k}{1-x^k}=\frac{x}{1-x}+\frac{x^2}{1-x^2}+\frac{x^3}{1-x^3}+\ldots$$

known under the name of Lambert's series. Expanding each of its terms into the geometrical progression

$$\frac{x^k}{1-x^k}=x^k+x^{2k}+x^{3k}+\ldots$$

we obtain the double sum $\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}x^{kl}$, in which for every natural number $n$ the power $x^n$ appears as many times as there are solutions of the equation $kl=n$ in natural numbers $k$ and $l$, i.e. $d(n)$ times. Hence (for $|x|<1$) we have

$$\sum_{k=1}^{\infty}\frac{x^k}{1-x^k}=\sum_{n=1}^{\infty}d(n)x^n.$$

We see that the function $d(n)$ is the coefficient at $x^n$ in the expansion of Lambert's series in a power series.

The function $d(n)$ is also the coefficient in the expansion of the square of the $\zeta$ function. For $s>1$ we consider an infinite series

$$\zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k^s}=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\ldots$$

(one proves in Analysis that the series is convergent for $s>1$). Now we apply the so-called *Dirichlet multiplication* to the product $\zeta(s)\zeta(s)$.

Dirichlet's multiplication is as follows: given two series $a_1+a_2+\ldots$ and $b_1+b_2+\ldots$, we multiply $(a_1+a_2+\ldots)$ by $(b_1+b_2+\ldots)$ and put together those products $a_k b_l$ for which the products of indices are equal, i.e. $(a_1+a_2+\ldots)(b_1+b_2+\ldots)=a_1b_1+(a_1b_2+a_2b_1)+(a_1b_3+a_3b_1)+(a_1b_4+ +a_2b_2+a_4b_1)+(a_1b_5+a_5b_1)+(a_1b_6+a_2b_3+a_3b_2+a_6b_1)+(a_1b_7+a_7b_1)+\ldots$ As can easily be seen this multiplication applied to $\zeta(s)$ gives

(15) $$\big(\zeta(s)\big)^2=\sum_{n=1}^{\infty}\frac{d(n)}{n^s}.$$

**§ 4. Sum of divisors.** The sum of the natural divisors of a natural number $n$ is denoted by $\sigma(n)$. It follows from theorem 1 that if (1) is the factorization into primes of the number $n$, then

(16) $$\sigma(n)=\sum q_1^{\lambda_1}q_2^{\lambda_2}\ldots q_k^{\lambda_k},$$

where the summation extends all over the systems of $k$ integers (4) satisfying inequalities (3). But, as one easily sees, each summand of (16) is obtained in the expansion of the product

$$(1+q_1+q_1^2+\ldots+q_1^{a_1})(1+q_2+q_2^2+\ldots+q_2^{a_2})\ldots(1+q_k+q_k^2+\ldots+q_k^{a_k})$$

precisely once.

On the other hand, each of the terms of the expansion of this product is one of the summands of the sum of (16). Hence

THEOREM 3. *The sum $\sigma(n)$ of the natural divisors of a natural number $n$ whose factorization into primes is $n=q_1^{a_1}q_2^{a_2}\ldots q_k^{a_k}$ is equal to*

(17) $$\sigma(n)=\frac{q_1^{a_1+1}-1}{q_1-1}\cdot\frac{q_2^{a_2+1}-1}{q_2-1}\cdots\frac{q_k^{a_k+1}-1}{q_k-1}.$$

In particular, $\sigma(100)=\dfrac{2^3-1}{2-1}\cdot\dfrac{5^3-1}{5-1}=7\cdot 31=217$.

It follows immediately from theorem 3 that for natural numbers $a, b$, with $(a,b)=1$ we have $\sigma(ab)=\sigma(a)\sigma(b)$. It is easy to see that, if $(a,b)>1$, then $\sigma(ab)<\sigma(a)\sigma(b)$. Using theorem 3, we can easily calculate $\sigma(1)=1$, $\sigma(2)=3$, $\sigma(3)=4$, $\sigma(4)=7$, $\sigma(5)=6$, $\sigma(6)=12$, $\sigma(7)=8$, $\sigma(8)=15$, $\sigma(9)=13$, $\sigma(10)=18$.



For $n>1$ we have $\sigma(n)>n$. It follows that $\sigma(n)>5$ for $n>4$. As one sees from the table presented above, $\sigma(n)$ takes for $n\leqslant 4$ the values $1, 3, 4$ and $7$. Therefore there is no natural number $n$ for which $\sigma(n)=5$.

THEOREM 4. *There exist infinitely many natural numbers which are not the values of the function $\sigma(x)$ for natural $x$.*

Proof. Let $n$ be a natural number $>9$ and let $k$ be a natural number such that

(18) $$\frac{n}{3}-1<k\leqslant\frac{n}{2}.$$

The number of the $k$'s for which (18) holds is clearly greater than $n/2-n/3=n/6$. In virtue of (18) we have

(19) $$2k\leqslant n \quad\text{and}\quad 3k+3>n,$$

and, since $n>9$, we have $3k>6$, which in consequence gives $k\geqslant 3$. Hence one sees that the number $2k$ has at least four different divisors: $1, 2, k, 2k$. Therefore $\sigma(2k)\geqslant 1+2+k+2k$, which, by (19), gives $\sigma(2k)>n$. Since the number of different natural numbers $k$ for which (18) and (19) and consequently the inequality $\sigma(2k)>n$ hold is greater than $n/6$, then among the numbers $\sigma(1), \sigma(2), \ldots, \sigma(n)$ there are more than $n/6$ numbers greater than $n$. Hence in the sequence $1, 2, \ldots, n$ there are more than $n/6$ natural numbers which cannot be the values of the function $\sigma(x)$ for $x\leqslant n$. These numbers cannot be the values of the function $\sigma(x)$ for $x>n$ either, because the numbers are $\leqslant n$ and $\sigma(x)\geqslant 1+x>n$ for $x>n$. Therefore for every natural number $n>9$ there are more than $n/6$ natural numbers in the sequence $1, 2, \ldots, n$ which cannot be the values of the function $\sigma(x)$ for natural values of $x$. This proves theorem 4.

Thus, there exist infinitely many natural numbers $m$ for which the equation $\sigma(x)=m$ is insolvable in natural numbers $x$. It can be proved that all numbers $m=3^k$ $(k>1)$ have this property (cf. Sierpiński [26]). There are 59 such numbers $m\leqslant 100$. These are 2, 5, 9, 10, 11, 17, 19, 21, 22, 23, 25, 26, 27, 29, 33, 34, 35, 37, 41, 43, 45, 46, 47, 49, 50, 51, 52, 53, 55, 58, 59, 61, 64, 65, 66, 67, 69, 70, 71, 73, 75, 76, 77, 79, 81, 82, 83, 85, 86, 87, 88, 89, 92, 94, 95, 97, 99, 100. Among the remaining natural numbers $m\leqslant 100$ there are 25 for which the equation $\sigma(x)=m$ has precisely one solution in natural numbers. These are $m=1$, 3, 4, 6, 7, 8, 13, 14, 15, 20, 28, 30, 36, 38, 39, 40, 44, 57, 62, 63, 68, 74, 78, 91, 93. This suggests the question whether there exist infinitely many natural numbers $m$ for which the equation $\sigma(x)=m$ has precisely one solution. The answer in the affirmative follows from the more general theorem given below, which, according to P. Erdös [18], p. 12, states that if for any given $k$ there exists a number $m$ such that the equation $\sigma(x)=m$ has precisely $k$

solutions, then there exist infinitely many such numbers $m$. It is much easier to prove, however, that there are infinitely many natural numbers $m$ for which the equation $\sigma(x) = m$ has more than one solution. To this class belong, for instance, the numbers $m = 3(5^k-1)$, where $k = 1, 2, \ldots$ The reason is that, in virtue of $\sigma(6) = \sigma(11) = 12$ and $(6\cdot 5^{k-1}) = (11\cdot 5^{k-1}) = 1$, we have $\sigma(6\cdot 5^{k-1}) = \sigma(11\cdot 5^{k-1}) = 3(5^k-1)$.

It is easy to prove that there exist infinitely many natural numbers $m$ for which the equation $\sigma(x) = m$ has more than two solutions. This property attaches, for instance, to the numbers $2(13^k-1)$, where $k = 1, 2, \ldots$ In fact, we have $\sigma(14\cdot 13^{k-1}) = \sigma(15\cdot 13^{k-1}) = \sigma(23\cdot 13^{k-1}) = 2(13^k-1)$.

