ON THE RUPTURE DEGREE OF A GRAPH

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Abstract: In a communication network, vulnerability measures the resistance of the network to disruption of operation after the failure of certain stations or communication links. If we think of a connected graph as model-ing a network, the *rupture degree* of a graph is one measure of *graph vulnerability* and it is defined by

$$r(G) = \max\{w(G-S) - |S| - m(G-S) : S \subset V(G), w(G-S) > 1\},$$

where $w(G-S)$ is the number of components of $G-S$ and $m(G-S)$ is the order of a largest component of $G-S$. In this paper, general results on the rupture degree of a graph are considered. Firstly, some bounds on the rupture degree are given. Further, rupture degree of a complete k-ary tree is calculated. Also several results are given about complete k-ary tree and graph operations. Finally, we give formulas for the rupture degree of the cartesian product of some special graphs.

Key words: *Networks, vulnerability, connectivity, rupture degree, complete k-ary tree*

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1. Introduction

In this paper, we consider simple connected graphs and assume G to be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. A network can be modeled by a graph G in which vertices represent the processing elements and edges represent the communication between them. Networks appear in many different applications and settings. The most common networks are telecommunication networks, computer networks, the internet, road and rail networks and other logistic networks [9]. A tree is a type of a graph that is used to model a network such as k-ary tree network, binary tree network, and n-tree network. The complete k-ary trees are widely used in supercomputers and they are also used to describe decision processes in tree networks and tree structures support various basic dynamic network operations including search, minimum, maximum, insert, and delete.

Network designers often build a network configuration around specific processing, performance and cost requirements. They also identify the critical points of failure and modify the design to eliminate them [9]. In a communication network,

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the vulnerability measures are essential to guide the designers in choosing an appropriate topology. They have an impact on solving difficult optimization problems for networks.

Furthermore, some networks can be modeled by graphs obtained by graph operations such as Cartesian product, composition, power, etc. The resultant graphs can be characterized in terms of the input graphs and extract information from the original graph and encode it into a new structure. For example the ladder graph $P_n \times K_2$ and cyclic ladder graph $C_n \times K_2$ are known as ladder networks.

All of the above motivated us to investigate the vulnerability of k-ary trees and some of the graphs obtained by graph operations.

In a communication network, vulnerability measures the resistance of the network to disruption of operation after the failure of certain stations or communication links. When a network begins losing stations or communication links there is, eventually, a loss in its effectiveness. Thus, a communication network must be constructed to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network. When any disruption happens in a network three questions are considered:

(1) What is the number of elements that are not functioning?

(2) What is the number of remaining connected subnetworks?

(3) What is the size of a largest remaining group within which mutual communication can still occur?

If we think of a connected graph G as modeling a network, then many graph theoretical parameters, such as connectivity (see [7]), integrity (see [2]), scattering number (see [6]), toughness (see [4]), tenacity (see [5]) and their edge-analogues, have been defined to obtain the answers to these questions. The definition of these parameters are given below.

Definition 1 ([7])**.** A separating set or a vertex cut $S \subset V(G)$ of a connected graph G is a set of vertices whose removal renders G disconnected such that is $G - S$ has more than one component.

Definition 2 ([3])**.** The connectivity of G, denoted $\kappa(G)$, is the minimal size of a vertex set S such that $G - S$ is disconnected or has only one vertex.

Let S be a vertex cut of a non-complete connected graph G. Throughout this paper for any graph $G - S$, $m(G - S)$ and $w(G - S)$, respectively, denote the order of the largest component and the number of components in $G - S$. Moreover, $|S|$ denotes the number of the elements in the set S.

Definition 3 ([2])**.** The integrity of a non-complete connected graph G is defined by

$$I(G) = \min\{|S| + m(G - S) : S \subset V(G), w(G - S) > 1\}.$$

Definition 4 ([6])**.** The scattering number of a non-complete connected graph G is defined by

$$s(G) = \max\{w(G - S) - |S| : S \subset V(G), w(G - S) > 1\}.$$

Definition 5 ([4])**.** The toughness of a non-complete connected graph G is defined by

$$t(G) = \min\{\frac{|S|}{w(G-S)} : S \subset V(G), w(G-S) > 1\}.$$

Definition 6 ([5]). The tenacity of a non-complete connected graph G is defined by

$$T(G) = \min\{(|S| + m(G-S))/w(G-S) : S \subset V(G), w(G-S) > 1\}.$$

These parameters have been used to measure the vulnerability of networks. In addition, the *rupture degree* was introduced as a measure of graph vulnerability by Yinkui Li, Shenggui Zhang and Xueliang Li (see [10, 11]). Formally, *the rupture degree* of a non-complete connected graph G is defined by

$$r(G) = \max\{w(G-S) - |S| - m(G-S) : S \subset V(G), w(G-S) > 1\}$$

and the rupture degree of K_n is defined as $1-n$. By the Definition 1, the set S must be vertex cut of a graph G.

