ON A PARTICULAR CLASS OF LATTICE-VALUED POSSIBILISTIC DISTRIBUTIONS

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Abstract: Investigated are possibilistic distributions taking as their values sequences from the infinite Cartesian product of identical copies of a fixed finite subset of the unit interval of real numbers. Uniform and lexicographic partial orderings on the space of these sequences are defined and the related complete lattices introduced. Lattice-valued entropy function is defined in the common pattern for both the orderings, naturally leading to different entropy values for the particular ordering applied in the case under consideration. The mappings on possibilistic distributions with uniform partial ordering under which the corresponding entropy values are conserved as well as approximations of possibilistic distributions with respect to this entropy function are also investigated.

Key words: *Possibilistic distribution, possibilistic measure, lattice-valued uncertainty degrees, complete lattice, Boolean ordering, lexicographic ordering, possibilistic entropy function*

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1. Introduction and Motivation

The oldest mathematical models of uncertainty quantification and processing were those dealing with uncertainty in the sense of randomness, in other terms, models based on probability theory and mathematical statistics. For two reasons, uncertainty and probability have been quantified by real numbers from the unit interval $[0, 1]$. First, this space of values was preferred in general because of the fact that in this interval a great number of mathematical operations and notions are definable and applicable, which may be used as an aid when processing uncertainty (i.e., randomness and probability in this case) values during various theoretical studies and practical applications. The other reason leading to preferences for unit interval as an appropriate space for probability values consists in the fact that relative

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frequences of the occurences of random events in sequences (statistically independent, let us say) random samples are quantified by real numbers from $[0,1]$ simply because they are defined in this way. Hence, the aim to apply relative frequences as reasonable estimations of the related probability values may be taken as a support for the idea to quantify also probability values within this interval.

Quantification of uncertainty degrees by real numbers (in particular, in $[0,1]$) implied that all such values (probability values or expected values, in particular) could be related to each other by binary relation "greater than", "smaller than" or "equal to" ("identical with"). However, the application of uncertainty degrees with incomparable values involved serious problems: a projection of these uncertainty degrees onto real line, or even unit interval, become possible only under very strong and, because of the nature of the problem to be solved, hard to accept further conditions. Remember, e.g., the case of uncertainty degrees quantified by various subsets of some space, so that these subsets need not be nested (i.e., need not be completely ordered by the relation of set inclusion).

A theoretical solution to these problems consisted in replacing the unit interval as the space for uncertainty degrees or values by a weaker structure. The well-known approach in this sense is that leading to lattice-valued possibilistic measures initiated by J. A. Goguen in [5] and theoretically pursued by G. DeCooman in [2]. Further weakening of the conditions imposed on the structures in which possibility degrees take their values would lead to values in lattices (not necessarily complete) and to various semilattices, almost-lattices, etc.

In this paper we will not follow the pattern of further weakening of the conditions imposed on the structure in which possibility degrees take their values. Rather we will consider lattice-valued possibilistic distributions and measures taking their values in a particular complete lattice defined by the infinite countable Cartesian product of identical finite complete lattices, each of them being defined by a finite subset Q of the unit interval $[0,1]$ such that $0,1 \in Q$ holds and equipped by the standard linear ordering $\leq$ on $[0,1]$. So, the support of our complete lattice will be defined by the set $Q^\infty = \{\langle x_1, x_2, \dots \rangle : x_i \in Q, i = 1, 2, \dots\}$. We may also write $Q^\infty = \mathbb{X}_{i=1}^{\infty} Q_i$, $Q_i = Q, i \in \mathcal{N} = \{1, 2, \dots\}$. Two partial orderings $\leq_{\mathcal{T}}$ and $\leq_{\mathcal{L}}$ will be defined on Q^∞, namely, for $\boldsymbol{x} = \langle x_1, x_2, \dots \rangle, \boldsymbol{y} = \langle y_1, y_2, \dots \rangle \in Q^\infty, \boldsymbol{x} \leq_{\mathcal{T}} \boldsymbol{y}$ holds iff $x_i \leq y_i$ is valid for each $i \in \mathcal{N}$, and $\boldsymbol{x} \leq_{\mathcal{L}} \boldsymbol{y}$ holds iff either $\boldsymbol{x} = \boldsymbol{y}$ or $x_{i_0} < y_{i_0}$ holds for $i_0 = \min\{i \in \mathcal{N} : x_i \neq y_i\}$. Both the relations $\leq_{\mathcal{T}}$ and $\leq_{\mathcal{L}}$ define partial orderings on Q^∞, the first may be called the *Boolean* one, and the other the lexicographical.

The related complete lattices are denoted by $\mathcal{T} = \langle Q^\infty, \leq_{\mathcal{T}} \rangle$ and $\mathcal{L} = \langle Q^\infty, \leq_{\mathcal{L}} \rangle$ and $\mathcal{T}$-possibilistic distributions ($\mathcal{L}$-possibilistic distributions, resp.) on a space Ω are defined, as mappings $\pi : \Omega \to Q^\infty$ such that $\bigvee^{\mathcal{T}} \{\pi(\omega) : \omega \in \Omega\} = \mathbf{1}_{\mathcal{T}}$ ($\bigvee^{\mathcal{L}} \{\pi(\omega) : \omega \in \Omega\} = \mathbf{1}_L$, resp.) holds, where $\mathbf{1}_{\mathcal{T}} = \langle 1, 1, \dots \rangle \in \Omega^\infty$ holds. In order to quantify the total amount of uncertainty contained in a $\mathcal{T}$- or $\mathcal{L}$-possibilistic distribution π on Ω we introduce a rather simple $\mathcal{T}$- or $\mathcal{L}$-valued possibilistic entropy function $I^{\mathcal{T}}$ or $I^{\mathcal{L}}$. This entropy function is nontrivial only in the case of unimodal possibilistic distribution π (i.e., if $\pi(\omega_0) = \mathbf{1}_{\mathcal{T}}$ for at most one $\omega_0 \in \Omega$), but if this condition is satisfied, the achieved results seem to be rather interesting and nontrivial. The reader is referred to more detailed explanations and examples in Sections 3 and 4.

Starting in Section 5, our analysis will be focused on $\mathcal{T}$-valued possibilistic distributions and measures. Proposed are real-valued characteristics of the $\mathcal{T}$-valued entropy function $I^{\mathcal{T}}$ which conserve at least some properties of Q^{∞}-valued (i.e., infinite vector valued) function $I^{\mathcal{T}}$, and these characteristics seem to be rather sensitive to differences between $\mathcal{T}$-possibilistic distributions. Analyzed are also possibilistic products of $\mathcal{T}$-possibilistic distributions and approximations of such distributions.

Let us summarize our intention again: to analyze a lattice-valued possibilistic distributions and measures the values of which are not, in general, comparable with each other, hence, which take their values in a complete lattice which is not linear, but which is supported by a structure rich enough to allow for the deduction of nontrivial and perhaps important and interesting results. A very sketched outlook to some possible ways of further analysis of the problems submitted below can be found in the concluding Section 9 of this paper.

The paper is written on an almost self-explanatory level, just some elementary preliminaries from set theory, Boolean algebras and lattice theory in the extent of introductory chapters of [1, 4] or [10] (or some more recent textbooks or monographs) seem to be recommended. Zadeh's pioneering ideas concerning fuzzy sets and possibilistic measures can be found in [12, 13], interesting discussions on mutual relations among various notions of uncertainty are presented in [3].

2. Two Particular Complete Lattices

Let $Q = Q_K = \{0 < \lambda_1 < \lambda_2 < \dots \lambda_K < 1\}, K = 0, 1, 2, \dots,$ denote a $K+2$-tuple of real numbers from the unit interval $[0, 1]$, where $\leq$ is the standard linear ordering on $[0, 1]$, let $Q^{\infty} = \{\boldsymbol{x} = \langle x_1, x_2, \dots \rangle : x_i \in Q$ for each $i \in \mathcal{N} = \{1, 2, \dots\}\}$ denote the set of all infinite sequences of real numbers from Q. Let us introduce the two following binary relations $\leq_T$ and $\leq_L$ on Q^{∞}.

(i) For each $\boldsymbol{x}, \boldsymbol{y} \in Q^{\infty}, \boldsymbol{x} = \langle x_1, x_2, \dots \rangle, \boldsymbol{y} = \langle y_1, y_2, \dots \rangle, \boldsymbol{x} \leq_T \boldsymbol{y}$ holds iff $x_i \leq y_i$ holds for each $i \in \mathcal{N}$.

(ii) For the same $\boldsymbol{x}, \boldsymbol{y} \in Q^{\infty}, \boldsymbol{x} \leq_L \boldsymbol{y}$ holds iff either $\boldsymbol{x} = \boldsymbol{y}$ (i.e., $x_i = y_i$ for each $i \in \mathcal{N}$), or iff $x_{i_0} < y_{i_0}$ holds for $i_0 = \min\{i \in \mathcal{N} : x_i \neq y_i\}$.

Lemma 2.1 Both the structures $\langle Q^{\infty}, \leq_T \rangle$ and $\langle Q^{\infty}, \leq_L \rangle$ define complete lattices on Q^{∞}.

Proof: First of all, let us prove that both $\leq_T$ and $\leq_L$ define partial orderings on Q^{∞}. For $\leq_T$, $\boldsymbol{x} \leq_T \boldsymbol{x}$ for each $\boldsymbol{x} \in Q^{\infty}$ (reflexivity) and the implication "if $\boldsymbol{x} \leq_T \boldsymbol{y}$ and $\boldsymbol{y} \leq_T \boldsymbol{x}$ hold together, then $\boldsymbol{x} = \boldsymbol{y}$" (antisymmetry) are obviously valid. If $\boldsymbol{x} \leq_T \boldsymbol{y}$ and $\boldsymbol{y} \leq_T \boldsymbol{z}$ holds for $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in Q^{\infty}$, then for each $i \in \mathcal{N}, x_i \leq y_i$ and $y_i \leq z_i$ holds, hence, $x_i \leq z_i$ follows, so that $\boldsymbol{x} \leq_T \boldsymbol{z}$ is valid (transitivity) and $\leq_T$ defines a partial ordering on Q^{∞}.

For $\leq_L, \boldsymbol{x} \leq_L \boldsymbol{x}$ holds for each $\boldsymbol{x} \in Q^{\infty}$ by definition. Supposing that $\boldsymbol{x}, \boldsymbol{y} \in Q^{\infty}, \boldsymbol{x} \neq \boldsymbol{y}$, but $\boldsymbol{x} \leq_L \boldsymbol{y}$ and $\boldsymbol{y} \leq_L \boldsymbol{x}$ hold together, we arrive at the conclusion that for $i_0 = \min\{i \in \mathcal{N} : x_i \neq y_i\}$ the inequalities $x_{i_0} < y_{i_0}$ and $y_{i_0} < x_{i_0}$ hold together – a contradiction, so that $\boldsymbol{x} = \boldsymbol{y}$ follows. Let $\boldsymbol{x} \leq_L \boldsymbol{y}$ and $\boldsymbol{y} \leq_L \boldsymbol{z}$ be the case for some $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in Q^\infty$. If $\boldsymbol{x} = \boldsymbol{y}$ or $\boldsymbol{y} = \boldsymbol{z}$, then $\boldsymbol{x} \leq_L \boldsymbol{z}$ reduces to $\boldsymbol{x} \leq_L \boldsymbol{y}$ or to $\boldsymbol{y} \leq_L \boldsymbol{z}$, so that $\boldsymbol{x} \leq_L \boldsymbol{z}$ is trivially valid. Let $\boldsymbol{x} <_L \boldsymbol{y}$ and $\boldsymbol{y} <_L \boldsymbol{z}$ hold, so that the values $i_0 = \min\{i \in \mathcal{N} : x_i \neq y_i\}$ and $j_0 = \min\{j \in \mathcal{N} : y_j \neq \boldsymbol{z}_j\}$ are defined. If $i_0 = j_0$, then $x_i = y_i = z_i$ holds for each $i < i_0$ and the inequality $x_{i_0} < y_{i_0} < z_{i_0}$ is valid, so that $\boldsymbol{x} <_l \boldsymbol{z}$ follows. If $i_0 < j_0$ is the case, then $x_{i_0} < y_{i_0} = z_{i_0}$ holds, hence, $x_{i_0} < z_{i_0}$ follows and the relation $\boldsymbol{x} <_L \boldsymbol{z}$ is proved.