It is not known whether for every natural number $k$ there exists a natural number $m_k$ for which the equation $\sigma(x) = m_k$ has precisely $k$ solutions in natural numbers $x$. This follows from the conjecture H (cf. Schinzel [15]). It can be proved that if $m_k$ denotes the least of the numbers for which $\sigma(x) = m_k$ has precisely $k$ solutions, then $m_1 = 1$, $m_2 = 12$, $m_3 = 24$, $m_4 = 96$, $m_5 = 72$, $m_6 = 168$, $m_7 = 240$, $m_8 = 432$, $m_9 = 360$, $m_{10} = 504$, $m_{11} = 576$, $m_{12} = 1512$, $m_{13} = 1080$, $m_{14} = 1008$, $m_{15} = 720$, $m_{16} = 2304$, $m_{17} = 3600$, $m_{18} = 5376$, $m_{19} = 2160$, $m_{20} = 1440$.

The equation $\sigma(x) = m$ has precisely three solutions in natural numbers for the following six natural numbers $m \leqslant 100$, namely 24, 42, 48, 60, 84, 90.

The equation $\sigma(x) = m$ has precisely four solutions only for one natural number $m \leqslant 100$, namely for $m = 96$. It has precisely five solutions also for one natural number $m \leqslant 100$, namely for 72.

There is no natural number $m \leqslant 100$ for which the equation $\sigma(x) = m$ has more than five solutions in natural numbers; however, H. J. Kanold [2] has proved that for every natural number $k$ there exists a natural number $m$ such that the equation $\sigma(x) = m$ has $\geqslant k$ solutions in natural numbers $x$. The equation $\sigma(n) = \sigma(n+1)$ has only 9 solutions for $n < 10000$. These are $n = 14$, 206, 957, 1334, 1364, 1634, 2685, 2974, 4364 (cf. Mąkowski [3]). We do not know whether there exist infinitely many natural numbers $n$ for which $\sigma(n) = \sigma(n+1)$.

A. Mąkowski has asked whether for every integer $k$ there exists a natural number $n$ such that $\sigma(n+1) - \sigma(n) = k$, and, more generally, whether for every natural number $m$ and every integer $k$ there exists a natural number $n$ such that $\sigma(n+m) - \sigma(n) = k$.

If $n$ and $n+2$ are twin prime numbers then $\sigma(n+2) = \sigma(n)+2$. This equation, however, is also satisfied by the number $n = 434$, though the numbers 434 and 436 are not prime. A similar situation occurs for $n = 8575$ and $n = 8825$.

E. Catalan has conjectured (cf. Dickson [3]) that if $f(n) = \sigma(n) - n$, then for natural numbers $n > 1$ the infinite sequence of consecutive



iterations of the functions $f$

$$n,\ f(n),\ ff(n),\ fff(n),\ \ldots$$

either is periodic or terminates at the number 1. This is true for all $n \leqslant 275$ (cf. Poulet [4]). According to L. Alaoglu and P. Erdös [2] not only is the proof of this conjecture unknown, but also it is difficult to verify the conjecture for particular natural values of $n$.

It is easy to see that for $n = 12496 = 2^4\cdot 11\cdot 71$ all the numbers $n$, $f(n)$, $ff(n)$, $fff(n)$, $ffff(n)$ are different and that $fffff(n) = n$, so the sequence is periodic. For $n = 12$, however, we have $f(12) = 16$, $f(16) = 15$, $f(15) = 9$, $f(9) = 4$, $f(4) = 3$, $f(3) = 1$, which shows that the sequence terminates at the number 1, which, of course, is also the case for a prime $n$, since then $f(n) = 1$. For $n = 100$ we have $f(100) = 117$, $f(117) = 65$, $f(65) = 19$, $f(19) = 1$. For $n = 6$, however, we have $f(n) = n$ and the sequence is trivially periodic, the period consisting of one term. For $n = 95$ we have $f(95) = 25$, $f(25) = 6$, $f(6) = 6$ and we see that the sequence is periodic from the fourth term onwards the period consisting of one term only. For $n = 220$ we have $f(220) = 284$, $f(284) = 220 = n$, and so the sequence is periodic from the very beginning onwards, the period consisting of two terms. In an unpublished typescript P. Poulet [4] has announced that for $n = 936$ the sequence 936, 1794, 2238, 2250, ..., 74, 40, 50, 43, 1 is obtained, consisting of 189 terms, the greatest of them being 33289162091526.

This suggests the question whether there exist arbitrarily long sequences $n, f(n), ff(n), \ldots$ which terminate at 1 and whether there exist infinitely many natural numbers $n$ for which the above sequence is periodic. The answer to this question is positive provided the conjecture that every even number greater than 6 is the sum of two different prime numbers is true.

In fact, suppose that this conjecture is true and let $2k-1$ denote an arbitrary odd number $> 7$. Then $2k-2 > 6$ and, according to the conjecture, there exist two different prime numbers $p$ and $q$, both odd of course, such that $2k-2 = p+q$. Hence $f(pq) = \sigma(pq) - pq = 1+p+q = 2k-1$. Since $p, q$ are two different odd primes, we have, say, $p > q$, and so $p \geqslant q+2$ and $q \geqslant 3$. Hence $pq \geqslant 3p = 2p+p \geqslant 2p+q+2 > p+q+1 = 2k-1$. Consequently $pq > 2k-1$. Therefore for every odd number $n > 7$ there exists an odd number $m > n$ such that $f(m) = n$. Let $m = g(n)$. Then the infinite increasing sequence $g(n), gg(n), \ldots$ is obtained. If for a natural number $k$ we put $n = g^k(11)$ we get the sequence $n = g^k(11)$, $f(n) = g^{k-1}(11)$, ..., $f^k(n) = 11$, $f(11) = 1$. We have thus formed a decreasing sequence $n, f(n), ff(n), \ldots$ of $k+2$ terms, the last term being equal to 1.

If for a natural number $k$ we put $n = g^k(25)$, we obtain the periodic sequence $n = g^k(25)$, $f(n) = g^{k-1}(25), \dots$, $f^k(n) = 25$, $f(25) = 6$, $f(6) = 6, 6, \dots$ with $k+1$ decreasing terms preceding the period.

There is another question which one may ask in this connection. This is whether there exist infinitely many different natural numbers for which the sequence $n, f(n), ff(n), \dots$ is periodic and has no terms preceding the period.

We have just proved that the conjecture that every even natural number $>6$ is a sum of two different prime numbers implies that every odd natural number $>7$ is a term of the sequence $f(n)$ $(n = 1, 2, \dots)$. Moreover, we have $f(3) = 1$, $f(4) = 3$, $f(8) = 7$. However, it is easy to prove that the number 5 does not occur in the sequence $f(n)$ $(n = 1, 2, 3, \dots)$. In fact, if for a natural number $n$ the equality $f(n) = \sigma(n) - n = 5$ could hold, then 5 would of course be a composite number (because $\sigma(1) - 1 = 0$ and, for a prime $n$, $\sigma(n) - n = 1$). So $n = ab$, where $1 < a \leqslant b < n$. Then, since 1, $b$ and $n$ would be different divisors of the number $n$, we would have $\sigma(n) > 1 + b + n$, whence $5 = \sigma(n) - n \geqslant 1 + b > b$, and so $b < 5$. Therefore we would have $n = ab$ with $1 < a \leqslant b \leqslant 4$. But, as can easily be verified, this is impossible, since there are no natural numbers $a, b$ having the above properties for which the equation $\sigma(n) = n + 5$ is satisfied.

Without the conjecture that every even natural number $>6$ is a sum of two different prime numbers we are unable to prove that every odd number different from 5 is for a suitably chosen natural number $n$ a term of the sequence $\sigma(n) - n$ $(n = 1, 2, \dots)$. P. Erdös has asked whether there exist infinitely many natural numbers which do not belong to this sequence.

It can be proved that the relation $m \mid \sigma(mn-1)$ holds for all natural numbers $n$ if and only if $m = 3, 4, 6, 8, 12$ or $24$ (cf. Gupta [1]).