The connectivity deals with the question (1). The toughness and the scattering number take account of questions (1) and (2). The integrity deals with the questions (1) and (3). The rupture degree is a measure which deals with the questions (1), (2), and (3). Therefore, the rupture degree gives us more knowledge about the network disruption. On the other hand, the tenacity is also a measure which deals with the questions (1), (2), and (3). But Zhang *et al.* show that there exist graphs G_1 and G_2 such that $T(G_1) = T(G_2)$ and $r(G_1) \neq r(G_2)$. That is, the rupture degree and tenacity differ in showing the vulnerability of networks (see [10]). Consequently the rupture degree is a better parameter to measure the vulnerability of some networks. Zhang *et al.* obtained several results on the rupture degree of a graph (see [10, 11]).

In Section 2, known results on the rupture degree are given. Section 3 gives some bounds on the rupture degree of a graph. In Section 4, rupture degree of a complete k-ary tree is calculated. Also some results are given about the complete binary tree and about the graphs obtained by some graph operations. In the final section, we give formulas for the rupture degree of the cartesian product of some special graphs.

2. Basic Results

Throughout this paper, we use Bondy and Murty ([3]) for terminology and notation. We denote the minimum vertex degree of a graph G by $\delta(G)$, the independence number of a graph G by $\alpha(G)$, the covering number of a graph G by $\beta(G)$, and the chromatic number of a graph G by $\chi(G)$. Moreover, we use $\lceil x \rceil$ for the smallest integer greater than x and $\lfloor x \rfloor$ for the greatest integer smaller than x.

Theorem 1 ([10]). *The rupture degree of*
(*a*) *the path graph* P_n $(n \geq 3)$ *is*

$$r(P_n) = \begin{cases} -1, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

(b) *the cycle graph C_n is*

$$r(C_n) = \begin{cases} -1, & \text{if } n \text{ is even,} \\ -2, & \text{if } n \text{ is odd.} \end{cases}$$

The next theorems give some bounds for the rupture degree.

Theorem 2 ([10]). *Let G be a non-complete connected graph of order n. Then*
(a) $r(G) \leq n - 2\delta - 1$,

(b) $3 - n \leq r(G) \leq n - 3$,

(c) $2\alpha(G) - n - 1 \leq r(G) \leq \dfrac{(\alpha(G))^2 - \kappa(G)(\alpha(G) - 1) - n}{\alpha(G)}$.

Theorem 3 ([11]). *Let G be a non-complete connected graph with the tenacity $T(G)$. Then $r(G) \leq \alpha(G)(1 - T(G))$.*

Definition 7 ([3]). Let G_1 and G_2 be two graphs with disjoint vertex sets V_1 and V_2. The *join operation* is denoted by G_1+G_2 and consists of $V(G_1) \cup V(G_2)$ vertices and the original edges $E(G_1)$, $E(G_2)$ and all edges joining $V(G_1)$ and $V(G_2)$.

The following theorem gives a result between the rupture degree and the join operation.

Theorem 4 ([10]). *Let G_1 and G_2 be two connected graphs of order n_1 and n_2, respectively. Then $r(G_1 + G_2) = max\{r(G_1) - n_2, r(G_2) - n_1\}$.*

Theorem 5 ([3]). *If G is a connected graph of order n, then $\alpha(G) + \beta(G) = n$.*

3. Bounds on the Rupture Degree

In this section, we give two upper bounds in the next theorems.