If $i_0 > j_0$ is the case, then $x_{j_0} = y_{j_0} < z_{j_i}$ holds, so that $\boldsymbol{x} <_L \boldsymbol{z}$ holds as well. Hence, the transitivity of the relation $\leq_L$ is proved, so that $\leq_L$ defines a partial ordering on Q^∞.

In order to prove that both $\langle Q^\infty, \leq_T\rangle$ and $\langle Q^\infty, \leq_L\rangle$ define complete lattices on Q^∞ we have to prove that for each $\emptyset \neq A \subset Q^\infty$ the supremum $\bigvee^T A$ and the infimum $\bigwedge^T A$ (induced by $\leq_T$ on $\mathcal{P}(Q^\infty)$), as well as the supremum $\bigvee^L A$ and the infimum $\bigwedge_L A$ (induced by $\leq_L$ on $\mathcal{P}(\Omega^\infty)$) are defined (written in more detail, $\bigvee^T A = \bigvee^T_{\boldsymbol{x}\in A} \boldsymbol{x}$, $\bigwedge^T A = \bigwedge^T_{\boldsymbol{x}\in A} \boldsymbol{x}$, and similarly for $\bigvee^L A$ and $\bigwedge^L A$).

Considering $\leq_T$ and $\emptyset \neq A \subset Q^\infty$, we may easily verify that

$$\bigvee^T A = \left\{\bigvee_{\boldsymbol{x}\in A} x_i\right\}_{i=1}^{\infty} = \left\langle \bigvee_{\boldsymbol{x}\in A} x_1, \bigvee_{\boldsymbol{x}\in A} x_2, \bigvee_{\boldsymbol{x}\in A} x_3, \dots \right\rangle \tag{2.1}$$

and

$$\bigwedge^T A = \left\{\bigwedge_{\boldsymbol{x}\in A} x_i\right\}_{i=1}^{\infty} = \left\langle \bigwedge_{\boldsymbol{x}\in A} x_1, \bigwedge_{\boldsymbol{x}\in A} x_2, \bigwedge_{\boldsymbol{x}\in A} x_3, \dots \right\rangle, \tag{2.2}$$

obviously, the supremum and infimum operations in $\bigvee_{\boldsymbol{x}\in A} x_i$ and $\bigwedge_{\boldsymbol{x}\in A} x_i$ are induced by the standard linear ordering in $[0,1]$, here reduced to $Q = \{0 < \lambda_1 < \lambda_2 < \dots < \lambda_K < 1\}$.

For $\leq_L$ we proceed as follows. Given $\emptyset \neq A \subset Q^\infty$, set $x_1^0 = \bigvee_{\boldsymbol{x}\in A} x_1$ and set $A_1 = \{\boldsymbol{x} \in A : x_1 = x_1^0\}$. As the set Q is finite, supremum (w.r.to $\leq$ on $[0,1]$) of each nonempty subset of Q is identical with some element of the subset of Q in question, so that the set A_1 is nonempty. Set $x_2^0 = \bigvee_{\boldsymbol{x}\in A_1} x_2$, set $A_2 = \{\boldsymbol{x} \in A_1 : x_2 = x_2^0\}$. Again, $A_2 \neq \emptyset$, so that we set $x_3^0 = \bigvee_{\boldsymbol{x}\in A_2} x_3$ and $A_3 = \{\boldsymbol{x} \in A_2 : x_3 = x_3^0\}$, etc. In general, if $A_n \subset A_{n-1} \subset \dots \subset A_1 \subset A$ and $x_1^0, x_2^0, \dots, x_n^0$ are defined, set $x_{n+1}^0 = \bigvee_{\boldsymbol{x}\in A_n} x_{n+1}$ and $A_{n+1} = \{\boldsymbol{x} \in A_n : x_{n+1} = x_{n+1}^0\}$.

As can be easily seen, for each $i \in \mathcal{N}$ the set $A_i \subset A$ is nonempty, hence, x_{i+1}^0 is defined. Consequently, the intersection $\bigcap_{i=1}^\infty A_i$ is nonempty and $\boldsymbol{x}^0 = \langle x_1^0, x_2^0, \dots\rangle$ is in this set. Let $\boldsymbol{x}^* \in Q^\infty$ be a sequence, different from $\boldsymbol{x}^0$, so that there exists $i_0 \in \mathcal{N}$ such that $x_{i_0}^0 \neq x_{i_0}^*$. In this case, however, $x_{i_0}^* \neq \bigvee_{\boldsymbol{x}\in A_{i_0-1}} x_{i_0}$, hence, $\boldsymbol{x}^*$ cannot be in A_{i_0}, so that $\boldsymbol{x}^*$ is not in $\bigcap_{i=1}^\infty A_i$. So, $A^\infty = \bigcap_{i=1}^\infty A_i = \{x^0\} = \{\langle x_1^0, x_2^0, \dots\rangle\}$ and $\boldsymbol{x}^* \leq_L \boldsymbol{x}^0$ holds for each $\boldsymbol{x}^* \in A$. Consequently, $\boldsymbol{x}^0 = \bigvee^L A$ follows. If A is finite, then $\boldsymbol{x}^0 \in A$ holds, for infinite A this need not be the case. For $\bigwedge^L A$ the construction is completely dual, setting $y_1^0 = \bigwedge_{\boldsymbol{x}\in A} x_1, B_1 = \{\boldsymbol{x} \in A : y_1 = y_1^0\}$, $y_2^0 = \bigwedge_{\boldsymbol{y}\in B_0} y_2$, $B_2 = \{\boldsymbol{y} \in B_1 : y_2 = y_2^0\}$, etc. The sequence $\boldsymbol{y}^0 = \langle y_1^0, y_2^0, \dots\rangle \in A$ then defines the infimum $\bigwedge_L A$. Hence, also $\langle Q^\infty, \leq_L\rangle$ defines a complete lattice and the assertion is proved. $\square$

The following remarks concerning both the complete lattices are perhaps worth being noted explicitly. Taking $K = O$, we obtain that $Q^\infty = Q_0^\infty = \{0,1\}^\infty$ and $\langle\{0,1\}^\infty, \leq_T\rangle$ is particular case of complete lattices under consideration. As can be easily seen, this complete lattice is isomorphic to the Boolean algebra of all subsets of $\mathcal{N}$ with respect to standard set operations and to the set inclusion as partial ordering, the only what is to be done is to identify subsets of $\mathcal{N}$ with their characteristic functions or identifiers (infinite $0-1$ sequences, in this particular case). Hence, as subsystems of the power-set $\mathcal{P}(\mathcal{N})$ are not closed w.r.to unions (i.e., suprema in $\langle\mathcal{P}(\mathcal{N}), \subseteq\rangle$) in the sense that for some $\mathcal{A} \subset \mathcal{P}(\mathcal{N})$ the set $\bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A$ is not in $\mathcal{A}$, the same is valid for each $\langle Q^\infty, \leq_T\rangle$ for no matter which $K \geq 0$. On the other side, as proved in Lemma 2.1, for $\leq_L$ and for finite set A the supremum $\bigvee^L A = \bigvee^L_{\boldsymbol{x} \in A} \boldsymbol{x}$ is always identical with some $\boldsymbol{x} \in A$ no matter which the value $K \subset 0, 1, 2, \dots$ may be. This fact is valid due to the finiteness of each set $Q = Q_K = \{0 < \lambda_1 < \lambda_2 < \cdots < \lambda_K < 1\}$. For infinite sets Q, say, $Q = \{0 < \lambda_1 < \lambda_2 < \cdots < 1\}$ the supremum value $\bigvee Q_0$ for $Q_0 \subset Q, Q_0 \neq Q$ need not be in Q, so that nor the sequence $\bigvee^L A$, even if perhaps defined, need not be in A. The partial ordering $\leq_L$ on Q^∞ is linear (complete) in the sense that for each $\boldsymbol{x}, \boldsymbol{y} \in Q^\infty, \boldsymbol{x} \neq \boldsymbol{y}$, either $\boldsymbol{x} <_L \boldsymbol{y}$ or $\boldsymbol{y} <_L \boldsymbol{x}$ holds. Indeed, let $i_0 = \min\{i \in \mathcal{N} : \boldsymbol{x}_i \neq y_i\}$; if $x_{i_0} < y_{i_0}$ holds, then $\boldsymbol{x} <_L \boldsymbol{y}$ is the case, if $y_{i_0} < x_{i_0}$ is valid, then $\boldsymbol{y} <_L \boldsymbol{x}$ is the case.

Lemma 2.2 For each $\emptyset \neq A \subset A^\infty$ the inequalities $\bigvee^L A \leq_T \bigvee^T A$ and $\bigvee^L A \leq_L \bigvee^T A$ are valid.

Proof: As shown in (2.1), for each $i \in \mathcal{N}$ the relation $(\bigvee^T A)_i = \bigvee_{\boldsymbol{x} \in A} x_i$ holds. As $(\bigvee^L A)_i$ is identical with x_i for some $\boldsymbol{x} \in A$, the relation $(\bigvee^T A)_i \geq (\bigvee^L A)_i$ holds for each $i \in \mathcal{N}$, hence, $\bigvee^L A \leq_T \bigvee^T A$ follows.

If $\bigvee^L A = \bigvee^T A$, also the second relation in Lemma 2.2 trivially holds. If $\bigvee^L A \neq \bigvee^T A$, set $i_0 = \min\{i \in \mathcal{N} : (\bigvee^L A)_i \neq (\bigvee^T A)_i\}$. Combining this fact with the first part of this proof, we obtain that $(\bigvee^L A)_{i_0} < (\bigvee^T A)_{i_0}$ holds, hence, $\bigvee^L A \leq_L \bigvee^T A$ follows. The assertion is proved. $\square$

Partial ordering $\leq_T$ will be called *Boolean,* as in the most simple case with $K = 0$, when $Q^\infty = \{0,1\}^\infty$ $\leq_T$ copies the inclusion in the Boolean algebra of all subsubsets of the set $\mathcal{N}$ of positive integers. Partial ordering $\leq_L$ will be called *lexicographical* and its inspiration by ordering of words in vocabularies or by ordering of binary or decadic codes of real numbers is obvious.

3. $\mathcal{T}$- and $\mathcal{L}$-Possibilistic Distributions and Possibilistic Entropy Function

Definition 3.1 Let $\mathcal{T} = \langle Q^\infty, \leq_T\rangle$ and $\mathcal{L} = \langle Q^\infty, \leq_L\rangle$ be the complete lattices defined in Section 2, let Ω be a nonempty set. A mapping $\pi : \Omega \to Q^\infty$ is called *$\mathcal{T}$-(valued) possibilistic distribution on* Ω, if $\bigvee^T_{\omega \in \Omega} \pi(\omega) = \langle 1, 1, \dots\rangle = \mathbf{1}_{\mathcal{T}}$. A mapping $\pi : \Omega \to Q^\infty$ is called *$\mathcal{L}$-valued possibilistic distribution on* Ω, if $\bigvee^L_{\omega \in \Omega} \pi(\omega) = \langle 1, 1, \dots\rangle = \mathbf{1}_{\mathcal{L}}$.