We do not know whether there exist infinitely many natural numbers $n$ for which $\sigma(n)$ is the square of a natural number. The positive answer to this question can easily be derived from conjecture H (Chapter III, § 8). In fact, let $f(x) = 2x^2 - 1$, the polynomial $f(x)$ is irreducible and, since $f(0) = -1$, it satisfies condition S formulated in Chapter III. Therefore, according to conjecture H, there exist infinitely many natural numbers $x$ for which $p = 2x^2 - 1$ is a prime number $>7$. For those $x$'s we have $\sigma(7p) = 8(p+1) = (4x)^2$. This proves that $\sigma(7p)$ is the square of a natural number. We know some solutions of the equation $\sigma(x^3) = y^2$ in natural numbers, e.g. $x = 7$, $y = 20$. We also know some of the solutions of the equation $\sigma(x^2) = y^3$ in natural numbers, e.g. $x = 2\cdot 3\cdot 11\cdot 653$, $y = 7\cdot 13\cdot 19$.



EXERCISES. **1.** Prove that the equality $\sigma(n) = n+1$ holds if and only if $n$ is a prime.

*Proof.* If $p$ is a prime number, then it has precisely two divisors, namely $p$ and 1. Therefore $\sigma(p) = p+1$. On the other hand, if $n$ is a composite number, i. e. if $n = ab$, where $a$ and $b$ are natural numbers $> 1$, then $1 < a < ab = n$ and consequently $n$ has at least three different natural divisors: 1, $a$ and $n$. Hence $\sigma(n) \geqslant 1 + a + n > n + 1$. Finally, if $n = 1$, then $\sigma(n) = 1 < n+1$.

**2.** Prove that for every natural number $m$ there exist natural numbers $x, y$ such that $x - y \geqslant m$ and $\sigma(x^2) = \sigma(y^2)$.

*Proof.* Let $n$ be an arbitrary natural number $> m$ such that $(n, 10) = 1$. For $x = 5n$, $y = 4n$ we have $x - y = n > m$ and $\sigma(x^2) = \sigma(y^2) = 31\sigma(n^2)$.

**3.** Find all the natural numbers whose divisors added up give odd sums.

*Solution.* Suppose that $n$ is a natural number such that $\sigma(n)$ is odd. Let $n = 2^{\alpha}k$, where $k$ is an odd number and $\alpha$ is a non-negative integer. We have $\sigma(n) = (2^{\alpha+1} - 1)\sigma(k)$ and consequently $\sigma(k)$ must be an odd number. Since $k$ is odd, each of its divisors must be odd and, since the sum of the divisors $\sigma(k)$ is odd, the number of the divisors $d(k)$ must also be odd.

Hence, as we have learned in § 1, $k$ must be the square of a natural number, i.e. $k = m^2$. Thus we see that $n = 2^{\alpha}m^2$. If $\alpha$ is even, that is if $\alpha = 2\beta$, then $n = (2^{\beta}m)^2$. If $\alpha$ is odd, then $\alpha = 2\beta + 1$ and so $n = 2(2^{\beta}m)^2$. Hence either $n = l^2$ or $n = 2l^2$, where $l$ is a natural number.

On the other hand, if $n = l^2$ or $n = 2l^2$, where $l$ is a natural number, then $n = 2^{\alpha} q_1^{\alpha_1} q_2^{\alpha_2} \dots q_k^{\alpha_k}$ is the factorization of $n$ into primes, $q_1, q_2, \dots, q_k$ being odd prime numbers. We then have $\sigma(n) = (2^{2\alpha+1} - 1)\sigma(q_1^{2\alpha_1}) \dots \sigma(q_k^{2\alpha_k})$ or $\sigma(n) = (2^{2\alpha+2} - 1)\sigma(q_1^{2\alpha_1}) \dots \sigma(q_k^{2\alpha_k})$. But since the number $\sigma(q_i^{2\alpha_i}) = 1 + q_i + q_i^2 + \dots + q_i^{2\alpha_i}$, as the sum of an odd number of summands, each of them odd, is odd, the number $\sigma(n)$ is odd. Therefore the answer is that $\sigma(n)$ is odd if and only if $n$ is either square or a square multiplied by 2.

**4.** Prove that if $n$ is a composite number, then $\sigma(n) > n + \sqrt{n}$.

*Proof.* Being composite, $n$ has a divisor $d$ such that $1 < d < n$. Hence $1 < n/d < n$. If $d \geqslant \sqrt{n}$, then $n/d \geqslant \sqrt{n}$. But since $n/d$ is also a divisor of $n$ (not necessarily different from $d$) and $1 < n/d < n$, we see that $\sigma(n) \geqslant n + \sqrt{n} + 1$, whence $\sigma(n) > n + \sqrt{n}$, which was to be proved.

*Remark.* As an easy consequence of the fact just proved, we note that $\lim_{n\to\infty}(\sigma(p_n+1) - \sigma(p_n)) = +\infty$ and that $\lim_{n\to\infty}(\sigma(p_n) - \sigma(p_n - 1)) = -\infty$.

**5.** Prove that for every natural number $k > 1$ the equation $\sigma(n) = n+k$ has a finite $\geqslant 0$ number of solutions.

*Proof.* If $\sigma(n) = n+k$, where $k$ is a natural number $> 1$, then $n$ must be a composite number and, according to exercise 4, $\sigma(n) > n + \sqrt{n}$, which proves that $n < k^2$.

In particular, the equation $\sigma(n) = n+2$ has no solutions and the equation $\sigma(n) = n+3$ has precisely one solution, namely $n = 4$.

**6.** Prove that $\lim_{n\to\infty} \frac{\sigma(n!)}{n!} = +\infty$.

*Proof.* It is easy to prove that $\sigma(m)/m$ is the sum of the reciprocals of the natural divisors of $m$. Since the divisors of the number $n!$ comprise at least the natural numbers $\leqslant n$, we see that

$$\frac{\sigma(n!)}{n!} \geqslant \frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}.$$

But, since

$$\lim_{n\to\infty}\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right) = +\infty, \quad \text{we have} \quad \lim_{n\to\infty}\frac{\sigma(n!)}{n!} = \infty.$$

**7.** L. Alaoglu and P. Erdös [1] call a natural number $n$ *superabundant* if $\sigma(n)/n > \sigma(k)/k$ whenever $k < n$. Prove that there are infinitely many such numbers.

Proof. Let $u_n = \sigma(n)/n$ for $n = 1, 2, \ldots$ It follows from exercise 6 that the infinite sequence $u_1, u_2, \ldots$ has no upper bound. Therefore, in order to prove the theorem it is sufficient to prove the following more general theorem:

*Every infinite sequence with real numbers as terms and with no upper bound contains infinitely many terms, each being greater than any of the preceding ones.*

In fact, suppose that a sequence $u_1, u_2, \ldots$ has no upper bound. Then we have $\lim_{n\to\infty} \max(u_1, u_2, \ldots, u_n) = +\infty$ and for each natural number $m$ there exists a natural number $l > m$ such that

$$a_l = \max(u_1, u_2, \ldots, u_l) > \max(u_1, u_2, \ldots, u_m).$$

In the sequence $u_1, u_2, \ldots, u_l$ there exist of course terms which are equal to $a_l$. Let $u_n$ denote the first of them. We then have $n > m$, $n \leqslant l$ and $u_n > u_k$ for $k < n$. Thus we have shown that for every natural number $> m$ there exists a natural number $n > m$ such that $u_n > u_k$ whenever $k < n$. The theorem is thus proved.

**8.** A. K. Srinivasan [1] calls a natural number $n$ a *practical number* if every natural number $\leqslant n$ is a sum of different divisors of the number $n$. Prove that for natural numbers $n > 1$ the number $2^{n-1}(2^n-1)$ is practical.

Proof. If $k$ is a natural number $\leqslant 2^n-1$, then, as we know, $k$ is a sum of different numbers of the sequence $1, 2, 2^2, \ldots, 2^{n-1}$. On the other hand, if $2^n-1 < k \leqslant 2^{n-1}(2^n-1)$, then $k = (2^n-1)t+r$, where $t$ is a natural number $\leqslant 2^{n-1}$ and $0 \leqslant r < 2^n-1$, so $t$ and $r$ are sums of different numbers of the sequence $1, 2, 2^2, \ldots \ldots, 2^{n-1}$. The proof follows at once.

