CONDITIONAL COMBINATIONS OF QUANTUM SYSTEMS

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Abstract: The paper presents a new method of conditional combination of quantum systems that takes into account the external environmental conditions. As a practical example of the method presented here, the well-known Bell states are modeled as conditional combination of two q-bits. Analogous approach can be applied in modeling conditional combinations of two and more quantum system sequences.

Key words: *Quantum informatics, quantum systems, q-bits, wave probabilistic functions*

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1. Introduction

The problem of complex probability functions has attracted considerable attention in quantum system theory [1] or in quantum informatics [2], where the wave probabilistic functions have been recognized as a necessary tool for information processing [7] and quantum system modeling [4].

One of the identified problems in quantum information science [2] was the problem of composition of complex quantum systems through the different quantum components. It was proven many times, e.g. in [1, 2], that fully entangled Bell states cannot be composed of two general q-bits.

The main goal of the paper is to analyze the quantum composition problem and to find a new method making it possible to compose fully entangled complex quantum systems by using its quantum components. We will introduce the transition matrix P that can add into composition rule a new piece of information about property of external environment. The external environment has, then, a direct impact on the quantum compositions. We, therefore, call this principle conditional composition of quantum systems because the composition rule is affected (conditioned) by external influence or constrains.

The paper is structured as follows: Section 2 defines the combination methods of quantum systems where Section 2.1 introduces the basic principle of conditional

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combination of quantum systems, Section 2.2 analyzes the Bell states as an example of conditional combination of quantum states, and Section 2.3 extends the presented method into general sequences of conditionally combined quantum systems together with an example of two q-bits sequences, given in Section 2.4. Section 3 presents our conclusions.

2. Combination Methods of Quantum Systems

Let us define n-level quantum state [1]:

$$|\psi\rangle = \psi\left(Y_1\right) \cdot |\,Y_1\rangle + \psi\left(Y_2\right) \cdot |\,Y_2\rangle + ... + \psi\left(Y_n\right) \cdot |\,Y_n\rangle\ , \tag{1}$$

where the symbol $|\cdot\rangle$ labels the system state and the probabilities of finding the state Y_z under the measurement are given as follows:

$$P\left(Y_z\right) = \left|\psi\left(Y_z\right)\right|^2 \tag{2}$$

$$\sum_{z=1}^{n} \left|\psi\left(Y_z\right)\right|^2 = 1. \tag{3}$$

In further parts, the functions $\psi\left(\cdot\right)$ are referred to as *wave probabilistic functions* [3] and equation (1) stresses the links among the pure quantum states $|Y_1\rangle, |Y_2\rangle\ , ..., |Y_n\rangle$.

Let us define the basic operations with wave probabilistic functions. We start with the definition of two quantum systems represented by two "bra-ket" forms:

$$|\psi_1\rangle = \psi_1\left(Y_{1,1}\right) \cdot |\,Y_{1,1}\rangle + \psi_1\left(Y_{1,2}\right) \cdot |\,Y_{1,2}\rangle + ... + \psi_1\left(Y_{1,n}\right) \cdot |\,Y_{1,n}\rangle \tag{4}$$

$$|\psi_2\rangle = \psi_2\left(Y_{2,1}\right) \cdot |\,Y_{2,1}\rangle + \psi_2\left(Y_{2,2}\right) \cdot |\,Y_{2,2}\rangle + ... + \psi_2\left(Y_{2,n}\right) \cdot |\,Y_{2,n}\rangle. \tag{5}$$