When seeking a lattice-valued modification of an entropy or uncertainty function applicable to possibilistic distributions $\pi : \Omega \to Q^{\infty}$, we take inspiration from the classical Shannon entropy function H. Given a finite or countable space $\Omega = \{\omega_1, \omega_2, \dots\}$ and a probability distrubution p on Ω, i.e., $p : \Omega \to [0,1]$ is such that $\sum_{i=1}^{\infty} p(\omega_i) = 1$, then Shannon entropy H of p is defined by (cf., e.g., [6])

$$\begin{aligned} H(p) &= -\sum_{i=1}^{\infty} p_i \log_2(p_i) = \sum_{i=1}^{\infty} p_i \log_2(1/p_i) = \\ &= \sum_{\omega\in\Omega} p(\omega) \log_2(1/p(\omega)), \end{aligned} \tag{3.1}$$

applying the convention $0 \log_L 0 = 0$. Hence, $H(p)$ is defined as the expected value of the decreasing (in $p(\omega)$) function $\log_2(1/p(\omega))$. Replacing this function by another nonincreasing function of $p(\omega)$, namely by the function $1 - p(\omega)$, we arrive at the function $\sum_{\omega\in\Omega} p(\omega) P(\Omega - \{\omega\})$, where P is the probability measure on $\mathcal{P}(\Omega)$ induced by p (cf. [9] and [11] for more detail).

In order to shift our model from probability distributions to $\mathcal{T}$ or $\mathcal{L}$-possibilistic distributions let us replace $P(\Omega - \{\omega\})$ by $\Pi^T(\Omega - \{\omega\}) = \bigvee^T_{\omega_0\in\Omega, \omega_0\neq\omega} \pi(\omega_0)$ (by $\Pi^L(\Omega - \{\omega\}) = \bigvee^L_{\omega_0\in\Omega,\omega_0\neq\omega} \pi(\omega_0)$, resp.) and the product by $\wedge^T$ ($\wedge^L$, resp.), so arriving at functions

$$I^T(\pi) = \bigvee\nolimits^T_{\omega\in\Omega} [\pi(\omega) \wedge^T \Pi^T(\Omega - \{\omega\})] \tag{3.2}$$

and

$$I^L(\pi) = \bigvee\nolimits^L_{\omega\in\Omega} [\pi(\omega) \wedge^L \Pi^L(\Omega - \{\omega\})]. \tag{3.3}$$

These quantifications of uncertainty are not too fine or flexible, if $I^T(\pi) = \mathbf{1}_{\mathcal{T}}$ (if $I^L(\pi) = \mathbf{1}$, resp.). Indeed, if $\pi(\omega_1) = \pi(\omega_2) = \mathbf{1}$ for $\omega_1, \omega_2 \in \Omega, \omega_1 \neq \omega_2$, then for each $\omega \in \Omega$ either ω_1 or ω_2 is in $\Omega - \{\omega\}$, so that $\Pi(\Omega - \{\omega\}) = \mathbf{1}$ and

$$I^T = \bigvee\nolimits^T_{\omega\in\Omega} [\pi(\omega) \wedge \Pi(\Omega - \{\omega\})] = \bigvee\nolimits^T \pi(\omega) = \mathbf{1} \tag{3.4}$$

holds, the relation $I^L(\pi) = \mathbf{1}$ being valid as well (a refinement of this entropy function being suggested in [7] and [8]). Nevertheless, let us apply I^T and I^L in what follows. $\mathcal{T}$- and $\mathcal{L}$-possibilistic distributions on Ω will be called *single*, if there is just one $\omega \in \Omega$ such that $\pi(\omega) = \mathbf{1}$.

Before going on with a deeper analysis of $\mathcal{T}$- and $\mathcal{L}$-possibilistic distributions, let us introduce a very simple example for illustration.

Let $K = 1$, so that $Q = Q_1 = \{0 < \lambda_1 < 1\}$, let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, let $\pi : \Omega \to Q^{\infty}$ be defined in this way:

$$\begin{aligned} \pi(\omega_1) &= \langle \lambda_1, \lambda_1, 1, 1, 1, \dots \rangle = \langle \lambda, \lambda, 1 \rangle, \\ \pi(\omega_2) &= \langle 1, 0, \lambda_1, 1, 1, \dots \rangle = \langle 1, 0, \lambda, \rangle, \\ \pi(\omega_3) &= \langle 1, 1, 1, 1, 1, \dots \rangle = \langle 1, 1, 1, \rangle. \end{aligned} \tag{3.5}$$

In order to simplify our notation we omit the values $(\pi(\omega_i))_j$ for $j \geq 4$, as these values are supposed to be 1, and we write λ instead of λ_1. Obviously,

$$\bigvee\nolimits_{\omega\in\Omega}^{T} \pi(\omega_1) = \pi(\omega_1) \vee^T \pi(\omega_2) \vee^T \pi(\omega_3) = \langle 1,1,1\rangle = \mathbf{1} \bigvee\nolimits_{\omega\in\Omega}^{L} \pi(\omega), \tag{3.6}$$

holds, so that π defines a $\mathcal{T}$-as well as an $\mathcal{L}$-possibilistic distribution on Ω. For $\pi^T(\Omega - \{\omega\})$ we obtain that

$$\begin{aligned}
\Pi^T(\Omega - \{\omega_1\}) &= \pi(\omega_2) \vee^T \pi(\omega_3) = \langle 1,0,\lambda\rangle \vee^T \langle 1,1,1\rangle = \langle 1,1,1\rangle,\\
\Pi^T(\Omega - \{\omega_2\}) &= \pi(\omega_1) \vee^T \pi(\omega_3) = \langle \lambda,\lambda,1\rangle \vee^T \langle 1,1,1\rangle = \langle 1,1,1\rangle,\\
\Pi^T(\Omega - \{\omega_3\}) &= \pi(\omega_1) \vee^T \pi(\omega_2) = \langle \lambda,\lambda,1\rangle \vee^T \langle 1,0,\lambda\rangle = \langle 1,\lambda,1\rangle.
\end{aligned} \tag{3.7}$$

Hence,

$$\begin{aligned}
I^T(\pi) &= \bigvee\nolimits_{i=1}^{T,3} [\pi(\omega_i) \wedge \Pi(\Omega - \{\omega_j\})] =\\
&= (\langle \lambda,\lambda,1\rangle \wedge^T \langle 1,1,1\rangle) \vee^T (\langle 1,0,\lambda\rangle \wedge \langle 1,1,1\rangle) \vee^T (\langle 1,1,1\rangle \wedge_T \langle 1,\lambda,1\rangle) =\\
&= \langle \lambda,\lambda,1\rangle \vee^T \langle 1,0,\lambda\rangle \vee^T \langle 1,\lambda,1\rangle = \langle 1,\lambda,1\rangle \neq \mathbf{1}.
\end{aligned} \tag{3.8}$$

For $\Pi^L(\Omega - \{\omega\})$ we obtain that

$$\begin{aligned}
&\Pi^L(\Omega - \{\omega_1\}) = \pi(\omega_2) \vee^L \pi(\omega_3) = \langle 1,0,\lambda\rangle \vee^L \langle 1,1,1\rangle = \langle 1,1,1\rangle,\\
&\Pi^L(\Omega - \{\omega_2\}) = \pi(\omega_1) \vee^L \pi(\omega_3) = \langle \lambda,\lambda,1\rangle \vee^L \langle 1,1,1\rangle = \langle 1,1,1\rangle,\\
&\Pi^L(\Omega - \{\omega_3\}) = \pi(\omega_1) \vee^L \pi(\omega_2) = \langle \lambda,\lambda,1\rangle \vee^L \langle 1,0,\lambda\rangle = \langle 1,0,\lambda\rangle.
\end{aligned} \tag{3.9}$$

Hence,

$$\begin{aligned}
I^L(\pi) &= \bigvee\nolimits_{i=1}^{L,3} [\pi(\omega_i) \wedge^L (\Omega - \{\omega_i\}) =\\
&= (\langle \lambda,\lambda,1\rangle \wedge^L \langle 1,1,1\rangle) \vee^L (\langle 1,0,\lambda\rangle \wedge^L \langle 1,1,1\rangle) \vee^L\\
\vee^L &\ (\langle 1,1,1\rangle \wedge^L (\langle 1,0,\lambda\rangle) =\\
&= \langle \lambda,\lambda,1\rangle \vee^L \langle 1,0,\lambda\rangle \vee^L \langle 1,0,\lambda\rangle = \angle 1,0,\lambda\rangle.
\end{aligned} \tag{3.10}$$

The inequalities $I^L(\pi) <_T I^T(\pi)$ as well as $I^L(\pi) \leq_L I^T(\pi)$ are obviously valid.

Lemma 3.1 Let $\mathcal{T} = \langle Q^\infty, \leq_T\rangle$ and $\mathcal{L} = \langle Q^\infty, \leq_L\rangle$ be the complete lattices defined in Section 2, let Ω be a nonempty set, let $\pi_1, \pi_2 : \Omega \to Q^\infty$ be two mappings.

(i) If π_1 is a $\mathcal{T}$-possibilistic distribution on Ω and if $\pi_1 \leq_T \pi_2$ holds, i.e., if $\pi_1(\omega) \leq_T \pi_2(\omega)$ holds for each $\omega \in \Omega$, then π_2 defines a $\mathcal{T}$-possibilistic distribution on Ω and the relation $I^T(\pi_1) \leq_T I^T(\pi_2)$ holds.

(ii) If π_1 is an $\mathcal{L}$-possibilistic distribution on Ω and if $\pi_1 \leq_L \pi_2$ holds, i.e., if $\pi_1(\omega) \leq_L \pi_2(\omega)$ holds for each $\omega \in \Omega$, then π_2 defines an $\mathcal{L}$-possibilistic distribution on Ω and the relation $I^L(\pi_1) \leq_L I^L(\pi_2)$ holds.

Proof: Consider the case (i). If $\pi_1(\omega) \leq^T \pi_2(\omega)$ holds for each $\omega \in \Omega$, then $\Pi_1^T(A) = \bigvee_{\omega \in A}^T \pi_1(\omega) \leq_T \bigvee_{\omega \in A}^T \pi_2(\omega) = \Pi_2^T(A)$ holds for each $\emptyset \neq A \subset \Omega$, in particular, $\Pi_1^T(\Omega - \{\omega\}) \leq_T \Pi_2^T(\Omega - \{\omega\})$ holds for each $\omega \in \Omega$. Moreover, if $\boldsymbol{x}_1 \leq_T \boldsymbol{x}_2$ and $\boldsymbol{y}_1 \leq_T \boldsymbol{y}_2$ holds for some $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2 \in Q^\infty$, then $\boldsymbol{x}_1 \wedge^T \boldsymbol{y}_1 \leq_T \boldsymbol{x}_2 \wedge^T \boldsymbol{y}_2$ holds as well. Applying these trivial inequalities to $\boldsymbol{x}_i = \pi_i(\omega)$ and $\boldsymbol{y}_i = \Pi_i^T(\Omega - \{\omega\})$, $i = 1, 2$, we obtain that

$$
\begin{aligned}
I^T(\pi_1) &= \bigvee\nolimits_{\omega \in \Omega}^T [\pi_1(\omega) \wedge \Pi_1^T(\Omega - \{\omega\})] \leq_T \bigvee\nolimits_{\omega \in \Omega}^T [\pi_2(\omega) \wedge \Pi_2^T(\Omega - \{\omega\})] = \\
&= I^T(\pi_2)
\end{aligned}
\tag{3.11}
$$

holds. Consequently, $\bigvee_{\omega \in \Omega}^T \pi(\omega) = \mathbf{1}$, so that π_2 defines a $\mathcal{T}$-possibilistic distribution on Ω.