Theorem 6. *Let G be a non-complete connected graph of order n. Then*

$$r(G) \leq \alpha(G) - \delta(G) - 1.$$

Proof. Let S be a vertex cut of G. Then, from the definition of integrity, we know that $I(G) \leq |S| + m(G - S)$. Moreover, $I(G) \geq \delta(G) + 1$ for any graph G (see [2]). Then we have

$$|S| + m(G - S) \geq \delta(G) + 1.$$

If we subtract both sides from $w(G - S)$, then we get

$$w(G - S) - |S| - m(G - S) \leq w(G - S) - \delta(G) - 1.$$

Since $w(G - S) \leq \alpha(G)$ for any graph G, we have

$$r(G) \leq \alpha(G) - \delta(G) - 1.$$

□

Remark 1. It is obvious that

$$\alpha(G) - \delta(G) - 1 \leq n - 2\delta(G) - 1$$

since $\delta(G) \leq \beta(G)$ for any graph G. Hence, the result in Theorem 6 is better than Theorem 2(a).

Theorem 7. *Let G be a non-complete connected graph of order n. Then*

$$r(G) \leq n + 1 - \frac{2n}{\alpha(G)}.$$

Proof. Let S be a vertex cut of G. From the definition of integrity we know that $I(G) \leq |S| + m(G - S)$. Moreover, $I(G) \geq \chi(G)$ for any graph G (see [1]). Therefore, $\chi(G) \leq |S| + m(G - S)$.

On the other hand, since $w(G - S) \leq n - |S| - m(G - S) + 1$ (see [12]) for any graph G of order n, we have

$$\begin{aligned} w(G - S) - |S| - m(G - S) &\leq n + 1 - 2(|S| + m(G - S)) \\ &\leq n + 1 - 2\chi(G). \end{aligned}$$

Since $\chi(G) \geq \dfrac{n}{\alpha(G)}$ for any graph G of order n (see [3]), we have

$$\begin{aligned} w(G - S) - |S| - m(G - S) &\leq n + 1 - \frac{2n}{\alpha(G)} \\ r(G) &\leq n + 1 - \frac{2n}{\alpha(G)}. \end{aligned}$$

□

Remark 2. If $\alpha(G) \leq \beta(G)$ for any graph G, then since $\alpha(G) + \beta(G) = n$

$$\begin{aligned} 2\alpha(G) &\leq n \\ 4 &\leq \frac{2n}{\alpha(G)} \\ n + 1 - \frac{2n}{\alpha(G)} &\leq n - 3. \end{aligned}$$

So we claim that the result in Theorem 7 is better than Theorem 2(b). The graph $G = K_1 + (K_p \cup qK_1)$ such that $q + 1 \leq p$ can be given as an example where p and q are positive integers.

4. Rupture Degree of Complete k-ary Trees

In this section, we consider the complete k-ary trees. The following theorem is about the rupture degree of a complete k-ary tree.

Definition 8. The complete k-ary tree $T_{k,d}$ of depth d is the rooted tree in which all vertices at level $d - 1$ or less have exactly k children, and all vertices at level d are leaves.

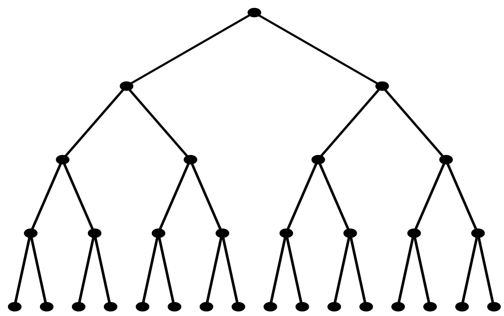

Fig. 1 $T_{2,4}$.

Fig. 1 shows a complete 2-ary tree $T_{2,4}$ of depth 4.

Theorem 8. *Let $T_{k,d}$ be a complete k-ary tree of depth d. Then*

$$r(T_{k,d}) = \frac{k^{d+1} + (-1)^d}{k+1} - 1.$$

Proof. A complete k-ary tree of depth d has $\frac{k^{d+1}-1}{k-1}$ vertices and the covering number of $T_{k,d}$ is

$$\beta(T_{k,d}) = \begin{cases} \dfrac{k^{d+1}-1}{k^2-1}, & \text{d is odd;} \\ \dfrac{k(k^d-1)}{k^2-1}, & \text{d is even.} \end{cases}$$

Let S be an arbitrary vertex cut of $T_{k,d}$ and $|S| = x$ be the number of vertices in S, in other words the number of vertices whose removal renders $T_{k,d}$ disconnected. Then we have two cases depending on S:

Case 1. Let S be a minimal vertex covering set of $T_{k,d}$. Then the cardinality of S gives the covering number and we have $|S| = \beta(T_{k,d})$ and hence $G - T_{k,d}$ has only isolated vertices and the order of the largest component is $m(T_{k,d} - S) = 1$. Therefore, the number of the components equals to the independence number and we get $w(T_{k,d} - S) = \frac{k(k^{d+1}-1)}{k^2-1}$ when d is odd and $w(T_{k,d} - S) = \frac{k^{d+2}-1}{k^2-1}$ when d is even. Thus,

$$r(T_{k,d}) = \begin{cases} \dfrac{k^{d+1}-k-2}{k+1}, & \text{d is odd;} \\ \dfrac{k^{d+1}-k}{k+1}, & \text{d is even.} \end{cases} \tag{1}$$

Case 2. If S is not a minimal vertex covering set of $T_{k,d}$, then we have two cases according to the cardinality of S.

(a) If $1 \leq x \leq \beta(T_{k,d}) - 1$, then $w(T_{k,d} - S) \leq kx + 1$ and $m(T_{k,d} - S) \geq 1$. Hence, $r(T_{k,d}) \leq \max_x \{kx + 1 - x - 1\} = \max_x \{(k-1)x\}$. The function $f(x) = (k-1)x$ is an increasing function and takes its maximum value at $x = \beta(T_{k,d}) - 1$. Then

$$r(T_{k,d}) \leq \begin{cases} \dfrac{k^{d+1} - k^2}{k+1}, & \text{d is odd;} \\ \dfrac{k^{d+1} - k^2 - k + 1}{k+1}, & \text{d is even.} \end{cases} \tag{2}$$

(b) If $\beta(T_{k,d}) \leq x \leq \frac{k^{d+1}-1}{k+1}$, then $w(T_{k,d} - S) \leq \frac{k^{d+1}-1}{k-1} - x$ and $m(T_{k,d} - S) \geq 1$. Hence, we have $r(T_{k,d}) \leq \max_x \left\{\dfrac{k^{d+1} - 1}{k-1} - x - x - 1\right\} = \max_x \left\{\dfrac{k^{d+1} - 1}{k-1} - 2x - 1\right\}$. The function $f(x) = \dfrac{k^{d+1} - 1}{k-1} - 2x - 1$ is a decreasing function and takes its maximum value at $x = \beta(T_{k,d})$. Hence,

$$r(T_{k,d}) \leq \begin{cases} \dfrac{k^{d+1} - k - 2}{k+1}, & \text{d is odd;} \\ \dfrac{k^{d+1} - k}{k+1}, & \text{d is even.} \end{cases} \tag{3}$$

By the definition of rupture degree if we take the maximum of (1), (2) and (3) and formulize it for the odd and even values of d, we get

$$r(T_{k,d}) = \frac{k^{d+1} + (-1)^d}{k+1} - 1.$$

□

Definition 9. The ath power of a graph G is a graph with the same set of vertices as G and an edge between two vertices if and only if there is a path in G of length at most a between them.

Now we consider a complete binary tree H_d such that it is a 2-ary complete tree $T_{2,d}$.

Theorem 9. *Let H_d^2 be the second power of a complete binary tree with depth d. Then*

$$r(H_d^2) = \begin{cases} 3 - 3 \times 2^{\frac{d-1}{2}}, & \textit{if d is odd} \\ 2 - 2^{\frac{d}{2}+1}, & \textit{if d is even} \end{cases}$$

Proof. A second power of a complete binary tree with depth d contains $2^0, 2^1, \ldots, 2^d$ vertices on $0th, 1st, \ldots, dth$ levels, respectively. Let S be a vertex cut of H_d^2 such that $w(H_d^2 - S) - |S| - m(H_d^2 - S) = r(H_d^2)$.
We have two cases for d:
Case 1. Let d be an odd integer. Then S must contain all the vertices on the xth and $(x+1)$th levels where $1 \leq x \leq \lfloor \dfrac{d}{2} \rfloor$. So $|S| = 2^x + 2^{x+1}$ where $1 \leq x \leq \lfloor \dfrac{d}{2} \rfloor$.