For a necessary and sufficient condition for a natural number $n$ to be a practical number, cf. Sierpiński [16]. See also Stewart [2].

10 is not a practical number, 100 and 1000 are.

**9.** Find a natural number $m$ for which the equation $\sigma(x) = m$ has more than a thousand solutions.

Solution. We use the following method, due to S. Mazur. Suppose that we have found $s$ triples of prime numbers $p_i, q_i, r_i$ $(i = 1, 2, \ldots, s)$, all of those $3s$ primes being different and, moreover,

$$(p_i+1)(q_i+1) = r_i+1, \quad i = 1, 2, \ldots, s. \tag{20}$$

Let

$$a_i^{(0)} = p_iq_i, \quad a_i^{(1)} = r_i, \quad i = 1, 2, \ldots, s. \tag{21}$$

For every sequence $a_1, a_2, \ldots, a_s$ consisting of $s$ numbers equal to 0 or 1 we put

$$n_{a_1, a_2, \ldots, a_s} = a_1^{(a_1)}a_2^{(a_2)}\ldots a_s^{(a_s)}. \tag{22}$$



Since the numbers $p_i, q_i, r_i$, $i = 1, 2, \ldots, s$ are different primes, conditions (21) and (22) give

$$\sigma(n_{a_1, a_2, \ldots, a_s}) = \sigma(a_1^{(a_1)})\sigma(a_2^{(a_2)})\ldots\sigma(a_s^{(a_s)}). \tag{23}$$

In virtue of (21) we have

$$\sigma(a_i^{(0)}) = (p_i+1)(q_i+1), \quad \sigma(a_i^{(1)}) = r_i+1, \quad i = 1, 2, \ldots, s,$$

and consequently, by (20), $\sigma(a_i^{(0)}) = \sigma(a_i^{(1)}) = \sigma(r_i)$, for $i = 1, 2, \ldots, s$, and so $\sigma(a_i^{(a_i)}) = \sigma(r_i)$, $i = 1, 2, \ldots, s$. Thus we see that formula (23) implies the equality

$$\sigma(n_{a_1, a_2, \ldots, a_s}) = \sigma(r_1)\sigma(r_2)\ldots\sigma(r_s) = \sigma(r_1r_2\ldots r_s)$$

for each of $2^s$ sequences $a_1, a_2, \ldots, a_s$.

The numbers $n_{a_1, a_2, \ldots, a_s}$, which are $2^s$ in number, are all different because, in view of (22) and (21), their factorizations into prime numbers are different. Thus we have obtained $2^s$ different natural numbers, each having the same sum of divisors.

Thus, in order to find, say, 1024 natural numbers the sums of the divisors of which are equal, it is sufficient to find 10 triples of prime numbers $p_i, q_i, r_i$ $(i = 1, 2, \ldots, 10)$ such that all thirty are different and equalities (20) hold for them. It is easy to check that the following triples satisfy these conditions.

$$2, 3, 11;\; 5, 7, 47;\; 13, 17, 251;\; 19, 23, 479;\; 29, 41, 1259;\; 31, 83, 2687;$$

$$43, 71, 3167;\; 59, 61, 3719;\; 53, 101, 5507;\; 83, 97, 8231.$$

It follows that for

$$m = 12\cdot 48\cdot 252\cdot 480\cdot 1260\cdot 2688\cdot 3168\cdot 3720\cdot 5508\cdot 8232$$

the equation $\sigma(x) = m$ has at least 1024 solutions in natural numbers $x$.

**§ 5. Perfect numbers.** There exist infinitely many natural numbers $n$ such that the sum of the divisors of $n$ excluding $n$ is less than $n$. Such are, for instance, all the prime numbers and their natural powers. There exist also infinitely many natural numbers $n$ such that the sum of the divisors of $n$ excluding $n$ is greater than $n$. For instance such are the numbers of the form $n = 2^k\cdot 3$, where $k = 2, 3, \ldots$ However, we do not know whether there exist infinitely many natural numbers $n$ such that the sum of the divisors of $n$ excluding $n$ is equal to $n$ itself. These are called *perfect numbers*.

There are 23 known perfect numbers. All of them are even and we do not know whether there exist any odd perfect numbers. It has been proved that if such a perfect number exists it must be greater than $10^{20}$ (cf. Kanold [3]) and must have at least six different prime factors (Gradstein [1], cf. Kühnel [1], Norton [1]). The greatest of the known perfect numbers is the number $2^{11212}(2^{11213}-1)$ which has 6751 digits. The least perfect number is the number $6 = 1+2+3$ and the next is $28 = 1+2+4+7+14$. The sum of the divisors of number $n$, each of them less than $n$, is of course the number $\sigma(n)-n$. Accordingly, a natural number is a perfect number if $\sigma(n)-n = n$, i.e. if it satisfies the equation

$$\sigma(n) = 2n. \tag{24}$$

THEOREM 5. *In order that an even number $n$ be perfect it is necessary and sufficient that it should be of the form $2^{s-1}(2^s-1)$, where $s$ is a natural number and $2^s-1$ is a prime.*

Proof. Let $n$ be an even perfect number. Then $n = 2^{s-1}l$, where $s > 1$ and $l$ is an odd number. Hence $\sigma(n) = (2^s-1)\sigma(l)$ and in virtue of (24), $(2^s-1)\sigma(l) = 2^s l$. Since $(2^s-1, 2^s) = 1$, we see that $\sigma(l) = 2^s q$, where $q$ is a natural number. Hence $(2^s-1)q = l$, which, in view of $\sigma(l) = 2^s q$, implies $\sigma(l) = l+q$. But, in virtue of $(2^s-1)q = l$, we have $q \mid l$ and $q < l$ (because $s > 1$). Consequently, the number $l$ has at least two different natural divisors, $q$ and $l$, and the formula $\sigma(l) = l+q$ proves that it has no other divisors. Consequently, we see that $q = 1$ and that $l$ is a prime number. But $l = (2^s-1)q = 2^s-1$. Therefore $n = 2^{s-1}l = 2^{s-1}(2^s-1)$, and so $2^s-1$ is a prime number. Thus we have proved the necessity of the condition.

In order to prove the sufficiency we suppose that $2^s-1$ is a prime number (of course an odd one). Further, let $n = 2^{s-1}(2^s-1)$. We have $\sigma(n) = (2^s-1)\sigma(2^s-1) = (2^s-1)2^s$ since $2^s-1$ is a prime number. So $\sigma(n) = 2n$, which proves that $n$ is a perfect number; this proves the sufficiency of the condition and, at the same time, completes the proof of the theorem.

It is easy to prove that, if $2^s-1$ is a prime number, then $s$ must be also a prime. In fact, if $s = ab$, where $a$ and $b$ are natural numbers $> 1$, then

$$2^s-1 = (2^a-1)(1+2^a+2^{2a}+\ldots+2^{(b-1)a}),$$

which shows that, since $a > 1$, i.e. $a \geqslant 2$, and thus $2^a-1 \geqslant 2^2-1 \geqslant 3$, the number $2^s-1$ is composite.

Thus theorem 5 implies the following

COROLLARY. *All the even perfect numbers are given by the formula $2^{p-1}(2^p-1)$, where $p$ and $2^p-1$ are prime numbers.*

Perfect numbers were investigated by Euclid, who discovered the following method of finding them:

"We calculate the consecutive sums of the series $1+2+4+16+$ $+32+\ldots$ If a sum turns out to be a prime, we multiply it by its last summand and obtain a perfect number".

Using theorem 5 we see that the method of Euclid indeed gives all even perfect numbers.



Now we are going to find some of the even perfect numbers. In order to do this, we let $p$ be consecutive prime numbers starting from number 2 and we look whether the number $2^p-1$ is prime or not. We see that for $p = 2, 3, 5, 7$ the numbers $2^p-1 = 3, 7, 31, 127$ are prime. This gives the first four perfect numbers, which were actually known in antiquity. They are $2(2^2-1) = 6$, $2^2(2^3-1) = 28$, $2^4(2^5-1) = 496$, $2^6(2^7-1) = 8128$. For $p = 11$ the number $2^{11}-1 = 23\cdot 89$ is composite, and so we do not obtain a perfect number.

It follows from theorem 5 that the task of finding even perfect numbers is the same as that of finding Mersenne's numbers defined as being prime numbers of the form $2^s-1$. We shall return to the latter problem in Chapter X.