Analyzing, in more detail, the proof just introduced we can easily observe that only the most elementary properties of partial orderings and related supremum and infimum operations were applied and these properties are common for both the partial orderings $\leq_T$ and $\leq_L$ and for the induced operations $\bigvee^T, \bigwedge^T$ and $\bigvee^L, \bigwedge^L$. Hence, (ii) can be proved by a routine rewriting of the proof above, just replacing $\mathcal{T}$ by $\mathcal{L}, \bigvee^T$ by $\bigvee^L$ and $\bigwedge^T$ by $\bigwedge^L$. The assertion is proved. □

4. Some Properties of the Possibilistic Entropy Function for $\mathcal{T}$-Possibilistic Distributions

Let Ω be a nonempty set and let $\pi : \Omega \to Q^\infty$ be a $\mathcal{T}$-possibilistic distribution on Ω, hence, $\pi(\omega) = \langle (\pi(\omega))_1, (\pi(\omega))_2, (\pi(\omega))_3, \dots \rangle, (\pi(\omega))_j \in Q = \{0 < \lambda_1 < \lambda_2 < \dots < \lambda_k < 1\}$ holds for each $\omega \in \Omega$ and each $j = 1, 2, \dots$, and $\bigvee_{\omega \in \Omega}^T \pi(\omega) = \mathbf{1} = \langle 1, 1, \dots \rangle$ holds. As, due to (2.1),

$$
\bigvee\nolimits_{\omega \in \Omega}^T \pi(\omega) = \Big\langle \bigvee_{\omega \in \Omega} (\pi(\omega))_1, \bigvee_{\omega \in \Omega} (\pi(\omega))_2, \dots \Big\rangle = \Big\langle \bigvee_{\omega \in \Omega} (\pi(\omega))_j \Big\rangle_{j=1}^\infty
\tag{4.1}
$$

trivially holds, $\pi : \Omega \to Q^\infty$ defines a $\mathcal{T}$-possibilistic distribution on Ω, if for each $j = 1, 2, \cdots \bigvee_{\omega \in \Omega} (\pi(\omega))_j = 1$ holds, here $\bigvee$ denotes the standard linear ordering on $[0, 1]$ reduced to $Q \subset [0, 1]$. As the set Q is well-ordered and finite, the relation $\bigvee_{\omega \in \Omega} (\pi(\omega))_j = 1$ holds iff there exists $\omega_j \in \Omega$ such that $(\pi(\omega_j))_j = 1$. To conclude, we arrive at the following simple assertion.

Lemma 4.1 A mapping $\pi : \Omega \to Q^\infty$ defines a $\mathcal{T}$-possibilistic distribution on Ω iff there exists, for each $j \in \mathcal{N}$, an element $\omega_j \in \Omega$ such that $(\pi(\omega_j))_j = 1$.

Consequently, if π is a $\mathcal{T}$-possibilistic distribution on Ω, then for each $j \in \mathcal{N}$ the mapping $(\pi(\cdot))_j : \Omega \to Q$ defines a real-valued possibilistic distribution on Ω and the mapping $\Pi_j : \mathcal{P}(\Omega) \to Q$, defined by $\Pi_j(A) = \bigvee_{\omega \in A} (\pi(\omega))_j$ for each $A \subset \Omega$, defines the real-valued possibilistic measure on $\mathcal{P}(\Omega)$ induced by $(\pi(\omega))_j$ on Ω. In particular,

$$\Pi_j(\Omega - \{\omega\}) = \bigvee_{\omega_1 \in \Omega, \omega_1 \neq \omega} (\pi(\omega_1))_j. \tag{4.2}$$

Applying, again, (2.1) we obtain, for each $A \subset \Omega$, that

$$\begin{aligned}\Pi(A) &= \bigvee\nolimits^T_{\omega \in A} \pi(\omega) = \bigvee\nolimits^T_{\omega \in A} \langle (\pi(\omega))_1, (\pi(\omega))_2, (\pi(\omega))_3, \ldots \rangle = \\ &= \Big\langle \bigvee_{\omega \in A} (\pi(\omega))_1, \bigvee_{\omega \in A} (\pi(\omega))_2, \bigvee_{\omega \in A} (\pi(\omega))_3, \ldots \Big\rangle \\ &= \langle \Pi_1(A), \Pi_2(A), \Pi_3(A), \ldots \rangle \end{aligned} \tag{4.3}$$

according to the definition of $\Pi_j(A)$ a few lines above. In particular, for $A = \Omega - \{\omega\}$, $\omega \in \Omega$, we obtain that

$$\begin{aligned}\Pi(\Omega - \{\omega\}) &= \Big\langle \bigvee_{\omega_1 \in \Omega, \omega_1 \neq \omega} (\pi(\omega_1))_1, \bigvee_{\omega_1 \in \Omega, \omega_1 \neq \omega} (\pi(\omega_1))_2, \ldots \Big\rangle \\ &= \langle \Pi_1(\Omega - \{\omega\}), \Pi_2(\Omega - \{\omega\}), \Pi_3(\Omega - \{\omega\}), \ldots \rangle. \end{aligned} \tag{4.4}$$

Consequently,

$$\begin{aligned}\pi(\omega) \wedge^T \Pi(\Omega - \{\omega\}) &= \langle (\pi(\omega))_1 \wedge \Pi_1(\Omega - \{\omega\}), (\pi(\omega))_2 \wedge \Pi_2(\Omega - \{\omega\}), \\ &\qquad (\pi(\omega))_3 \wedge \Pi_3(\Omega - \{\omega\}) \ldots \rangle, \end{aligned} \tag{4.5}$$

and, according to the relation (3.2),

$$\begin{aligned} I^T(\pi) &= \bigvee\nolimits^T_{\omega \in \Omega} (\pi(\omega) \wedge \Pi(\Omega - \{\omega\})) = \\ &= \Big\langle \bigvee_{\omega \in \Omega} ((\pi(\omega))_1 \wedge \Pi_1(\Omega - \{\omega\})), \bigvee_{\omega \in \Omega} ((\pi(\omega))_2 \wedge \Pi_2(\Omega - \{\omega\})), \\ &\qquad \bigvee_{\omega \in \Omega} ((\pi(\omega))_3 \wedge \Pi_3(\Omega - \{\omega\})), \ldots \Big\rangle = \\ &= \langle I_1(\pi), I_2(\pi), \ldots \rangle, \end{aligned} \tag{4.6}$$

where $I_j(\pi) = \bigvee_{\omega \in \Omega} ((\pi(\omega))_j \wedge \Pi_j(\Omega - \{\omega\})), j = 1, 2, \ldots$.

Lemma 4.2 Let $\pi : \Omega \to Q^\infty$ define a $\mathcal{T}$-possibilistic distribution on a nonempty space Ω, let the $\mathcal{T}$-valued entropy function $I^T(\pi)$ be defined by (3.2), let the Q-valued entropies $I_j(\pi), j = 1, 2, \ldots$, be defined by (4.2). For each $j = 1, 2, \ldots$, let ω_j be an element of Ω such that $(\pi(\omega))_j = 1$. Then, the value $\Pi_j(\Omega - \{\omega_j\})$ is uniquely defined, i.e., does not depend on possible free choice of ω_j, and the relation

$$I_j(\pi) = \Pi_j(\Omega - \{\omega_j\}) \tag{4.7}$$

holds.

Proof: As proved above, for each $j \in \mathcal{N}$ the mapping $(\pi(\cdot))_j : \Omega \to Q$ defines a Q-valued possibilistic distribution on Ω so that, due to the fact that Q is finite and $q_K < 1$ holds, there exists at least one $\omega_j \in \Omega$ such that $\pi(\omega_j)_j = 1$ holds. Let us analyze the two cases.

(i) There exist at least *two* $\omega_j^1, \omega_j^2 \in \Omega$ such that $(\pi(\omega_j^1))_j = (\pi(\omega_j^2))_j = 1$ is valid. Hence, at least one of the values $(\pi(\omega_j^1))_j$ or $(\pi(\omega_j^2))_j$ (or both) are in the set of values from Q such that $\bigvee_{\omega^* \in \Omega, \omega^* \neq \omega} (\pi(\omega^*))_j = \Pi_j(\Omega - \{\omega\})$, so that the identity $\Pi_j(\Omega - \{\omega\}) = 1$ for each $\omega \in \Omega$ follows.

Consequently, the relation

$$\begin{aligned} I_j(\pi) &= \bigvee_{\omega \in \Omega} ((\pi(\omega))_j \wedge \Pi_j(\Omega - \{\omega\})) = \bigvee_{\omega \in \Omega} ((\pi(\omega))_j \wedge 1) = \\ &= \bigvee_{\omega \in \Omega} (\pi(\omega))_j = 1 - \Pi_j(\Omega - \{\omega_j\}) \end{aligned} \tag{4.8}$$

holds for both $\omega_j = \omega_j^1, \omega_j^2$ (as a matter of fact, for each ω_j such that $(\pi(\omega_j))_j = 1$).

(ii) Let there exist just one $\omega_j \in \Omega$ such that $(\pi(\omega_j))_j = 1$. As $(\pi(\cdot))_j$ defines a Q-valued possibilistic distribution on Ω, Π_j is the corresponding possibilistic measure on $\mathcal{P}(\Omega)$ and $\{\omega\} \cup (\Omega - \{\omega\}) = \Omega$ holds, we obtain that the relation $(\pi(\omega))_j \vee \Pi_j(\Omega - \{\omega\}) = 1$ is valid for each $\omega \in \Omega$, where $\vee$ denotes the standard supremum in $\langle [0,1], \leq \rangle$. So, if $(\pi(\omega))_j < 1$ is the case, then $\Pi_j(\Omega - \{\omega\}) = 1$ follows, so that

$$\begin{aligned} I_j(\pi) &= \bigvee_{\omega \in \Omega} ((\pi(\omega))_j \wedge \Pi_j(\Omega - \{\omega\})) = \\ &= \bigvee_{\omega \in \Omega, \omega \neq \omega_j} ((\pi(\omega))_j \wedge \Pi_j(\Omega - \{\omega\})) \vee ((\pi(\omega_j))_j \wedge \Pi(\Omega - \{\omega_j\})) = \\ &= \left(\bigvee_{\omega \in \Omega, \omega \neq \omega_j} [(\pi(\omega_j))_j \wedge 1] \right) \vee [1 \wedge \Pi_j(\Omega - \{\omega_j\})] = \\ &= (\Pi_j(\Omega - \{\omega_j\})) \vee (\Pi_j(\Omega - \{\omega_j\}) = \Pi_j(\Omega - \{\omega_j\}). \end{aligned} \tag{4.9}$$

The assertion is proved. □

Corollary 4.1 Let the notations and conditions of Lemma 4.2 hold, let there exist just one $\omega_j \in \Omega$ such that $(\pi(\omega_j))_j = 1$ holds. Then, the inequality $I_j(\pi) < 1$ follows.

Proof: Indeed, according to (4.7),

$$\begin{aligned} I_j(\pi) &= \Pi_j(\Omega - \{\omega_j\}) = \bigvee_{\omega \in \Omega, \omega \neq \omega_j} (\pi(\omega))_j = \bigvee \{((\pi(\omega))_j : \omega \in \Omega, (\pi(\omega))_j < 1\} = \\ &= \bigvee \{(\pi(\omega))_j : \omega \in \Omega, (\pi(\omega))_j \leq q_K\} \leq \lambda_K < 1 \end{aligned} \tag{4.10}$$

holds. As a trivial consequence we obtain that

$$I(\pi) = \bigvee\nolimits_{\omega\in\Omega}^{T}(\pi(\omega) \wedge^{T} \Pi(\Omega - \{\omega\})) = \langle I_1(\pi), I_2(\pi), I_3(\pi), \dots\rangle < \mathbf{1} \tag{4.11}$$

follows, as $I_j(\pi) < 1$ is the case.