Hence, $w(H_d^2 - S) = 2^{x+1} + 1$ and $m(H_d^2 - S) = \sum_{t=1}^{d-(x+1)} 2^t = 2^{d-x} - 2$, where $1 \leq x \leq \lfloor \frac{d}{2} \rfloor$. Then we have

$$r(H_d^2) = \max_{1 \leq x \leq \frac{d}{2}} \{(2^{x+1} + 1) - (2^x + 2^{x+1}) - (2^{d-x} - 2)\} = \max_{1 \leq x \leq \frac{d}{2}} \{3 - 2^x - 2^{d-x}\}.$$

The function $3 - 2^x - 2^{d-x}$ takes its maximum value at $x = \frac{d-1}{2}$ when d is odd. Hence, if we substitute the maximum value in the function $3 - 2^x - 2^{d-x}$, then we have

$$r(H_d^2) = 3 - 3 \times 2^{\frac{d-1}{2}}. \tag{4}$$

Case 2. If d is an even integer, in addition to *Case 1*, S must also contain the vertex on the $0th$ level. So $|S| = 2^{x+1} + 2^x + 2^0$ where $1 \leq x \leq \lfloor \frac{d}{2} \rfloor$. Hence, $w(H_d^2 - S) = 2^{x+1} + 1$ and $m(H_d^2 - S) = 2^x - 2$, where $1 \leq x \leq \lfloor \frac{d}{2} \rfloor$.

$$r(H_d^2) = \max_{1 \leq x \leq \frac{d}{2}} \{(2^{x+1} + 1) - (2^{x+1} + 2^x + 1) - (2^x - 2)\} = \max_{1 \leq x \leq \frac{d}{2}} \{2 - 2^{x+1}\}.$$

Since the function $2 - 2^{x+1}$ is a decreasing function where $1 \leq x \leq \frac{d}{2}$, it takes its maximum value at $x = \frac{d}{2}$ when d is even. Hence, if we substitute the maximum value in the function $2 - 2^{x+1}$, then we have

$$r(H_d^2) = 2 - 2^{\frac{d}{2}+1}. \tag{5}$$

The proof is completed by the equalities (4) and (5). $\square$

Definition 10. The composition $G = G_1[G_2]$ of graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph with vertex set $V(G_1) \times V(G_2)$ and $u = (u_1, u_2)$ adjacent to $v = (v_1, v_2)$, whenever [u_1 adj v_1] or [$u_1 = v_1$ and u_2 adj v_2]. It is also called the graph lexicographic product.

Theorem 10. *Let H_{d_1} and H_{d_2} be complete binary trees with depth d_1 and d_2, respectively. Then*

$$r(H_{d_1}[H_{d_2}]) = \begin{cases} -\frac{1}{9}\ (2^{d_1+1} - 1)(2^{d_2+1} - 1) - 1, & \text{if } d_1 \text{ and } d_2 \text{ are odd;} \\ -\frac{2}{9}\ (8 - 5.2^{d_1} - 5.2^{d_2} + 2^{d_1+d_2+1}), & \text{if } d_1 \text{ and } d_2 \text{ are even;} \\ -\frac{2}{9}\ (7 - 5.2^{d_1} - 2^{d_2} + 2^{d_1+d_2+1}), & \text{if } d_1 \text{ is odd and } d_2 \text{ is even;} \\ -\frac{2}{9}\ (7 - 2^{d_1} - 5.2^{d_2} + 2^{d_1+d_2+1}), & \text{if } d_1 \text{ is even and } d_2 \text{ is odd.} \end{cases}$$

Proof. The proof is similar to that of Theorem 8. $\square$

5. Rupture Degree of Cartesian Product of Some Special Graphs

In the previous section, we give a general formula for the rupture degree of a complete k-ary tree of depth d and in the next theorem we determine the rupture degree of the Cartesian product of a complete graph with n vertices and a complete k-ary tree with depth d.

Definition 11. The Cartesian product $G = G_1 \times G_2$ of graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph with vertex set $V(G_1) \times V(G_2)$ and $u = (u_1, u_2)$ adjacent with $v = (v_1, v_2)$, whenever $[u_1 = v_1$ and u_2 adj $v_2]$ or $[u_2 = v_2$ and u_1 adj $v_1]$.