We denote by $V(x)$, $x$ being a real number, the number of perfect numbers $\leqslant x$. B. Hornfeck and E. Wirsing [1] have proved that $\lim\limits_{x\to\infty}\frac{\log V(x)}{\log x} = 0$ and E. Wirsing [1] has proved that there exists a natural number $A$ such that $V(x) < Ae^{A(\log x)/\log\log x}$.

We do not know whether there exist infinitely many natural numbers $n$ such that $n \mid \sigma(n)$, or whether there exist odd natural numbers with this property. It has been proved that there are no such numbers $n$ with $n < 10^{20}$ (Kanold [3]).

Natural numbers $n$ such that $\sigma(n) = mn$, where $m$ is a natural number $>1$, are called *$P_m$ numbers* or *multiply perfect numbers.* These numbers were investigated by Wersenne, Fermat, Descartes, Legendre, and others.

Accordingly, $P_2$ numbers are perfect numbers. P. Poulet [1], (pp. 9-27) has found 334 $P_m$ numbers with $m \leqslant 8$.

In 1953 B. Franqui and M. Garcia [1] obtained 63 new $P_m$ numbers (cf. also Franqui and Garcia [2], Brown [1] and [2]). The numbers $P_3$ were investigated by R. Steuerwald [1].

P. Cattaneo [1] has called a number *quasi-perfect* if it is equal to the sum of its own non-trivial natural divisors, i.e. the divisors different from 1 and the number itself. Accordingly, quasi-perfect numbers are those natural numbers $n$ for which $\sigma(n) = 2n+1$. We do not know whether there are any such numbers. However, it is easy to prove that there exist infinitely many natural numbers $n$ such that $\sigma(n) = 2n-1$. For instance, such are all the numbers $2^k$ with $k = 0, 1, 2, \ldots$ A. Mąkowski [5] has investigated the solutions of the equation $\sigma(n) = 2n+2$ in natural numbers. He has noticed that, if $2^k-3$ is a prime number, then $n = 2^{k-1}(2^k-3)$ is a solution of this equation. The numbers $2^k-3$ are prime for the following values of $k < 24$: $k = 2, 3, 4, 5, 6, 9, 10, 12, 14, 20, 22$. The equation has also other solutions, e.g. $n = 650$.

EXERCISES. **1.** Prove that there exist infinitely many odd natural numbers $n$ such that $\sigma(n) > 2n$.

Proof. Such are for instance the numbers $n = 945m$, where $m$ is a natural number which is not divisible by $2, 3, 5, 7$. Since $945 = 3^3\cdot 5\cdot 7$, $(m, 945) = 1$ and so $\sigma(n) = \sigma(945)\sigma(m) \geqslant \sigma(945)m = 1920m > 2n$. Since $m$ is not divisible by 2, $n$ is an odd number.

It can be proved that 945 is the least odd natural number for which $\sigma(n) > 2n$ holds.

**2.** Find all the natural numbers $n$ such that $n$ is equal to the product of all the natural divisors of $n$ excluding $n$.

Solution. Let $Q_n$ denote the product of all the natural divisors of number $n$. We are looking for natural numbers $n$ such that $Q_n/n = n$, i.e. for numbers $n$ for which $Q_n = n^2$. If $d_1, d_2, \ldots, d_s$ are all the natural divisors of numbers $n$ (which are $s = d(n)$ in number), then the numbers $n/d_1, n/d_2, \ldots, n/d_s$ are also natural divisors of the number $n$. It follows that $Q_n = d_1 d_2\ldots d_s = n^s/Q_n$, and so $Q_n = n^{s/2} = n^{d(n)/2}$. Since $Q_n = n^2$, we see that $n^2 = n^{d(n)/2}$, whence $d(n) = 4$, and, as can easily be verified, the converse is also true: if $d(n) = 4$, then $Q_n = n^2$. Therefore, in order that a natural number be equal to the product of the natural divisors of $n$ excluding $n$, it is necessary and sufficient that $n$ have precisely four natural divisors. It follows from the formula for the number of divisors given by (5) that, provided (1) is the factorization of $n$ into primes, the equality

$$(a_1+1)(a_2+1)\ldots(a_k+1) = 4$$

holds. Since the exponents $a_1, a_2, \ldots, a_k$ are natural numbers, the above formula is valid only in the case where $k < 2$, i.e. for $k = 1$ or $k = 2$. If $k = 1$, then $a_1+1 = 4$, whence $a_1 = 3$ and $n$ turns out to be the cube of a prime number. If $k = 2$, then $a_1 = a_2 = 1$ and $n$ turns out to be a product of two different primes. Thus we see that every natural number $n$ which is the product of its own divisors less than $n$ is either the cube of a prime numbers or the product of two different primes. The following are all the numbers of this kind that are less than 30: 6, 8, 10, 14, 15, 21, 22, 26, 27.

**3.** Prove the following theorem of Descartes (mentioned in a letter to Mersenne of 15th November 1638):

1° If $n$ is a $P_3$ number and is not divisible by 3, then $3n$ is a $P_4$ number.

2° If a number $n$ is divisible by 3 but not divisible either by 5 or by 9 and, moreover, if it is a $P_3$ number, then $45n$ is $P_4$.

3° If a number $n$ is not divisible by 3 and if $3n$ is a $P_{4k}$ number, then $n$ is a $P_{3k}$ number.

Proof. 1° If $n$ is a $P_3$ number, then $\sigma(n) = 3n$ and if $n$ is not divisible by 3, then $\sigma(3n) = \sigma(3)\sigma(n) = 4\cdot 3n$ and consequently $3n$ is a $P_4$ number.

2° If $n$ is a $P_3$ number and $n = 3k$, where $k$ is divisible neither by 3 nor by 5, then $\sigma(45n) = \sigma(3^3\cdot 5k) = \sigma(3^3)\sigma(5)\sigma(k) = 40\cdot 6\cdot\sigma(k)$. But, in virtue of $n = 3k$ and $k$ not being divisible by 3, we have $\sigma(n) = \sigma(3)\sigma(k) = 4\sigma(k)$. Consequently, $\sigma(45n) = 60\cdot 4\sigma(k) = 60\sigma(n)$. Hence, in view of $n$ being a $P_3$ number $\sigma(n) = 3n$, we see that $\sigma(45n) = 180n = 4\cdot 45n$, which proves that $45n$ is a $P_4$ number.

3° If $n$ is not divisible by 3 and if $3n$ is a $P_{4k}$ number, then $\sigma(3n) = 4k\cdot 3n$, which implies that $\sigma(3n) = \sigma(3)\sigma(n) = 4\sigma(n)$, whence $\sigma(n) = 3kn$, which proves that $n$ is a $P_{3k}$ number.



**4.** Prove that 120 and 672 are $P_3$ numbers, the number $2^5\cdot 3^3\cdot 5\cdot 7$ is a $P_4$ number, and $2^7\cdot 3^4\cdot 5\cdot 7\cdot 11^2\cdot 17\cdot 19$ is a $P_5$ number.

The proof follows at once if we look at the factorizations into primes of the numbers $120 = 2^3\cdot 3\cdot 5$ and $672 = 2^5\cdot 3\cdot 7$. It can be proved that 120 is the least $P_3$ number.

**5.** Prove that, if $\sigma(n) = 5n$, then $n$ has more than 5 different prime factors.

Proof. Suppose that (1) is the factorization of $n$ into primes. Then, by (17), one has

$$\sigma(n) < \frac{q_1^{a_1+1}q_2^{a_2+1}\ldots q_k^{a_k+1}}{(q_1-1)(q_2-1)\ldots(q_k-1)} = \frac{q_1}{q_1-1}\cdot\frac{q_2}{q_2-1}\cdot\ldots\cdot\frac{q_k}{q_k-1}\cdot n.$$

If $k \leqslant 5$, then we would have

$$\sigma(n) < \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{11}{10}n = \frac{77}{16}n < 5n$$

which contradicts the equality $\sigma(n) = 5n$.

**6.** Prove the following theorem of Mersenne. If $n$ is not divisible by 5 and it is a $P_5$ number, then $5n$ is a $P_6$ number.

The proof is straightforward.

**§ 6. Amicable numbers.** Two natural numbers are called *amicable numbers* if each of them is equal to the sum of all the natural divisors of the other except the number itself. It is easy to see that in order that two natural numbers $n, m$ be amicable it is necessary and sufficient that $\sigma(m) = \sigma(n) = m+n$.