To summarize intuitively the results: if there are at least two occurrences of 1 in $\{(\pi(\omega))_j : \omega \in \Omega\}$, then $I_j(\pi) = 1$, if there is just one occurrence of 1 in $\{(\pi(\omega))_j : \omega \in \Omega\}$ and $(\pi(\omega))_j = 0$ for each other $\omega \in \Omega$, then $I_j(\pi) = 0$, and $0 < I_j(\pi) < 1$ holds otherwise, i.e., if there is just one occurrence of 1 in $\{(\pi(\omega))_j : \omega \in \Omega\}$, but $0 < (\pi(\omega))_j < 1$ holds for at least one $\omega \in \Omega$. Consequently, if there are $\omega_1, \omega_2 \in \Omega$ such that $\pi(\omega_1) = \pi(\omega_2) = \langle 1, 1, \dots\rangle = \mathbf{1}$ and $\omega_1 \neq \omega_2$, then $I_j(\pi) = 1$ for each $j = 1, 2, \dots$, hence $I^T(\pi) = \mathbf{1}_{\mathcal{T}}$. □

5. Real-Valued Embeddings of $\mathcal{T}$-Valued Sequences in General and Entropy Functions in Particular

We still keep in mind and firmly support all the arguments imposed by numerous and highly appreciated authors in favor of the idea to quantify uncertainty (in the sense of vagueness) by non-numerical fuzziness degrees, in particular, by fuzzy subsets of the set of positive integers, as introduced and analyzed above. Nevertheless, we do not eliminate from consideration the application of natural and real numbers, as well as the rich and deeply analyzed mathematical structures over them, as appropriate tools when processing the non-numerical objects and structures under consideration. In particular, let us define and investigate the following mapping ascribing real numbers from $[0, 1]$ to Q^∞-valued sequences. For each $\boldsymbol{x} = \langle \boldsymbol{x}_1, \boldsymbol{x}_2, \dots\rangle \in Q^\infty$ set $c(\boldsymbol{x}) = \sum_{i=1}^{\infty} \boldsymbol{x}_i 2^{-i} \in [0, 1]$. Obviously, $c(\boldsymbol{x})$ takes its minimum value 0 iff $\boldsymbol{x} = \langle 0, 0, \dots\rangle = \emptyset_{\mathcal{T}}$ and $c(\boldsymbol{x})$ takes its maximum value 1 iff $\boldsymbol{x} = \langle 1, 1, \dots\rangle = \mathbf{1}_{\mathcal{T}}$ holds. The mapping c is consistent with the partial ordering $\leq_{\mathcal{T}}$ on Q^∞ in the sense that $c(\boldsymbol{x}) \leq c(\boldsymbol{y})$ holds for each $\boldsymbol{x}, \boldsymbol{y} \in Q^\infty$ such that $\boldsymbol{x} \leq_{\mathcal{T}} \boldsymbol{y}$ (i.e., $\boldsymbol{x}_i \leq \boldsymbol{y}_i$ for every $i \in \mathcal{N}$) is the case.

As analyzed above, given a $\mathcal{T}$-valued possibilistic distribution on a nonempty space Ω, its possibilistic entropy

$$I^T(\pi) = \langle I_1(\pi), I_2(\pi), \dots\rangle = \langle \Pi_1(\Omega - \{\omega_1\}), \Pi_2(\Omega - \{\omega_2\}), \dots\rangle, \tag{5.1}$$

where, for each $j \in \mathcal{N}, (\pi(\omega_j))_j = 1$ holds, takes also its values in Q^∞, so that the value $c(I^T(\pi)) = \sum_{i=1}^{\infty} I_j(\pi)2^{-i}$ is defined and may be taken as a real-valued characteristic of the entropy value in question. Hence, when transforming $\mathcal{T}$-possibilistic distribution π into a new possibilistic distribution π^*, we may calculate and compare both the values $c(I^T(\pi))$ and $c(I^T(\pi^*))$ and we may draw some conclusions from this comparison. Let us illustrate this idea on the following very simple example.

Let $K = 1$, so that $Q = \{0, \lambda, 1\}$ is the case (the index in λ, being omitted as superfluous), let $\Omega = \{\omega_1, \omega_2\}$, and let $\pi : \Omega \to Q^\infty$ be defined in the following way.

$$\begin{aligned} \pi(\omega_1) &= \langle 1, \lambda, 0, 0, 1, 1, 1, \lambda, 0, 0, 1, 1, \dots \rangle, \\ \pi(\omega_2) &= \langle 0, 1, 1, 1, \lambda, 0, 0, 1, 1, 1, \lambda, 0, \dots \rangle, \end{aligned} \tag{5.2}$$

hence $\pi(\omega_1)$ is defined by the infinite sequential repetition of the six-tuples $\langle 1, \lambda, 0, 0, 1, 1\rangle$, and $\pi(\omega_2)$ is defined by the infinite sequential repetition of the six-tuples $\langle 0, 1, 1, 1, \lambda, 0\rangle$. As can be easily verified, for each $j = 1, 2, \dots$ either $((\pi(\omega_1))_j$ or $((\pi(\omega_2))_j$ equals 1, so that $\pi(\omega_1) \vee^T \pi(\omega)2) = \langle 1, 1, 1, \dots \rangle = \mathbf{1}$ is the case and π defines a $\mathcal{T}$-possibilistic distribution on Ω for $Q^\infty = \langle 0, \lambda, 1\}^\infty$.

For $I_j(\pi)$ we obtain that

$$\begin{aligned} I_1(\pi) &= \Pi_1(\Omega - \{\omega_1\}) = \Pi_1(\{\omega_2\}) = (\pi(\omega_2))_1 = 0, \text{ as } (\pi(\omega_1))_1 = 1, \\ I_2(\pi) &= \Pi_2(\Omega - \{\omega_2\}) = \Pi_2(\{\omega_1\}) = (\pi(\omega_1))_2 = \lambda, \text{ as } (\pi(\omega_2))_2 = 1, \\ I_3(\pi) &= \Pi_3(\Omega - \{\omega_2\}) = \Pi_3(\{\omega_1\}) = (\pi(\omega_1))_3 = 0, \text{ as } (\pi(\omega_2))_3 = 1. \end{aligned} \tag{5.3}$$

The values $I_j(\pi)$ do not depend on the ordering of the elements of Ω, so that $I_4(\pi) = I_1(\pi) = 0$, $I_5(\pi) = I_2(\pi) = \lambda$, $I_6(\pi) = I_3(\pi) = 0$, and for greater indices j the values $I_j(\pi)$ repeat in regular cycles. Hence,

$$I^T(\pi) = \langle I_1(\pi), I_2(\pi), \dots \rangle = \langle 0, \lambda, 0, 0, \lambda, 0, 0, \lambda, 0 \dots \rangle, \tag{5.4}$$

so that $I_j(\pi) = \lambda$ for $j = 2 + 3i, i = 0, 1, 2, \dots, I_j(\pi) = 0$ otherwise. Hence,

$$\begin{aligned} c(I^T(\pi)) &= \sum_{j=1}^{\infty} (I_j(\pi)) 2^{-j} = \lambda \sum_{i=0}^{\infty} (2^{-2}) 2^{-3i} = (\lambda/4) \sum_{i=0}^{\infty} (1/8)^i = \\ &= (\lambda/4)/(1 - (1/8)) = (\lambda/4)(8/7) = (2/7)\lambda. \end{aligned} \tag{5.5}$$

Let us consider the modification of $\mathcal{T}$-possibilistic distribution π resulting when replacing in both the sequences $\pi(\omega_1), \pi(\omega_2) \in Q^\infty$ all the occurrences of λ by 1, hence, applying the notation introduced above, when replacing each $\pi(\omega_i), i = 1, 2$, by $\pi^+(\omega_i)$. As $\pi(\omega_i) \leq \pi^+(\omega_i)$ obviously holds, π^+ defines a Q^∞-valued possibilistic distribution on Ω (for $Q = \{1, \lambda, 0\}$, as above). Hence,

$$\begin{aligned} \pi^+(\omega_1) &= \langle 1, 1, 0, 0, 1, 1, 1, 1, 0, 0, 1, 1, \dots \rangle, \\ \pi^+(\omega_2) &= \langle 0, 1, 1, 1, 1, 0, 0, 1, 1, 1, 1, 0, \dots \rangle. \end{aligned} \tag{5.6}$$

Checking the calculations from above, leading from $\pi(\omega_1)$ and $\pi(\omega_2)$ to $I^T(\pi)$ and $c(I^T(\pi))$, we can easily observe that the corresponding calculation for $I^T(\pi^+)$ and $c(I^T(\pi^+))$ consists in simple replacement of λ by 1, so that we arrive at the result that

$$I^T(\pi^+) = \langle I_1(\pi^+), I_2(\pi^+), \dots, \rangle = \langle 0, 1, 0, 0, 1, 0, \dots \rangle, \tag{5.7}$$

so that

$$c(I^T(\pi^+)) = \sum_{i=0}^{\infty}(2^{-2}) \cdot 2^{-3i} = (1/4)\sum_{i=0}^{\infty}(1/8)^i = 2/7. \tag{5.8}$$

As an illustration of the sensitivity of the mapping $c(I^T(\pi))$ when aiming to distinguish among different Q^∞-valued possibilistic distributions on Ω perhaps still another example is worth being introduced explicitly. Consider the possibilistic distribution π^1 on $\Omega = \{\omega_1, \omega_2\}$ defined as follows.

$$\begin{aligned}
\pi^1(\omega_1) &= \langle(\pi^1(\omega_1))_1, (\pi^1(\omega_1))_2, \dots\rangle = \langle 1, \lambda_1, 1, \lambda_1, 1, \lambda_1, \dots\rangle, \\
\pi^1(\omega_2) &= \langle(\pi^1(\omega_2))_1, (\pi^1(\omega_2))_2, \dots\rangle = \langle \lambda_2, 1, \lambda_2, 1, \lambda_2, 1, \dots\rangle.
\end{aligned} \tag{5.9}$$

Hence, $i_j(\pi^1) = \Pi_j(\Omega - \{\omega_j\}) = \lambda_2$, if $j = 1, 3, 5, 7, \dots, I_j(\pi^1) = \lambda_1$, if $j = 2, 4, 6, \dots$ So,

$$\begin{aligned}
c(I^T(\pi^1)) &= \sum_{j=1}^{\infty}(I_j(\pi^1))2^{-j} = \sum_{j=1, j \text{ odd}}^{\infty} \lambda_2 2^{-j} + \sum_{j=1, j \text{ even}}^{\infty} \lambda_1 2^{-j} = \\
&= \sum_{j=0}^{\infty} \lambda_2 2^{-(2j+1)} + \sum_{j=1}^{\infty} \lambda_1 2^{-2j} = \lambda_2(1/2)\sum_{j=0}^{\infty} 2^{-2j} + \lambda_1(1/4)\sum_{j=0}^{\infty} 2^{-2j} = \\
&= (\lambda_2/2)\sum_{j=0}^{\infty}(1/4)^j + (\lambda_1/4)\sum_{j=0}^{\infty}(1/4)^j = (\lambda_2/2)(1/(1-(1/4))) + \\
&+ (\lambda_1/4)(1/(1-(1/4))) = (\lambda_2/2)(4/3) + (\lambda_1/4)(4/3) = (2/3)\lambda_2 + (1/3)\lambda_1.
\end{aligned} \tag{5.10}$$

Let π^2 be the possibilistic distribution on $\Omega = \{\omega_1, \omega_2\}$ defined in the same way as π^1, just with the roles of the values λ_1 and λ_2 interchanged, so that $c(I^T(\pi^2)) = (2/3)\lambda_1 + (1/3)\lambda_2$. So, e.g., if $\lambda_1 = 1/3$ and $\lambda_2 = 2/3$, then $c(I^T(\pi^1)) = (2/3)(2/3) + (1/3)(1/3) = 5/9$, and $c(I^T(\pi^2)) = (2/3)(1/3) + (1/3)(2/9) = 4/9$. Obviously, if $\lambda_1 = \lambda_2 = \lambda$, then $c(I^T(\pi^1)) = c(I^T(\pi^2)) = \lambda$.