Combination of the two quantum systems (4) and (5) yields the common wave probabilistic function that is defined with the help of the Kronecker product [2]:

$$\begin{array}{l} |\psi_{1,2}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = \psi_1\left(Y_{1,1}\right) \cdot \psi_2\left(Y_{2,1}\right) \cdot |Y_{1,1}\ Y_{2,1}\rangle + ... \\ +\psi_1\left(Y_{1,1}\right) \cdot \psi_2\left(Y_{2,n}\right) \cdot |Y_{1,1}\ Y_{2,n}\rangle + \\ +... + \psi_1\left(Y_{1,n}\right) \cdot \psi_2\left(Y_{2,1}\right) \cdot |Y_{1,n}\ Y_{2,1}\rangle + ... \\ +\psi_1\left(Y_{1,n}\right) \cdot \psi_2\left(Y_{2,n}\right) \cdot |Y_{1,n}\ Y_{2,n}\rangle \end{array} \tag{6}$$

where the probability of finding system (6) in state $|Y_{1,i}\ Y_{2,z}\rangle$ can be computed as

$$P\left(Y_{1,i}, Y_{2,z}\right) = \left|\psi\left(Y_{1,i}\right) \cdot \psi\left(Y_{2,z}\right)\right|^2 \tag{7}$$

under normalization conditions

$$\sum_{i=1}^{n} \sum_{z=1}^{n} \left|\psi\left(Y_{1,i}\right) \cdot \psi\left(Y_{2,z}\right)\right|^2 = 1. \tag{8}$$

In (4-8), the procedure for the combination/joining of two wave probabilistic functions is shown. In the same way, the joining of more than two functions can be generalized.

2.1 Conditional combination of quantum systems

Let us rewrite equation (6) representing the well-known combination of two quantum states (4) and (5) into matrix form in the following way:

$$|\psi_{1,2}\rangle = \begin{bmatrix} |Y_{1,1}\rangle & |Y_{1,2}\rangle & . & |Y_{1,n}\rangle \end{bmatrix} \cdot \begin{bmatrix} \psi(Y_{1,1}) & 0 & . & 0 \\ 0 & \psi(Y_{1,2}) & . & 0 \\ . & . & . & . \\ 0 & 0 & . & \psi(Y_{1,n}) \end{bmatrix} \cdot$$
$$\cdot \begin{bmatrix} 1 & 1 & . & 1 \\ 1 & 1 & . & . \\ . & . & 1 & 1 \\ 1 & . & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \psi(Y_{2,1}) & 0 & . & 0 \\ 0 & \psi(Y_{2,2}) & . & 0 \\ . & . & . & . \\ 0 & 0 & . & \psi(Y_{2,n}) \end{bmatrix} \cdot \begin{bmatrix} |Y_{2,1}\rangle \\ |Y_{2,2}\rangle \\ . \\ |Y_{2,n}\rangle \end{bmatrix} \tag{9}$$

The weighted states given in (4) or (5) can be represented in matrix form by row or column vectors as follows:

$$|\psi_{1,2}\rangle = \begin{bmatrix} \psi(Y_{1,1}) \cdot |Y_{1,1}\rangle & \psi(Y_{1,2}) \cdot |Y_{1,2}\rangle & \psi(Y_{1,n}) \cdot |Y_{1,n}\rangle \end{bmatrix} \cdot P \cdot \begin{bmatrix} \psi(Y_{2,1}) \cdot |Y_{2,1}\rangle \\ \psi(Y_{2,1}) \cdot |Y_{2,1}\rangle \\ . \\ \psi(Y_{2,1}) \cdot |Y_{2,1}\rangle \end{bmatrix} \tag{10}$$

The newly introduced matrix P is given for standard combination of quantum systems (6) as:

$$P = \begin{bmatrix} 1 & 1 & . & 1 \\ 1 & 1 & . & . \\ . & . & 1 & 1 \\ 1 & . & 1 & 1 \end{bmatrix} \tag{11}$$

The component of matrix P on i, j-position can be interpreted as the transition probability between states $|Y_{1,i}\rangle$ and $|Y_{2,j}\rangle$ caused by external condition. In standard combination of quantum states (6) we suppose that all transitions between quantum states given in (4) and (5) are not affected by external environment (are not conditioned) and then the matrix P has the form given in (11).