The first pair of amicable numbers, 220 and 284, was found by Pythagoras. The pair $2^4\cdot 23\cdot 47$ and $2^4\cdot 1151$ was discovered by Fermat, the pair $2^7\cdot 191\cdot 383$ and $2^7\cdot 73727$ by Descartes. As many as 59 pairs of amicable numbers were found by Euler, among them the pair $2^3\cdot 17\cdot 79$ and $2^3\cdot 23\cdot 59$ and the pair $2^3\cdot 19\cdot 41$ and $2^5\cdot 199$. A longer paper devoted to the amicable numbers has been written by E. B. Escott [2]: he has presented a list of 390 pairs of amicable numbers found in the last 25 centuries. A considerable number of new pairs of amicable numbers, among them the pair $2^9\cdot 239$ and $2^4\cdot 43\cdot 197$ and the pair $2\cdot 5^3\cdot 13\cdot 89\cdot 113$ and $2\cdot 5\cdot 13\cdot 379\cdot 701$, have been found by P. Poulet [3].

We know pairs of amicable numbers which are all odd, e.g. the pair $3^3\cdot 5\cdot 7\cdot 11$, $3\cdot 5\cdot 7\cdot 139$. But we do not know any pair with one of the numbers odd and the other even. Neither do we know whether there exist infinitely many pairs of amicable numbers.

The notion of a pair of amicable numbers has been generalized to the notion of a $k$-tuplet of amicable numbers. The notion is due to L. E. Dickson, who calls a $k$-tuplet of natural numbers $n_1, n_2, \ldots, n_k$ a *$k$-tuplet of amicable numbers* if

$$\sigma(n_1) = \sigma(n_2) = \ldots = \sigma(n_k) = n_1+n_2+\ldots+n_k$$

(Dickson [2], cf. also Mason [1]).

A. Mąkowski [3] has found the following triples of amicable numbers: $2^2\cdot 3^2\cdot 5\cdot 11$, $2^5\cdot 3^2\cdot 7$, $2^2\cdot 3^2\cdot 71$ and $2^3\cdot 3\cdot 5\cdot 13$, $2^2\cdot 3\cdot 5\cdot 29$, $2^2\cdot 3\cdot 5\cdot 29$ (in the second triple two of the numbers are equal). There exist triples for which all three numbers are equal, e.g. $n_1 = n_2 = n_3 = 120$.

A different definition of a $k$-tuplet of amicable numbers has been given by B. F. Yanney [1]. The definition is as follows: a $k$-tuplet of natural numbers $n_1, n_2, \ldots, n_k$ is called a *$k$-tuplet of amicable numbers* if

$$n_1+n_2+\ldots+n_k+\sigma(n_i) = \sigma(n_1)+\sigma(n_2)+\ldots+\sigma(n_k), \quad i = 1, 2, \ldots, k,$$

this being clearly equivalent to the condition

$$n_1+n_2+\ldots+n_k = (k-1)\sigma(n_i) \quad \text{for} \quad i = 1, 2, \ldots, k.$$

For $k = 2$ both definitions reduce to the ordinary definition of a pair of amicable numbers.

For $k > 2$, however, the definitions no longer coincide and a $k$-tuplet, which is a $k$-tuplet of amicable numbers according to one definition is not a $k$-tuplet of amicable numbers according to the other. An example of a triple which is a triple of amicable numbers according to the definition of Yanney is the triple 308, 455, 581.

We have $308 = 2^2\cdot 7\cdot 11$, $455 = 5\cdot 7\cdot 13$, $581 = 7\cdot 83$, so $\sigma(n_1) = \sigma(n_2) = \sigma(n_3) = 672$ and $n_1+n_2+n_3 = 1344 = 2\cdot 672$.

It is not known whether there are pairs of relatively prime amicable numbers. H. J. Kanold [1] has proved that if in a pair $m_1, m_2$ of amicable numbers the numbers $m_1, m_2$ are relatively prime, then each of them must be greater than $10^{23}$ and the number $m_1 m_2$ must have more than 20 prime factors.

P. Erdös [16] has proved that, if $A(x)$ is the number of the pairs of amicable numbers $\leqslant x$, then $\lim\limits_{x\to\infty} A(x)/x = 0$.

**§ 7. The sum** $\sigma(1)+\sigma(2)+\ldots+\sigma(n)$**.** In this section we are going to find the formula for the sum

$$S(x) = \sigma(1)+\sigma(2)+\ldots+\sigma([x]), \tag{25}$$

where $x$ is a real number $\geqslant 1$.

Let $n$ be a natural number. The number $n$ is a term of the sum $\sigma(k)$ if and only if $n$ is a divisor of the number $k$. Therefore, in order to calculate the number of the summands $\sigma(k)$ in the sum $S(x)$ in which $n$ appears as a summand, it is sufficient to find the number of the $k$'s $\leqslant x$ which are divisible by $n$. But those are the numbers $k$ for which $k = nl \leqslant x$, where $l$ is a natural number satisfying of course the inequality $l \leqslant x/n$. Clearly, the number of $l$'s is $[x/n]$. Accordingly, a natural number $n$ is a summand of the sum of $\sigma(k)$ for $[x/n]$ different natural numbers $k \leqslant x$. From this we infer that



$$S(x) = \sum_{n=1}^{[x]} n\left[\frac{x}{n}\right]. \tag{26}$$

There is another methods of finding sum (25). In fact, the number $\sigma(k)$ can be thought of as the sum of natural numbers $n$ satisfying the equation

$$mn = k,$$

where $m$ is a natural number. Therefore sum (25) can be regarded as the sum of the numbers $n$ for which there exist natural numbers $m$ such that $mn \leqslant x$. Then for a fixed number $m$ number $n$ can be any of the numbers

$$1, 2, 3, \ldots, \left[\frac{x}{m}\right],$$

the sum of those being equal to

$$1+2+\ldots+\left[\frac{x}{m}\right] = \frac{1}{2}\left[\frac{x}{m}\right]^2 + \frac{1}{2}\left[\frac{x}{m}\right].$$

Consequently, if we let $m$ to take all the possible values for which the inequality $mn \leqslant x$ can be satisfied, the sum of all $n$'s, i.e. the sum $S(x)$, is equal to

$$S(x) = \frac{1}{2}\sum_{m=1}^{[x]}\left[\frac{x}{m}\right]^2 + \frac{1}{2}\sum_{m=1}^{[x]}\left[\frac{x}{m}\right]. \tag{27}$$

Comparing (26) and (27) we find the identity

$$\sum_{n=1}^{[x]} n\left[\frac{x}{n}\right] = \frac{1}{2}\sum_{m=1}^{[x]}\left[\frac{x}{m}\right]^2 + \frac{1}{2}\sum_{m=1}^{[x]}\left[\frac{x}{m}\right]$$

which is of some interest in itself. Clearly, it can also be written in the form

$$\sum_{n=1}^{[x]}\left[\frac{x}{n}\right]^2 = \sum_{n=1}^{[x]}(2n-1)\left[\frac{x}{n}\right].$$

Neither of the formulas (26), (27) is of any practical use for finding the numerical values of the sum $S(x)$ for a given number $x$. A formula more suitable for this purpose is to be found in a similar way as formula (13) was found and is as follows:

$$(28)\qquad S(x) = \frac{1}{2}\left(\sum_{n=1}^{[\sqrt{x}]}\left[\frac{x}{n}\right]^2 + \sum_{n=1}^{[\sqrt{x}]}(2n+1)\left[\frac{x}{n}\right] - [\sqrt{x}]^3 - [\sqrt{x}]^2\right).$$

For instance, with the use of this formula we can easily calculate $S(100) = 8249$.

Now, if in (28) we drop the symbol [ ] and replace the sum $\sum_{n=1}^{[\sqrt{x}]} 1/n^2$ by the sum of the infinite series $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$, each time calculating the error it involves, then we obtain the value $\pi^2 x^2/12$ as an approximation of the sum $S(x)$, the error being not greater than $Ax\sqrt{x}$, where $A$ is a positive constant independent of $x$.

**§ 8. The numbers $\sigma(n)$ as coefficients of various expansions.** The function $\sigma(n)$ (similarly to the function $d(n)$; cf. § 3) appears as the coefficient in various expansions in infinite series.