6. Entropy-Value-Preserving Operations over Q^∞-Valued Possibilistic Distributions

As above, let us consider the set $Q = Q_K = \{0 < \lambda_1 < \lambda_2 < \dots < \lambda_K < 1\}$, $K \geq 0$, and the space Q^∞ of all infinite sequences of elements from Q. Q^∞-valued possibilistic distribution on a nonempty space Ω is a mapping $\pi : \Omega \to Q^\infty$ such that $\bigvee_{\omega\in\Omega}^{T} \pi(\omega) = \langle 1, 1, 1, \dots\rangle = \mathbf{1}_{\mathcal{T}}$. Here $\mathcal{T} = \langle Q^\infty, \leq_T\rangle$ is the complete lattice with the partial ordering $\leq_T$ on Q^∞ defined by the Cartesian product of the ordering $0 < \lambda_1 < \lambda_2 < \dots \lambda_K < 1$ on Q, so that, for each $\boldsymbol{x} = \langle x_1, x_2, \dots\rangle, \boldsymbol{y} = \langle y_1, y_2, \dots\rangle \in Q^\infty, \boldsymbol{x} \leq_T \boldsymbol{y}$ holds iff $x_j \leq y_j$ is the case for each $j = 1, 2, \dots$

Lemma 6.1 For each Q^∞-valued possibilistic distribution π defined on a nonempty space Ω there exists Q^∞-valued possibilistic distribution π^* on a two-element space $\Omega^* = \{\omega^1, \omega^2\}$ such that $I^T(\pi) = I^T(\pi^*)$ holds in the sense of equality relation on Q^∞, so that $I_j(\pi) = I_j(\pi^*)$ is valid for each $j \in \mathcal{N} = \{1, 2, \dots\}$.

Proof: According to the definitions and results introduced above,

$$I^T(\pi) = \langle I_1(\pi), I_2(\pi), \dots \rangle = \langle \Pi(\Omega - \{\omega_1\}), \Pi(\Omega - \{\omega_2\}), \dots \rangle, \tag{6.1}$$

where, for each $j \in \mathcal{N}, \omega_j \in \Omega$ is such an element of Ω that $(\pi(\omega_j))_j = 1$ holds. At least one such element always exists for each $j \in \mathcal{N}$, if there are two or more such elements, the choice of one of them is irrelevant, as in such a case $\Pi(\Omega - \{\omega_j\}) = 1$ holds.

Considering $\Omega^* = \{\omega^1, \omega^2\}$, let $\pi^* : \Omega^* \to Q^\infty$ be defined in this way.

$$\pi^*(\omega^1) = \langle (\pi^*(\omega^1))_1, (\pi^*(\omega^1))_2, (\pi^*(\omega^1))_3, \dots \rangle, \tag{6.2}$$

where $(\pi^*(\omega^1))_j = 1$, if $j = 1, 3, 5, \dots$ (i.e., for j odd), and $(\pi^*(\omega^1))_j = \Pi_j(\Omega - \{\omega_j\})$, if $j = 2, 4, 6, \dots$ (i.e., for j even). Dually, $(\pi^*(\omega^2))_j = \Pi_j(\Omega - \{\omega_j\})$, if j is odd, and $(\pi^*(\omega^2))_j = 1$, if j is even.

As can be easily checked, $\pi^*(\omega^1) \vee^T \pi^*(\omega^2) = \langle 1, 1, \dots \rangle = \mathbf{1}_{\mathcal{T}}$, so that π^* defines a Q^∞-valued possibilistic distribution on Ω^*. Moreover

$$\begin{aligned} I^T(\pi^*) &= \langle \Pi_1^*(\Omega^* - \{\omega^{1,*}\}), \Pi_2^*(\Omega^* - \{\omega^{2,*}\}), \dots \rangle \\ &= \langle \Pi_1(\Omega - \{\omega_1\}), \Pi_2(\Omega - \{\omega_2\}), \dots \rangle \\ &= \langle I_1(\pi), I_2(\pi), \dots \rangle, \end{aligned} \tag{6.3}$$

where, for each $j \in \mathcal{N}, \omega^{j,*}$ is an element from Ω^* such that $(\pi^*(\omega^{j,*})) = 1$; such $\omega^{j,*} \in \Omega^*$ always exists. Hence, $I_j(\pi) = I_j(\pi^*)$ holds for each $j \in \mathcal{N}$ and the assertion is proved. $\Box$

Lemma 6.2 Let $Q^\infty, \Omega, \Omega^*$, and π be as in Lemma 6.1, let $\lambda = c(I^T(\pi)) = \sum_{i=1}^{\infty}(I_j(\pi))2^{-j}$ be defined. Then, there exists a $\{0, 1, \lambda\}^\infty$-valued possibilistic distribution on Ω^* such that $c(I^T(\pi)) = c(I^T(\pi^*))$ holds.

Remark 1 The shift from Lemma 6.1 to Lemma 6.2 seems to be evident. In Lemma 6.1, when seeking a simplified possibilistic distribution with the same entropy value, we still have to keep the whole space Q^∞ as the source for our possibility degrees, on the other side, the entropy value of the simplified Q^∞-valued possibilistic distribution is identical with the entropy value $I^T(\pi)$ of the original possibilistic distribution in the sense of sequential identity, i.e., each pairs of sequence members are identical. On the other side, when seeking the approximating distribution and its entropy value just in the more simple space $\{0, 1, \lambda\}^\infty$, i.e., in Q^∞ with Q reduced to $\{0, \lambda, 1\}$, we have succeeded just in the sense that the real-valued mapping c ascribes the same value to both the possibilistic distributions π and π^*.

Proof: (of Lemma 6.2) Given Q^∞-valued possibilistic distribution π on Ω, denote by λ the real value

$$c(I^T(\pi)) = \sum_{j=1}^{\infty}(I_j(\pi))2^{-j} = \sum_{j=1}^{\infty}(\Pi_j(\Omega - \{\omega_j\}))2^{-j} \tag{6.4}$$

for ω_j such that $(\pi(\omega_j))_j = 1$. Define π^* on $\Omega^* = \{\omega_1, \omega_2\}$ in this way: $(\pi(\omega^1))_j = 1$ for $j = 1, 3, 5, \ldots, (\pi(\omega^1))_j = \lambda$ for $j = 2, 4, 6, \ldots,$ and dually $(\pi(\omega^2))_j = \lambda$ for $j = 1, 3, 5, \ldots,$ $(\pi(\omega^2))_j = 1$ for $j = 2, 4, 6, \ldots$ Obviously, $\pi^*(\omega_1) \vee^T \pi^*(\omega^2) = \langle 1, 1, 1, \ldots \rangle = \mathbf{1}_{\mathcal{T}}$, so that π^* defines a $\{0, 1, \lambda\}^\infty$-valued possibilistic distribution on Ω^*. Moreover, $I_j(\pi^*) = \Pi^*(\Omega^* - \{\omega^{*,j}\}) = \lambda$ holds for each $j = 1, 2, \ldots,$ here $\omega^{*,j} \in \Omega^*$ is such an element that $(\pi^*(\omega^{*,j}))_j = 1$ holds. Consequently, we obtain that the relation

$$c(I^T(\pi^*)) = \sum_{j=1}^{\infty}(I_j(\pi^*))2^{-j} = \sum_{j=1}^{\infty}\Pi^*(\Omega^* - \{\omega^{*,j}\})2^{-j} = \sum_{j=1}^{\infty}\lambda 2^{-j} = \lambda \tag{6.5}$$

holds. The assertion is proved. □

7. Approximations of Q^∞-Valued Possibilistic Distributions

Let $\pi : \Omega \to Q^\infty$ be a Q^∞-valued possibilistic distribution on Ω, so that $\bigvee_{\omega\in\Omega}^T \pi(\omega) = \langle 1, 1, 1, \ldots \rangle = \mathbf{1}_{\mathcal{T}}$ holds. Let for each $i \in \mathcal{N} = \{1, 2, \ldots\}$ the pair $0 \le \alpha_i \le \beta_i \le 1$ of real numbers be given, let R denote the system $\{\langle \alpha_i, \beta_i \rangle : i \in \mathcal{N}\}$. Define the mapping $\pi^R : \Omega \to Q^\infty$, i.e., $\pi^R(\omega) = \langle (\pi^R(\omega))_1, (\pi^R(\omega))_2, \ldots \rangle$, in this way:

$$\begin{array}{ll} (i) & \text{if } (\pi(\omega))_i < \alpha_i, \text{ then } (\pi^R(\omega))_i = 0, \\ (ii) & \text{if } (\pi(\omega))_i > \beta_i, \text{ then } (\pi^R(\omega))_i = 1, \\ (iii) & (\pi^R(\omega))_i = (\pi(\omega))_i \text{ otherwise, i.e., if } \alpha_i \le (\pi(\omega))_i =\le \beta_i \text{ holds.} \end{array} \tag{7.1}$$

Lemma 7.1 For each Q^∞-valued possibilistic distribution π on Ω and for each system R, π^R defines a Q^∞-valued possibilistic distribution on Ω.

Proof: According to our former results, the only item we have to prove is the relation $\bigvee_{\omega\in\Omega}(\pi^R(\omega))_i = 1$ for each $i \in \mathcal{N}$. As $\bigvee_{\omega\in\Omega}(\pi(\omega))_i = 1$ holds for each $i \in \mathcal{N}$ and $\bigvee_{j=1}^K \lambda_j = \lambda_K < 1$ holds in Q, we obtain that, for each $j \in \mathcal{N}$, there exists $\omega_j \in \Omega$ such that $(\pi(\omega_j)_j = 1$ holds. However, in this case (7.1) yields that $(\pi^R(\omega_j))_j = 1$, hence, $\bigvee_{\omega\in\Omega}(\pi^R(\omega))_j = 1$, so that $\bigvee_{\omega\in\Omega}\pi^R(\omega) = \langle 1, 1, \ldots \rangle = \mathbf{1}_{\mathcal{T}}$ follows and the assertion is proved. □

Theorem 7.1 Under the notations and conditions of Lemma 7.1 the following inequalities are valid:

$$\begin{array}{lll} (i) & \text{if } I_j(\pi) \in \langle \alpha_j, \beta_j \rangle \text{ holds, then } I_j(\pi^R) = I_j(\pi), & \\ (ii) & \text{if } I_j(\pi) < \alpha_j, \text{ then } I_j(\pi^R) = 0, & \\ (iii) & \text{if } I_j(\pi) > \beta_j, \text{ then } I_j(\pi^R) = 1. & \end{array} \tag{7.2}$$

Proof: Let $\alpha_j \leq I_j(\pi) \leq \beta_j$ be the case. As $I_j(\pi) = \Pi_j(\Omega - \{\omega_j\})$ holds, where $(\pi(\omega_j))_j = 1$, we obtain that $I_j(\pi) = \bigvee_{\omega \in \Omega, \omega \neq \omega_j} (\pi(\omega))_j \leq \beta_i$ follows. As the values in Q are such that $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_K < 1$ holds, there exists $\omega_j^* \in \Omega, \omega_j^* \neq \omega_j$, such that

$$\alpha_j \leq I_j(\pi) = (\pi(\omega_j^*))_j \leq \beta_j \tag{7.3}$$

follows. Hence, for all $\omega \in \Omega$, either $(\pi^R(\omega))_j = 0$ (if $(\pi(\omega))_j < \alpha_j$ holds), or $(\pi^R(\omega))_j = (\pi(\omega))_j$ (if $(\pi(\omega))_j \geq \alpha_j$ holds), and there is at least one $\omega \in \Omega$, namely $\omega = \omega_j^*$, satisfying this last condition. Consequently,

$$I_j(\pi) = \bigvee_{\omega \in \Omega, \omega \neq \omega_j} (\pi(\omega))_j = \bigvee_{\omega \in \Omega, \omega \neq \omega_j} (\pi^R(\omega))_j = I_j(\pi^R) \tag{7.4}$$

holds, and the case (i) in (7.2) is proved.