In case there exist some external conditions or some transitions of states are not allowed, matrix P will model this situation and the form (10) with general matrix P will represent the *conditional combination of quantum states.*

2.2 Bell states as conditional combination of quantum states

The *Bell states* are a concept in quantum informatics and represent the simplest possible examples of quantum entanglement between two q-bits [5]. The qubits are usually thought to be spatially separated. They exhibit, nevertheless, perfect correlations which cannot be explained without quantum mechanics. The Copenhagen interpretation [8] originated from the consequences of a thought experiment introduced in 1935 by Einstein, Podolsky, and Rosen [2]. The thought experiment involves two quantum systems that interact with each other and are then separated so that they interact no longer. Then, the value (in quantum physics position or momentum) of one of the systems is measured, and due to the known relationship between the measured value of the first quantum system and the value of the second one, the observer is aware of the value in the second system. A measurement is made on the second system, and again, due to the relationship between the two systems, this value can then be known in the first quantum system [9].

Let us define two q-bits represented by two "bra-ket" forms:

$$|\psi_1\rangle = \alpha_0 \cdot |0\rangle_1 + \alpha_1 \cdot |1\rangle_1 \tag{12}$$

$$|\psi_2\rangle = \beta_0 \cdot |0\rangle_2 + \beta_1 \cdot |1\rangle_2 . \tag{13}$$

Combination of the two q-bits (12) and (13) yields the common wave probabilistic function that is defined with the help of the Kronecker product:

$$|\psi_{1,2}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = \alpha_0 \cdot \beta_0 \cdot |0,0\rangle + \alpha_0 \cdot \beta_1 \cdot |0,1\rangle + \alpha_1 \cdot \beta_0 \cdot |1,0\rangle + \alpha_1 \cdot \beta_1 \cdot |1,1\rangle \tag{14}$$

It is evident that the complex numbers $\alpha_0, \alpha_1, \beta_0, \beta_1$ in equation (14) that reach the well-known Bell state cannot be found [5]:

$$|\psi_{1,2}\rangle_B = \gamma_1 \cdot |0,1\rangle + \gamma_2 \cdot |1,0\rangle . \tag{15}$$

If we use the conditioned combination (9) together with the general transition matrix P defined as:

$$P = \begin{bmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{bmatrix} \tag{16}$$

we can then write the conditional combination of q-bits (12) and (13) in the following form:

$$\begin{array}{l} |\psi_{1,2}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = p_{1,1} \cdot \alpha_0 \cdot \beta_0 \cdot |0,0\rangle + \\ + p_{1,2} \cdot \alpha_0 \cdot \beta_1 \cdot |0,1\rangle + p_{2,1} \cdot \alpha_1 \cdot \beta_0 \cdot |1,0\rangle + p_{2,2} \cdot \alpha_1 \cdot \beta_1 \cdot |1,1\rangle \end{array} \tag{17}$$

It can be easily understood that the following conditions must be fulfilled to achieve the Bell state (15):

$$\begin{array}{l} p_{1,1} = 0 \\ p_{2,2} = 0 \\ p_{1,2} \cdot \alpha_0 \cdot \beta_1 = \gamma_1 \\ p_{2,1} \cdot \alpha_1 \cdot \beta_0 = \gamma_2 \end{array} \tag{18}$$

Basic assumption for the Bell state is that transitions $p_{1,1}$ and $p_{2,2}$ are denied. So the Bell states could be seen as practical examples of conditional combination of two q-bits introduced in this paper.