As is known from Analysis, the iterated series

$$(29)\qquad \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} kx^{kl}$$

is absolutely convergent for $|x| < 1$. Reducing it to an ordinary series, for a fixed value of $n$ we put together the summands in which $x^n$ appears. Then the coefficients of the $k$th summands are the factors of the number $n = kl$. Consequently, the sum (29) turns into the sum $\sum_{n=1}^{\infty}\sigma(n)x^n$.

On the other hand, since $\sum_{l=1}^{\infty} kx^{kl} = kx^k/(1-x^k)$, we see that sum (29) is equal to the sum $\sum_{k=1}^{\infty} kx^k/(1-x^k)$. Thus we arrive at the formula

$$\sum_{k=1}^{\infty}\frac{kx^k}{1-x^k} = \sum_{n=1}^{\infty}\sigma(n)x^n, \qquad |x| < 1.$$

Since (29) is absolutely convergent for $|x| < 1$, we may interchange the elements of the series in such a way that, applying the identity $\sum_{k=1}^{\infty} kx^{kl} = x^l/(1-x^l)^2$, where $|x| < 1$, we obtain the formula

$$\sum_{l=1}^{\infty}\frac{x^l}{(1-x^l)^2} = \sum_{n=1}^{\infty}\sigma(n)x^n \quad \text{for} \quad |x| < 1.$$



In § 3 we have introduced Dirichlet's multiplication of two infinite series $a_1+a_2+\ldots$ and $b_1+b_2\ldots$ We apply it here to the case where $a_k = 1/k^{s-1}$, $b_l = 1/l^s$, $k$ and $l$ being natural numbers and $s$ being a real number $> 2$. We then have

$$a_k b_l = \frac{1}{k^{s-1}}\cdot\frac{1}{l^s} = \frac{k}{(kl)^s}.$$

Now, putting together the products $a_k b_l$ for which $kl$ is equal to a given natural number $n$, we see that their numerators are equal to the natural divisors $k$ of the number $n$; the sum of those being, clearly, $\sigma(n)/n^s$. Hence

$$\zeta(s-1)\zeta(s) = \sum_{n=1}^{\infty}\frac{\sigma(n)}{n^s} \quad \text{for} \quad s > 2.$$

**§ 9. Sums of summands depending on the natural divisors of a natural number $n$.** Let $f(n)$ be an arbitrary function defined for every natural number $n$. If $d_1, d_2, \ldots, d_s$ are all the divisors of a natural number $n$, then the sum

$$f(d_1)+f(d_2)+\ldots+f(d_s)$$

is denoted simply by

$$\sum_{d|n} f(d)$$

and called the sum of the summands $f(d)$ with $d$ ranging over the natural divisors of the number $n$. In particular, we have

$$\sum_{d|n} 1 = d(n), \qquad \sum_{d|n} d = \sigma(n) \qquad \text{but also} \qquad \sum_{d|n}\frac{n}{d} = \sigma(n).$$

For a given function $f(n)$ defined for natural numbers $n$ we write

$$F(n) = \sum_{d|n} f(d).$$

Let us find the sum

$$\sum_{n=1}^{[x]} F(x) = \sum_{n=1}^{[x]}\sum_{d|n} f(d)$$

for the real values $x \geqslant 1$.

The sum on the right-hand side of the last formula comprises the summands $f(k)$, where $k$ are natural numbers $\leqslant x$.

For a given natural number $k \leqslant x$ the summand $f(k)$ appears in the sum $\sum_{d|n} f(d)$ if and only if $k$ is a divisor of number $n$. (Clearly, it appears at most once). The number of such natural numbers $n \leqslant x$ is of course $\left[\frac{x}{k}\right]$. Consequently, the number of the summands $f(k)$ in the double sum is $\left[\frac{x}{k}\right]$, whence

$$\sum_{n=1}^{[x]} F(n) = \sum_{k=1}^{[x]} f(k)\left[\frac{x}{k}\right]. \tag{30}$$

In particular, if $f(n) = n^s$, where $s$ is a fixed integer, then $F(n)$ is the sum of the $s$th powers of the natural divisors of the natural number $n$. This sum is sometimes denoted by $\sigma_s(n)$. Then formula (30) gives

$$\sum_{n=1}^{[x]} \sigma_s(n) = \sum_{k=1}^{[x]} k^s\left[\frac{x}{k}\right].$$

We have of course $\sigma_0(n) = d(n)$, $\sigma_1(n) = \sigma(n)$ for $n = 1, 2, \ldots$ and we see that formulae (11) and (26) are particular cases of the last formula.

**§ 10. Möbius function.** Under this name we mean the arithmetical function $\mu(n)$ defined by the conditions

1° $\mu(1) = 1$,

2° $\mu(n) = 0$ if the natural number $n$ is divisible by the square of a natural number $> 1$,

3° $\mu(n) = (-1)^k$ if the natural number $n$ is the product of $k$ different prime factors.

Accordingly, $\mu(1) = 1$, $\mu(2) = \mu(3) = -1$, $\mu(4) = 0$, $\mu(5) = -1$, $\mu(6) = 1$, $\mu(7) = -1$, $\mu(8) = \mu(9) = 0$, $\mu(10) = 1$.

Now we are going to show a certain property of the function $\mu(n)$. Let $n$ be a natural number $> 1$ whose factorization into prime numbers is as in (1). Consider the product

$$(1-q_1^s)(1-q_2^s)\ldots(1-q_k^s), \tag{31}$$

where $s$ is a given integer.

The expansion of product (31) consists of the number 1 and the numbers $\pm d^s$, where $d$ is a divisor of number $n$, being a product of different prime factors; the sign $+$ or $-$ at each of the numbers appears according to whether the number is the product of an even or of an odd number of prime factors. In virtue of property 3° of the definition of the Möbius function, we see that the coefficient $\pm$ at $d^s$ is equal to $\mu(d)$.

If, in addition, we note that $\mu(1)\cdot 1^s = 1$ and that, in order that the number $d$ be equal to 1 or to the product of different prime numbers, it is necessary and sufficient that the number $n$ have no divisor which



is the square of a natural number $> 1$, then, by property 2°, we see that the product (31) is equal to the sum

$$\sum_{d|n} \mu(d) d^s.$$

That is

$$(1-q_1^s)(1-q_2^s)\ldots(1-q_k^s) = \sum_{d|n} \mu(d)\cdot d^s,$$

whence for $s = 0$ we obtain

$$\sum_{d|n} \mu(d) = 0 \tag{32}$$

for every natural number $n > 1$. Clearly, for $n = 1$, we have $\sum_{d|1} \mu(d) = \mu(1) = 1$. We see that, if $F(n) = \sum_{d|n} \mu(d)$, then $F(1) = 1$ and $F(n) = 0$ for natural numbers $n > 1$. Consequently, formula (30) gives

$$\sum_{k=1}^{[x]} \mu(k)\left[\frac{x}{k}\right] = 1 \quad \text{for} \quad x \geqslant 1. \tag{33}$$

Since the inequalities $0 \leqslant t-[t] < 1$ hold for all real numbers $t$ and since $|\mu(k)| \leqslant 1$ for natural numbers $k$, we see that $\left|\mu(k)\left[\frac{x}{k}\right] - \mu(k)\frac{x}{k}\right| < 1$ is valid whenever $x$ is a real number $\geqslant 1$ and $k$ is a natural number. From this we deduce that, if we drop the symbol [ ] in each of the summands of (33), then the error thus obtained is less than 1 and in the first summand is equal precisely to $x-[x]$. Thus, since there are $[x]-1$ summands in the sum excluding the first term, we have

$$\left|\sum_{k=1}^{[x]} \mu(k)\left[\frac{x}{k}\right] - x\sum_{k=1}^{[x]} \frac{\mu(k)}{k}\right| < x-[x]+[x]-1 = x-1,$$

whence, by (33), we obtain

$$\left|1 - x\sum_{k=1}^{[x]} \frac{\mu(k)}{k}\right| < x-1,$$

and this implies $\left|x\sum_{k=1}^{[x]} \mu(k)/k\right| \leqslant x$ and consequently $\left|\sum_{k=1}^{[x]} \mu(k)/k\right| \leqslant 1$. This proves that the module of each of the partial sums of the infinite series

$$\frac{\mu(1)}{1} + \frac{\mu(2)}{2} + \frac{\mu(3)}{3} + \ldots \tag{34}$$

is $\leqslant 1$. As was proved by H. v. Mangoldt in 1897, sum (34) is equal to 0. This had already been conjectured by Euler in 1748.