Let $I_j(\pi) < \alpha_j$ be the case. Then, however, for ω_j^* defined by (7.3), the inequality $(\pi(\omega_j^*))_j < \alpha_j$ holds, hence, $(\pi(\omega))_j < \alpha_j$ holds for each $\omega \in \Omega, \omega \neq \omega_j$. Consequently, $(\pi^R(\omega))_j = 0$ holds for each $\omega \in \Omega, \omega \neq \omega_j$, so that $I_j(\pi^R) = \bigvee_{\omega \in \Omega, \omega \not\in'} (\pi^R(\omega))_j = 0$ follows. Dually, let $I_j(\pi) > \beta_j$ hold, then the element $\omega_j^* \in \Omega$ such that $\Pi_j(\Omega - \{\omega_j\}) = (\pi(\omega_j^*))_j$ holds must satisfy the inequality $(\pi(\omega_j^*))_j > \beta_j$, so that $(\pi^R(\omega_j^*))_j = 1$ follows. Consequently, $I_j(\pi^R) = \bigvee_{\omega \in \Omega, \omega \neq \omega_j} (\pi^R(\omega))_j = 1$ holds and the assertion is proved. □

8. Cartesian Products of Q^∞-Valued Possibilistic Distributions

Theorem 8.1 Let Ω_1, Ω_2 be nonempty spaces, let $Q = \{0, \lambda_1, \lambda_2, \ldots, \lambda_K, 1\}, 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_K < 1$. For both $i = 1, 2$, let $\pi^i : \Omega_i \to Q^\infty$ be a Q^∞-valued possibilistic distribution on Ω_i, i.e., let $\bigvee_{\omega \in \Omega_i}^{T} \pi^i(\omega) = \langle 1, 1, \ldots \rangle = \mathbf{1}_{\mathcal{T}}$ hold, where $\mathcal{T} = \langle Q^\infty, \leq_{\mathcal{T}} \rangle$. Let $\pi^{12} : \Omega_1 \times \Omega_2 \to Q^\infty$ be defined so that

$$\pi^{12}(\omega_1, \omega_2) = \pi^1(\omega_1) \wedge_T \pi^2(\omega_2) \tag{8.1}$$

holds for each $\langle \omega_1, \omega_2 \rangle \in Q_1 \times Q_2$. Then, π^{12} defines a Ω^∞-valued possibilistic distribution on $\Omega_1 \times \Omega_2$, so that the relation

$$\bigvee\nolimits_{\langle \omega_1, \omega_2 \rangle \in \Omega_1 \times \Omega_2}^{T} \pi^{12}(\omega_1, \omega_2) = \langle 1, 1, \ldots \rangle = \mathbf{1}_{\mathcal{T}} \tag{8.2}$$

holds.

Proof: We have to prove that

$$\bigvee\nolimits^{T}_{\langle\omega_1,\omega_2\rangle\in\Omega_1\times\Omega_2}\pi^{12}(\omega_1,\omega_2) \quad = \quad \bigvee\nolimits^{T}_{\omega_1\in\Omega_1}\left(\bigvee\nolimits^{T}_{\omega_2\in\Omega_1}(\pi^1(\omega_1)\wedge_{\mathcal{T}}\pi^2(\omega_2))\right) = \\ = \quad \langle 1,1,\dots\rangle = \mathbf{1}_{\mathcal{T}} \tag{8.3}$$

holds. Hence, as π^1 defines a Q^∞-possibilistic distribution on Ω_1, it suffices to prove that

$$\bigvee\nolimits^{T}_{\omega_2\in\Omega_2}(\pi^1(\omega_1)\wedge\pi^2(\omega_2)) = \pi^1(\omega_1) \tag{8.4}$$

holds for each $\omega_1\in\Omega_1$. For each $\langle\omega_1,\omega_2\rangle\in\Omega_1\times\Omega_2, \pi^{12}(\omega_1,\omega_2)=\pi^1(\omega_1)\wedge\pi^2(\omega_2)$ is in Q^∞, so that

$$\begin{aligned} &\pi^{12}(\omega_1,\omega_2) = \langle(\pi^{12}(\omega_1,\omega_2))_1,(\pi^{12}(\omega_1,\omega_2))_2,(\pi^{12}(\omega_1,\omega_2))_3,\dots\rangle = \\ = \quad &\langle(\pi^1(\omega_1)\wedge_{\mathcal{T}}\pi^2(\omega_2))_1,(\pi^1(\omega_1)\wedge_{\mathcal{T}}\pi^2(\omega_2))_2,(\pi^1(\omega_1)\wedge_{\mathcal{T}}\pi^2(\omega_2))_3,\dots\rangle = \\ = \quad &\langle(\pi^1(\omega_1))_1\wedge(\pi^2(\omega_2))_1,(\pi^1(\omega_1))_2\wedge(\pi^2(\omega_2))_2,(\pi^1(\omega_1))_3\wedge(\pi^2(\omega_2))_3,\dots\rangle. \end{aligned} \tag{8.5}$$

Hence,

$$\bigvee\nolimits^{T}_{\omega_2\in\Omega_2}\pi^{12}(\omega_1,\omega_2) = \langle\bigvee_{\omega_2\in\Omega_2}((\pi^1(\omega_1))_1\wedge(\pi^2(\omega_2))_1), \\ \bigvee_{\omega_2\in\Omega_2}((\pi^1(\omega_1))_2\wedge(\pi^2(\omega_2))_2),\bigvee_{\omega_2\in\Omega_2}((\pi^1(\omega_1))_3\wedge(\pi^2(\omega_2))_3),\dots\rangle. \tag{8.6}$$

As π^2 defines a Q^∞-valued possibilistic distribution on Ω_2, so that $\bigvee^{T}_{\omega_2\in\Omega_2}\pi^2(\omega^2)=\langle 1,1,\dots\rangle=\mathbf{1}_{\mathcal{T}}$ holds, we obtain that $\bigvee_{\omega_2\in\Omega_2}(\pi^2(\omega_2))_j=1$ is the case for each $j\in\mathcal{N}=1,2,\dots$ Hence, as $Q=\{0<\lambda_1<\lambda_2<\cdots<\lambda_K\leq 1\}$ holds, for each $j\in\mathcal{N}$ there exists $\omega_{2,j}\in\Omega_2$ such that $(\pi^2(\omega_{2,j}))_j=1$. Consequently, for each $j\in\mathcal{N}$ there exists $\omega_{2,j}\in\Omega_2$ such that $(\pi^1(\omega_1))_j\wedge(\pi^2(\omega_{2,j}))_j=(\pi^1(\omega_1))_j\wedge 1=(\pi^1(\omega_1))_j$. Hence,

$$\bigvee_{\omega_2\in\Omega_2}[(\pi^1(\omega_1))_j\wedge(\pi^2(\omega_2))_j] = \bigvee_{\omega_2\in\Omega_2}((\pi^1(\omega_1))_j\wedge 1 = (\pi^1(\omega_1))_j. \tag{8.7}$$

As this is valid for each $j\in\mathcal{N}$, we obtain that for each $\omega_1\in\Omega$

$$\bigvee_{\omega_2\in\Omega_2}(\pi^1(\omega_1)\wedge_{\mathcal{T}}\pi^2(\omega_2)) = \pi(\omega_1) \tag{8.8}$$

holds. The assertion is proved. □

Theorem 8.2 Let the notations and conditions of Theorem 8.1 hold. Then, there exists, for each $j\in\mathcal{N}=\{1,2,\dots\}$, an element $\langle\omega_{j,1},\omega_{j,2}\rangle\in\Omega_1\times\Omega_2$ such that $(\pi^{12}(\omega_{j,1},\omega_{j,2}))_j=1$.

Proof: As proved in Theorem 8.1, the relation

$$\begin{aligned}
&\bigvee\nolimits^{T}_{\langle\omega_1,\omega_2\rangle\in\Omega_1\times\Omega_2}\pi^{12}(\omega_1,\omega_2)=\mathbf{1}_{\mathcal{T}}=\langle 1,1,\dots\rangle=\\
=\;&\left\langle\left(\bigvee\nolimits^{T}_{\langle\omega_1,\omega_2\rangle\in\Omega_1\times\Omega_2}\pi^{12}(\omega_1,\omega_2)\right)_1,\left(\bigvee\nolimits^{T}_{\langle\omega_1,\omega_2\rangle\in\Omega_1\times\Omega_2}\pi^{12}(\omega_1,\omega_2)\right)_2,\dots\right\rangle=\\
=\;&\left\langle\bigvee_{\langle\omega_1,\omega_2\rangle\in\Omega_1\times\Omega_2}\pi^{12}(\omega_1,\omega_2))_1,\bigvee_{\langle\omega_1,\omega_2\rangle\in\Omega_1\times\Omega_2}\pi^{12}(\omega_1,\omega_2))_2,\dots\right\rangle \qquad (8.9)
\end{aligned}$$

holds, as π^{12} defines a Q^{∞}-possibilistic distribution on $\Omega_1\times\Omega_2$. Hence,

$$\bigvee_{\langle\omega_1,\omega_2\rangle\in\Omega_1\times\Omega_2}(\pi^{12}(\omega_1,\omega_2))_j=1 \qquad (8.10)$$

holds for each $j\in\mathcal{N}$. However, for each $j\in\mathcal{N}$ and each $\langle\omega_1,\omega_2\rangle\in\Omega_1\times\Omega_2$, the value $(\pi^{12}(\omega_1,\omega_2))_j$ is in Q, so that either $(\pi^{12}(\omega_1,\omega_2))_j\leq\lambda_K<1$ or $(\pi^{12}(\omega_1,\omega_2))_j=1$ is the case. So, the relation (8.10) is fulfilled only if there exists a pair $\langle\omega_{j,1},\omega_{j,2}\rangle\in\Omega_1\times\Omega_2$, such that $(\pi^{12}(\omega_{j,1},\omega_{j,2}))_j=1$, hence, as $(\pi^{12}(\omega_{j,1},\omega_{j,2}))_j,=(\pi^{1}(\omega_{j,1}))_j\wedge(\pi^{2}(\omega_{j,2}))_j$ holds, $(\pi^{1}(\omega_{j,1}))_j=(\pi^{2}(\omega_{j,2}))_j=1$ follows. Consequently, for each $j\in\mathcal{N}$, the relation

$$\begin{aligned}
&\{\langle\omega_1,\omega_2\rangle\in\Omega_1\times\Omega_2:(\pi^{12}(\omega_1,\omega_2))_j=1\}=\\
=\;&\{\omega_1\in\Omega_1:(\pi^{1}(\omega_1))_j=1\}\times\{\omega_2\in\Omega_2:(\pi^{2}(\omega_2))_j=1\} \qquad (8.11)
\end{aligned}$$

is valid. The assertion is proved. □

For the value of the entropy function I^T defined by (3.2) and applied to Q^{∞}-valued possibilistic distribution π^{12} defined on the Cartesian product $\Omega_1\times\Omega_2$ we obtain, applying (4.4) and (4.7), that

$$I^T(\pi^{12})=\langle I_1(\pi^{12}),I_2(\pi^{12}),\dots\rangle, \qquad (8.12)$$

where, for each $j\in\mathcal{N}$,

$$I_j(\pi^{12})=\Pi^{12}_j((\Omega_1\times\Omega_2)-\{\langle\omega_{j,1},\omega_{j,2}\rangle\}). \qquad (8.13)$$

Here $\langle\omega_{j,1},\omega_{j,2}\rangle\in\Omega_1\times\Omega_2$ is such that $(\pi^{12}(\omega_{j,1},\omega_{j,2}))_j=1$, and Π^{12}_j is the Q-valued possibilistic measure on $\mathcal{P}(\Omega_1\times\Omega_2)$ induced by the possibilistic distribution π^{12}_j ascribing to each $\langle\omega_1,\omega_2\rangle\in\Omega_1\times\Omega_2$ the value $\pi^{12}(\omega_1,\omega_2))_j=(\pi^{1}(\omega_1))_j\wedge(\pi^{2}(\omega_2))_j\in Q$.