Theorem 11. *Let $T_{k,d}$ be a complete k-ary tree with depth d and K_n be a complete graph with n vertices where $k > n$. Then*

$$r(K_n \times T_{k,d}) = \begin{cases} \dfrac{k^{d+2} - k^{d+1}n - k^2n + 2n - k}{k^2 - 1}, & \text{if } d \text{ is odd} \\ \dfrac{k^{d+2} - k^{d+1}n - k^2n + kn + n - 1}{k^2 - 1}, & \text{if } d \text{ is even.} \end{cases}$$

Proof. $K_n \times T_{k,d}$, the Cartesian product of K_n and $T_{k,d}$, contains n copies of $T_{k,d}$. $\{1^i, 2^i, \ldots, (\frac{k^{d+1}-1}{k-1})^i\}$ is used for labeling each $T_{k,d}$, where $1 \le i \le n$, and the same numbered vertices of each copies form a copy of complete graph K_n. $K_n \times T_{k,d}$ contains $n\frac{k^{d+1}-1}{k-1}$ vertices since there are $k^0 n, k^1 n, \ldots, k^d n$ vertices on $0th, 1st, \ldots, dth$ levels, respectively. Let S be an arbitrary vertex cut of $K_n \times T_{k,d}$ and let $|S| = x$ be the number of vertices whose removal renders the graph disconnected.
Then we have two cases according to the cardinality of S:

Case 1. If $1 \le x \le n\beta(T_{k,d}) - 1$, then removing n vertices reveals at most $k+1$ components and so removing x vertices reveals at most $\dfrac{kx}{n} + 1$ components and each component has at least n vertices, i.e. $w((K_n \times T_{k,d}) - S) \le \dfrac{kx}{n} + 1$ and $m((K_n \times T_{k,d}) - S) > n$. Hence, $r(K_n \times T_{k,d}) < \max\limits_x \left\{ \dfrac{kx}{n} + 1 - x - n \right\} = \max\limits_x \left\{ x\left(\dfrac{k}{n} - 1\right) + 1 - n \right\}$. The function $f(x) = x(\frac{k}{n} - 1) + 1 - n$ is an increasing function and takes its maximum value at $x = n\beta(T_{k,d}) - 1$. Then

$$r(K_n \times T_{k,d}) < \begin{cases} \frac{k^{d+2} - k^{d+1}n - k^2n + 2n - k - k^3/n + 2k^2 + k/n - 2}{k^2-1}, & \text{if } d \text{ is odd} \\ \frac{k^{d+2} - k^{d+1}n - k^2n + n + nk - k^3/n + k^2 + k/n - 2}{k^2-1}, & \text{if } d \text{ is even.} \end{cases} \tag{6}$$

Case 2. If $n\beta(T_{k,d}) \le x \le n\dfrac{k^{d+1} - 1}{k - 1}$, then removing vertices renders the graph having components at most as the independence number of $T_{k,d}$ and so $m((K_n \times T_{k,d}) - S) \ge n - \left\lfloor \dfrac{x - n\beta(T_{k,d})}{\alpha(T_{k,d})} \right\rfloor$, $w((K_n \times T_{k,d}) - S) \le \dfrac{k^{d+2} - k}{k^2 - 1}$ when d is odd and $w((K_n \times T_{k,d}) - S) \leq \dfrac{k^{d+2}-1}{k^2-1}$ when d is even.

If d is odd, then we have $r(K_n \times T_{k,d}) \leq \max\limits_{x} \left\{ \dfrac{k^{d+2}-k}{k^2-1} - x - \left(n - \dfrac{x - n\beta(T_{k,d})}{\alpha(T_{k,d})} \right) \right\}$.

The function $f(x) = \dfrac{k^{d+2}-k}{k^2-1} - x - \left(n - \dfrac{x - n\beta(T_{k,d})}{\alpha(T_{k,d})} \right)$ is a decreasing function and takes its maximum value at $x = n\beta(T_{k,d})$.