2.3 Sequences of conditionally combined quantum systems

Let us suppose that there exist two sequences of N same quantum systems, each of them having n quantum states. Both sequences can be written in S_1, S_2 vector forms that can be decomposed into matrix representations:

$$S_1 = \left[\psi\left(Y_{1,1}\right)\cdot\left|Y_{1,1}\right\rangle_1 + ... + \psi\left(Y_{1,n}\right)\cdot\left|Y_{1,n}\right\rangle_1 ,..\psi\left(Y_{1,1}\right)\cdot\left|Y_{1,1}\right\rangle_N + \right.$$
$$\left. + ... + \psi\left(Y_{1,n}\right)\cdot\left|Y_{1,n}\right\rangle_N\right] =$$
$$= \left[\begin{array}{cccc}\psi\left(Y_{1,1}\right) & . & . & \psi\left(Y_{1,n}\right)\end{array}\right]\cdot\left[\begin{array}{cccc}\left|Y_{1,1}\right\rangle_1 & . & \left|Y_{1,1}\right\rangle_{N-1} & \left|Y_{1,1}\right\rangle_N \\ . & . & . & . \\ . & . & . & . \\ \left|Y_{1,n}\right\rangle_1 & . & \left|Y_{1,n}\right\rangle_{N-1} & \left|Y_{1,n}\right\rangle_N\end{array}\right] \tag{19}$$

$$S_2 = \left[\psi\left(Y_{2,1}\right)\cdot\left|Y_{2,1}\right\rangle_1 + ... + \psi\left(Y_{2,n}\right)\cdot\left|Y_{2,n}\right\rangle_1 ,..\psi\left(Y_{2,1}\right)\cdot\left|Y_{2,1}\right\rangle_N + \right.$$
$$\left. + ... + \psi\left(Y_{2,n}\right)\cdot\left|Y_{2,n}\right\rangle_N\right] =$$
$$= \left[\begin{array}{cccc}\psi\left(Y_{2,1}\right) & . & . & \psi\left(Y_{2,n}\right)\end{array}\right]\cdot\left[\begin{array}{cccc}\left|Y_{2,1}\right\rangle_1 & . & \left|Y_{2,1}\right\rangle_{N-1} & \left|Y_{2,1}\right\rangle_N \\ . & . & . & . \\ . & . & . & . \\ \left|Y_{2,n}\right\rangle_1 & . & \left|Y_{2,n}\right\rangle_{N-1} & \left|Y_{2,n}\right\rangle_N\end{array}\right] \tag{20}$$

The sequence of quantum systems (19) or (20) was first applied in quantum modeling in [4, 6] where a set of same quantum systems was supposed and the phase parameters defined the link between different quantum systems.

The question is how the set of quantum systems looks like if we extend the dimensionality of the problem, and we will suppose to have available both set of quantum systems (19) and (20) in the same space.

The general combination of both sequences (19) and (20) can be modeled in matrix representation as:

$$S = S_1 \cdot P \cdot S_2^T \tag{21}$$

where P is N-by-N transition matrix determining the quantum mixture rule or, in other words, conditional combinations among the sequences (19) and (20).

If matrix P is, for example, diagonal one, there will be possible only the combinations between i-th quantum system in sequence (19) and i-th quantum system in sequence (20):

$$\begin{aligned}\left|\psi_{1,2}\right\rangle_{i,i} = {} & \psi_1\left(Y_{1,1}\right)\cdot\psi_2\left(Y_{2,1}\right)\cdot\left|Y_{1,1}\ Y_{2,1}\right\rangle_{i,i} + \\ & + ... + \psi_1\left(Y_{1,1}\right)\cdot\psi_2\left(Y_{2,n}\right)\cdot\left|Y_{1,1}\ Y_{2,n}\right\rangle_{i,i} + \\ & + ... + \psi_1\left(Y_{1,n}\right)\cdot\psi_2\left(Y_{2,1}\right)\cdot\left|Y_{1,n}\ Y_{2,1}\right\rangle_{i,i} + \\ & + ... + \psi_1\left(Y_{1,n}\right)\cdot\psi_2\left(Y_{2,n}\right)\cdot\left|Y_{1,n}\ Y_{2,n}\right\rangle_{i,i} .\end{aligned} \tag{22}$$

It means that there will be available only N combined quantum systems (22).