Theorem 8.3 Let the notations and conditions of Theorem 8.1 hold. Then, for both $i=1,2$, the relation

$$I^T(\pi^{12})=\langle I_1(\pi^{12}),I_2(\pi^{12}),\dots\rangle\geq_{\mathcal{T}}\langle I_1(\pi^{i}),I_2(\pi^{i}),\dots\rangle=I^T(\pi^{i}) \qquad (8.14)$$

holds, i.e., for each $j\in\mathcal{N}$ and for both $i=1,2$ the inequality

$$I_j(\pi^{12}) = \Pi_j^{12}(\Omega_1 \times \Omega_2) - \{\langle \omega_{j,1}, \omega_{j,2} \rangle\}) \geq I_j(\pi^i) = \Pi_j^i(\Omega_i - \{\omega_{j,i}\}) \tag{8.15}$$

is valid. Here $\langle \omega_{j,1}, \omega_{j,2} \rangle \in \Omega_1 \times \Omega_2$ is such that $(\pi^{12}(\omega_{j,1}, \omega_{j,2}))_j = 1$, so that $(\pi^1(\omega_{j,1}))_j = \pi^2(\omega_{j,2}))_j = 1$ follows.

Proof: Let us prove that for $i = 1$ and for each $\omega_1 \in \Omega_1$ the equality

$$\Pi_j^1(\Omega_1 - \{\omega_1\}) = \Pi_j^{12}((\Omega_1 \times \Omega_2) - (\{\omega_1\} \times \Omega_2)) \tag{8.16}$$

holds, the proof for $i = 2$ being analogous. So,

$$\Pi_j^1(\Omega_1\{\omega_1\}) = \bigvee_{\omega_1^* \in \Omega_1, \omega_1^* \neq \omega_1} (\pi^1(\omega_1^*))_j, \tag{8.17}$$

$$\begin{aligned}
&\Pi_j^{12}((\Omega_1 \times \Omega_2) - (\{\omega_1\} \times \Omega_2)) = \bigvee_{\langle \omega_1', \omega_2' \rangle \in \Omega_1 \times \Omega_2, \omega_1' \neq \omega_1} (\pi^{12}(\omega_1', \omega_2'))_j = \\
= &\bigvee_{\langle \omega_1', \omega_2' \rangle \in \Omega_1 \times \Omega_2, \omega_1' \neq \omega_1} ((\pi^1(\omega_j'))_j \wedge (\pi^2(\omega_2'))_j) = \\
= &\bigvee_{\langle \omega_1', \omega_2' \rangle \in \Omega_1 \times \Omega_2, \omega_1' \neq \omega_1} ((\pi^1(\omega_1'))_j \wedge 1 = \bigvee_{\omega' \in \Omega_1, \omega' \neq \omega_1} (\pi^1(\omega_1'))_j = \\
= &\Pi_j^1(\Omega_1 - \{\omega_1\}).
\end{aligned} \tag{8.18}$$

As the inclusions $\{\langle \omega_1, \omega_2 \rangle\} \in \{\omega_1\} \times \Omega_2$, hence, also $(\Omega_1 \times \Omega_2) - \{\langle \omega_1, \omega_2 \rangle\} \supset (\Omega_1 + \Omega_2) - (\{\omega_1\} \times \Omega_2)$ are trivially valid, the inequality

$$\begin{aligned}
&\Pi^{12}((\Omega_1 \times \Omega_2) - \{\langle \omega_1, \omega_2 \rangle\}) \geq \Pi_j^{12}((\Omega_1 \times \Omega_2) - (\{\omega_1\} \times \Omega_2)) = \\
= &\Pi_j^1(\Omega_1 - \{\omega_1\})
\end{aligned} \tag{8.19}$$

follows and the assertion is proved. □

Lemma 8.1 For each $j \in \mathcal{N}$, if $I_j(\pi^{12}) = 1$, then either $I_j(\pi^1) = 1$ or $I_j(\pi_2) = 1$ (or both) is valid.

Proof: Let $I_j(\pi^{12}) = 1$. Applying (8.13) we obtain that

$$\Pi_j^{12}((\Omega_1 \times \Omega_2) - \{\langle \omega_{j,1}, \omega_{j,2} \rangle\}) = 1 \tag{8.20}$$

holds, where $\langle \omega_{j,1}, \omega_{j,2} \rangle \in \Omega_1 \times \Omega_2$ is such that $(\pi^{12}(\omega_{j,1}, \omega_{j,2}))_j = 1$, as for the values $\pi^{12}(\omega_1, \omega_2)$ either $\pi^{12}(\omega_1, \omega_2) \leq \lambda_K < 1$ or $\pi^{12}(\omega_1, \omega_2) = 1$ holds, the relation (8.20) may be valid only if there exists $\langle \omega_{j,1}^*, \omega_{j,2}^* \rangle \in \Omega_1 \times \Omega_2, \langle \omega_1^*, \omega_2^* \rangle \neq \langle \omega_{j,1}, \omega_{j,2} \rangle$, hence, $\langle \omega_{j,1}^*, \omega_{j,2}^* \rangle \in (\Omega_1 \times \Omega_2) - \{\langle \omega_{j,1}, \omega_{j,2} \rangle\}$, such that

$$(\pi^{12}(\omega_{j,1}^*, \omega_{j,2}^*))_j = (\pi^1(\omega_{j,1}^*))_j \wedge (\pi^2(\omega_{j,2}^*))_j = 1 \tag{8.21}$$

holds, consequently, $(\pi^1(\omega_{j,1}^*))_j = (\pi^2(\omega_{j,2}^*))_j = 1$. According to the property $(\pi^{12}(\omega_{j,1}, \omega_{j,2}))_j = 1$ imposed on $\langle \omega_{j,1}, \omega_{j,2} \rangle \in \langle \Omega_1 \times \Omega_2 \rangle$ by (8.19) we obtain that also $(\pi^1(\omega_{j,1}))_j = (\pi^2(\omega_{j,2}))_j = 1$ holds. As $\langle \omega_{j,1}, \omega_{j,2} \rangle \neq \langle \omega^*_{j,1}, \omega^*_{j,2} \rangle$ is the case, then either $\omega_{j,1} \neq \omega^*_{j,1}$ or $\omega_{j,2} \neq \omega^*_{j,2}$ (or both) follows, without any loss of generality let us consider the case when $\omega_{j,1} \neq \omega^*_{j,1}$ is valid. However, in this case there are different elements $\omega_{j,1}, \omega^*_{j,1}$, in Ω_1 such that $(\pi^1(\omega_{j,1}))_j = (\pi^1(\omega^*_{j,1}))_j = 1$ holds, so that $I_j(\pi^1) = \Pi^1_j(\Omega_1 - \{\omega_{j,1}\}) = 1$ follows. The case with $(\pi^2(\omega_{j,2}))_j = (\pi^2(\omega^*_{j,2}))_j = 1$ is analogous, so that the proof is completed. $\square$

9. Conclusions

We have investigated, in the sections above, possibilistic distributions and measures taking their values in the space of infinite sequences of real numbers from a finite fixed set $Q \subset [0,1]$ equipped by the standard linear ordering $\leq$. Introduced were two orderings on Q^∞, the Boolean partial ordering $\leq \mathcal{T}$ and the lexicographic ordering $\leq_\mathcal{L}$, which has been proved to be linear, i.e., for each $\boldsymbol{x}, \boldsymbol{y} \in Q^\infty$ either $\boldsymbol{x} \leq_\mathcal{L} \boldsymbol{y}$ or $\boldsymbol{y} \leq_\mathcal{L} \boldsymbol{x}$ holds. Due to the strong assumptions imposed on $\mathcal{T} = \langle Q^\infty, \leq_\mathcal{T} \rangle$ $(= \langle \boldsymbol{x}_{i=1}^\infty Q_i, \leq_\mathcal{T} \rangle)$ according to which all Q_i are identical and finite, so that the supremum of each $A \subset Q$ is defined in A, we have deduced some nontrivial results dealing with possibilistic distributions and measures taking their values in the complete lattice $\mathcal{T} = \langle Q^\infty, \leq_\mathcal{T} \rangle$.

As a matter of fact, both the complete lattices $\mathcal{T} = \langle Q^\infty, \leq_\mathcal{T} \rangle$ and $\mathcal{L} = \langle Q^\infty, \leq_\mathcal{L} \rangle$ possess reasonable and interesting interpretations. In both cases the possibility degrees ascribed to an element $\omega \in \Omega$ of the basic space Ω are combined from possibility degrees ascribed to various aspects from which the degree of possibility of $\omega \in \Omega$ may be evaluated. In the case of structure $\mathcal{T}$, when the partial possibility degrees are combined by the partial ordering $\leq_\mathcal{T}$, these particular possibility degrees are taken as rather independent of each other in the sense that the total degree of possibility ascribed to some $\omega_1 \in \Omega$ is taken as greater than or equal to the total degree of possibility ascribed to some $\omega_2 \in \Omega$ if and only if this inequality is valid for each particular aspect from which the degrees of possibility of the elements of Ω are evaluated. On the other side, the lexicographic ordering corresponds to the case when different aspects are ordered according to their decreasing importance or weight of these aspects when combining them together, so that the ordering of the combined degrees of possibility for various $\omega \in \Omega$ is determined by the ordering with respect to the most important aspect in which the possibility degrees for compared ω_1 and ω_2 differ form each other. A more detailed analysis of possibilistic distributions and measures with values in the lexicographic complete lattice $\langle Q^\infty, \leq_\mathcal{L} \rangle = \mathcal{L}$ seems to be useful and promising and should be considered as a very intuitive first step when going on, in some future paper, with the investigations presented here.

In this paper, we investigated possibilistic distributions and measures taking as their values infinite sequences of real numbers, each of these numbers being taken from the same, finite, and by the standard ordering over real numbers equipped set $Q \subset [0,1]$. Keeping in mind the character of the space Q^∞ as the Cartesian product $\mathbb{X}_{i=1}^\infty Q_i$, what should be considered during a future analysis is the case when $Q_i \subset [0,1]$ are not identical, when these sets are perhaps infinite, and some other generalizations of this kind. The achieved results should be compared with those obtained in the most simple case of identical and finite sets Q_i as investigated above. Let us hope we will be allowed to focus our attention on some of these problems at an appropriate future occasion.

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