Now, we apply Dirichlet's multiplication to the series $\sum_{k=1}^{\infty}\mu(k)/k^s$ and $\sum_{l=1}^{\infty}1/l^s$, where $s$ is a natural number $>1$. In virtue of $\mu(1)=1$ and formula (32) we obtain

$$\sum_{k=1}^{\infty}\frac{\mu(k)}{k^s}\cdot\sum_{l=1}^{\infty}\frac{1}{l^s}=1,$$

i.e. the formula

$$\sum_{k=1}^{\infty}\frac{\mu(k)}{k^s}=\frac{1}{\zeta(s)},$$

$s$ being a real number $>1$. In particular, since, as is known from Analysis, $\zeta(2)=\pi^2/6$, the last equality implies

$$\sum_{k=1}^{\infty}\frac{\mu(k)}{k^2}=\frac{6}{\pi^2}.$$

In this connection we observe, that it is easy to prove the equality

$$\sum_{k=1}^{\infty}\frac{\mu^2(k)}{k^s}=\frac{\zeta(s)}{\zeta(2s)},$$

where $s$ is a real number $>1$.

Reducing the iterated series $\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\mu(k)x^{kl}$ to an ordinary series by the method we applied previously to the series (29), for $|x|<1$ we obtain the formula

$$\sum_{n=1}^{\infty}\frac{\mu(n)x^n}{1-x^n}=x.$$

THEOREM 6. *For every arithmetical function $F(n)$ there exists only one arithmetical function $f(n)$ such that the equality*

(35) $$F(n)=\sum_{d|n}f(d)$$

*holds for all natural numbers $n$.*



Proof. If, for $n=1,2,\ldots$, formula (35) is valid, then the following infinite sequence of equality holds:

(36)
$$\begin{aligned}F(1)&=f(1),\\F(2)&=f(1)+f(2),\\F(3)&=f(1)+f(3),\\F(4)&=f(1)+f(2)+f(4),\\F(5)&=f(1)+f(5),\\F(6)&=f(1)+f(2)+f(3)+f(6)\\&\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

The first equality gives $f(1)=F(1)$. So $f(2)$ can be calculated from the second equality. Then, since $f(1)$ and $f(2)$ have already been found, $f(3)$ can be calculated from the third equality and so on. The $n$th equality gives the value of $f(n)$, provided the values of $f(k)$ for $k<n$, have already been found from the previous equalities. Therefore we see that if there exists a function satisfying formula (35), then there is only one such function. On the other hand, it is easy to see that, calculating the values $f(1),f(2),\ldots$ from (36), successively, we obtain a function $f(n)$ satisfying all the equalities of (36) and, consequently, satisfying (35).

The theorem is thus proved.

Equation (36) enables us to find the values $f(n)$ provided $F(1),F(2),\ldots$ $\ldots,F(n)$ are known. There exists also a general formula for the function $f(n)$, namely

(37) $$f(n)=\sum_{d|n}\mu(d)F\left(\frac{n}{d}\right)$$

which can alternatively be written in the form

(38) $$f(n)=\sum_{d|n}\mu\left(\frac{n}{d}\right)F(d)$$

or in the form

(39) $$f(n)=\sum_{kl=n}\mu(k)F(l),$$

where the summation is over all the pairs $k,l$ of the natural numbers for which $kl=n$.

In order to prove these formulae it is, of course, sufficient to prove that the function defined by (39) satisfies formula (35) for every natural number $n$.

In fact, (39) implies that

$$\sum_{d|n} f(d) = \sum_{d|n} \sum_{kl=d} \mu(k)F(l) = \sum_{lk|n} \mu(k)F(l)$$
$$= \sum_{l|n} F(l) \sum_{k|n/l} \mu(k) = F(n)$$

because, by the properties of the function $\mu$ stated above, $\sum_{k|n/l} \mu(k)$ is different from zero (and thus equal to 1) only if $n/l = 1$, i.e. $l = n$.

In particular, for $F(1) = 1$ and $F(n) = 0$, $n = 2, 3, \ldots$, theorem 6 implies that there exists precisely one function $f$, namely the Möbius function, $\mu(n) = f(n)$, for which the following conditions are satisfied,

$$f(1) = 1, \quad \sum_{d|n} f(d) = 0 \quad \text{for} \quad n = 2, 3, \ldots$$

**§ 11. Liouville function** $\lambda(n)$. This is the arithmetical function defined by the conditions

1° $\lambda(1) = 1$,

2° $\lambda(n) = (-1)^{a_1+a_2+\ldots+a_k}$ provided the factorization of $n$ into prime numbers is of the form (1).

We have $\lambda(1) = 1$, $\lambda(2) = \lambda(3) = -1$, $\lambda(4) = 1$, $\lambda(5) = -1$, $\lambda(6) = 1$, $\lambda(7) = \lambda(8) = -1$, $\lambda(9) = \lambda(10) = 1$.

Suppose that for a natural number $n > 1$ the factorization of $n$ into primes is as in (1).

Consider the product

$$\prod_{i=1}^{k} (1 - q_i^s + q_i^{2s} - q_i^{3s} + \ldots + (-1)^{a_i} q_i^{a_i s}),$$

where $s$ is an arbitrary integer. Expanding this product, we obtain the algebraic sum of the summands $(q_1^{\lambda_1} q_2^{\lambda_2} \ldots q_k^{\lambda_k})^s$, each multiplied by $(-1)^{\lambda_1+\ldots+\lambda_k} = \lambda(q_1^{\lambda_1} q_2^{\lambda_2} \ldots q_k^{\lambda_k})$, where the summation is all over the set of the divisors $d = q_1^{\lambda_1} q_2^{\lambda_2} \ldots q_k^{\lambda_k}$ of number $n$. Consequently, the product is equal to the sum $\sum_{d|n} \lambda(d) d^s$.

On the other hand, the formula for the sum of a geometric progression gives

$$1 - q_i^s + q_i^{2s} - q_i^{3s} + \ldots + (-1)^{a_i} q_i^{a_i s} = \frac{1 + (-1)^{a_i} q_i^{(a_i+1)s}}{1 + q_i^s}.$$

Applying this to each of the factors of the product we get

$$\prod_{i=1}^{k} \frac{1 + (-1)^{a_i} q_i^{(a_i+1)s}}{1 + q_i^s} = \sum_{p|n} \lambda(d) d^s.$$



In particular, for $s = 0$ we obtain the formula

$$(40) \qquad \frac{1 + (-1)^{a_1}}{2} \cdot \frac{1 + (-1)^{a_2}}{2} \cdots \frac{1 + (-1)^{a_k}}{2} = \sum_{d|n} \lambda(d).$$

The number $\frac{1 + (-1)^{a}}{2}$ is equal to zero or to one, depending on whether $a$ is odd or even. It follows that the left-hand side of formula (40) is different from zero, and thus equal to 1, if and only if all the exponents $a_1, a_2, \ldots, a_k$ are even, i.e. if $n$ is the square of a natural number. Thus we have proved the following

THEOREM 7. *The sum* $\sum_{d|n} \lambda(d)$ *is equal either to* 0 *or—in the case where* $n$ *is the square of a natural number—to* 1.

Although in the proof of theorem 7 we assumed $n > 1$, the theorem is true for $n = 1$, since $\lambda(1) = 1$.

Let $F(n) = \sum_{d|n} \lambda(d)$. Consequently, $F(n) = 1$ holds for any $n$ which is the square of natural numbers and $F(n) = 0$ otherwise. In virtue of (30) (for $f(k) = \lambda(k)$) we obtain

$$\sum_{k=1}^{[x]} \lambda(k) \left[\frac{x}{k}\right] = \sum_{n=1}^{[x]} F(n)$$

whenever $x \geq 1$. The sum on the right-hand side of this equality consists of as many summands equal to 1 as there are natural numbers $\leq x$ which are squares. Consequently the sum is equal to $[\sqrt{x}]$. Hence

$$\sum_{k=1}^{[x]} \lambda(k) \left[\frac{x}{k}\right] = [\sqrt{x}] \quad \text{for} \quad x \geq 1.$$