If matrix P has non-diagonal elements, it means that there are also possible combinations between i-th quantum system in (19) along with j-th quantum system in (20). Under the normalization condition, the absolute values of matrix P elements gives us information about the probability of finding such a combination within all the combined quantum systems. Phases of matrix P elements represent the dependences among set of combined quantum systems.

2.4 Example of conditionally combined two q-bits sequences

Let us define two N-dimensional sequences of q-bits with respect to equations (19) and (20) as follows:

$$S_1 = \begin{bmatrix} \alpha_0 & \alpha_1 \end{bmatrix} \cdot \begin{bmatrix} |0\rangle_{1,1} & . & . & |0\rangle_{1,N} \\ |1\rangle_{1,1} & . & . & |1\rangle_{1,N} \end{bmatrix} \tag{23}$$

$$S_2 = \begin{bmatrix} \beta_0 & \beta_1 \end{bmatrix} \cdot \begin{bmatrix} |0\rangle_{2,1} & . & . & |0\rangle_{2,N} \\ |1\rangle_{2,1} & . & . & |1\rangle_{2,N} \end{bmatrix} \tag{24}$$

It is supposed, by example, that the environmental combination expressed through matrix P is as follows:

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & . & 0 \\ . & 0 & . & 1 \\ 0 & . & 0 & 1 \end{bmatrix}. \tag{25}$$

Then, the conditional combination of these two sequences S_1, S_2 yields into the new sequence of combined quantum systems:

$$\begin{aligned} S \;=\; & \Big[\Big(\alpha_0 |0\rangle_{1,1} + \alpha_1 |1\rangle_{1,1}\Big) \cdot \Big(\beta_0 |0\rangle_{2,1} + \beta_1 |1\rangle_{2,1}\Big) \\ & \Big(\alpha_0 |0\rangle_{1,1} + \alpha_1 |1\rangle_{1,1} + \alpha_0 |0\rangle_{1,2} + \alpha_1 |1\rangle_{1,2}\Big) \cdot \Big(\beta_0 |0\rangle_{2,2} + \beta_1 |1\rangle_{2,2}\Big). \\ & ..\Big(\alpha_0 |0\rangle_{1,N-1} + \alpha_1 |1\rangle_{1,N-1} + \alpha_0 |0\rangle_{1,N} + \alpha_1 |1\rangle_{1,N}\Big) \cdot \\ & \Big(\beta_0 |0\rangle_{2,N} + \beta_1 |1\rangle_{2,N}\Big)\Big]. \end{aligned} \tag{26}$$

It is evident that components of the sequence (26) reach various combinations of different order of q-bits from sequences S_1, S_2.

3. Conclusion

Many monographs are devoted to the areas of quantum physics or quantum informatics, e.g. [1], there is a given practical example that the whole system is more complex than composition of its different parts. As an example, the well-known Bell state [5] is presented. If we take two q-bits, it is not mathematically possible to compose them in a way to reach the Bell state. Nevertheless, the Bell state really exists and can be prepared on quantum level, and its existence was experimentally proven on many occasions [2].

In this paper, we analyzed this problem and tried to find the principles that can open the door to more sophisticated quantum system compositions that should overcome the above mentioned bottleneck. The composition method can make it possible to put together also fully entangled states like, e.g. the Bell states, from basic quantum components.

We analyzed the standard combination method presented, e.g. in [4], and decomposed it into matrix form. As the by-product of this decomposition the transition matrix P was introduced. The transition matrix P can model external environmental conditions, e.g. states of which it cannot be combined (the exclusion principle) or which states are preferred to be combined (the gravitational force), etc. All these features add new information to the quantum combination rules, and can play a significant role in modeling complex information circuits / systems [7, 10]. As a suitable HW tool seems to be FPGA [11].

The presented methods can be also extended into more complex quantum systems, such as, e.g. sequences of quantum systems, etc. In these examples, the transition matrix P can model the combination rule among different quantum components to achieve a complex whole.

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