If d is even, then $r(K_n \times T_{k,d}) \leq \max\limits_{x} \left\{ \dfrac{k^{d+2}-1}{k^2-1} - x - \left(n - \dfrac{x - n\beta(T_{k,d})}{\alpha(T_{k,d})} \right) \right\}$. The function $f(x) = \dfrac{k^{d+2}-1}{k^2-1} - x - \left(n - \dfrac{x - n\beta(T_{k,d})}{\alpha(T_{k,d})} \right)$ is a decreasing function and takes its maximum value at $x = n\beta(T_{k,d})$. Thus, we get

$$r(K_n \times T_{k,d}) \leq \begin{cases} \dfrac{k^{d+2} - k^{d+1}n - k^2 n + 2n - k}{k^2-1}, & \text{if } d \text{ is odd} \\ \dfrac{k^{d+2} - k^{d+1}n - k^2 n + kn + n - 1}{k^2-1}, & \text{if } d \text{ is even.} \end{cases} \tag{7}$$

It is easy to see that there is a vertex cut S^* of $K_n \times T_{k,d}$ such that $|S^*| = n\beta(T_{k,d})$, $m((K_n \times T_{k,d}) - S^*) = n$ and $w((K_n \times T_{k,d}) - S^*) = \dfrac{k^{d+2}-k}{k^2-1}$ when d is odd and $w((K_n \times T_{k,d}) - S^*) = \dfrac{k^{d+2}-1}{k^2-1}$ when d is even. Therefore,

$$r(K_n \times T_{k,d}) = \begin{cases} \dfrac{k^{d+2} - k^{d+1}n - k^2 n + 2n - k}{k^2-1}, & \text{if } d \text{ is odd} \\ \dfrac{k^{d+2} - k^{d+1}n - k^2 n + kn + n - 1}{k^2-1}, & \text{if } d \text{ is even.} \end{cases} \tag{8}$$

By the definition of rupture degree if we take the maximum of (6), (7), and (8) we get

$$r(K_n \times T_{k,d}) = \begin{cases} \dfrac{k^{d+2} - k^{d+1}n - k^2 n + 2n - k}{k^2-1}, & \text{if } d \text{ is odd} \\ \dfrac{k^{d+2} - k^{d+1}n - k^2 n + kn + n - 1}{k^2-1}, & \text{if } d \text{ is even.} \end{cases}$$

□

Next, we give the rupture degree of the Cartesian product of two special graphs. Let C_n be a cycle graph of order n. A cycle graph with an even number of vertices is called an even cycle. Similarly, a cycle graph with an odd number of vertices is called an odd cycle. We know that if C_n is an even cycle graph of order n, then $r(C_n \times K_2) = -1$ (see [11]). In the following theorem, we calculate the rupture degree of the Cartesian product of an odd cycle graph C_n and a complete graph K_2.

Theorem 12. *Let C_n be an odd cycle graph of order n. Then $r(C_n \times K_2) = -3$.*

Proof. Let S be an arbitrary vertex cut of $C_n \times K_2$ and set $|S| = x$. If $3 \le x \le n$, then $w((C_n \times K_2) - S) \le x - 1$. Therefore, we have $m((C_n \times K_2) - S) \ge \left\lceil \frac{2n-x}{x-1} \right\rceil$. Hence,

$$\begin{aligned} w((C_n \times K_2) - S) - |S| - m((C_n \times K_2) - S) &\le x - 1 - x - \left\lceil \frac{2n-x}{x-1} \right\rceil \\ &\le -1 - \left\lceil \frac{2n-n}{n-1} \right\rceil \le -3. \end{aligned}$$

If $x \ge n+1$, then $w((C_n \times K_2) - S) \le 2n - x$ and $m((C_n \times K_2) - S) \ge 1$. Then

$$\begin{aligned} w((C_n \times K_2) - S) - |S| - m((C_n \times K_2) - S) &\le 2n - x - x - 1 \\ &\le 2n - 2(n+1) - 1 \le -3. \end{aligned}$$

By the choice of S, we have

$$r(C_n \times K_2) \le -3. \tag{9}$$

It is easy to see that there is a vertex cut S^* of $C_n \times K_2$ such that $|S^*| = n+1$, $w((C_n \times K_2) - S^*) = n - 1$ and $m((C_n \times K_2) - S^*) = 1$. Then from the definition of rupture degree we have

$$\begin{aligned} r(C_n \times K_2) &\ge w((C_n \times K_2) - S^*) - |S^*| - m((C_n \times K_2) - S^*) \\ &= n - 1 - (n+1) - 1 = -3. \end{aligned}$$

Hence,

$$r(C_n \times K_2) \ge -3. \tag{10}$$

The proof is completed by (9) and (10). □

Now we consider the graph $P_n \times K_2$ for $n \ge 2$.

Theorem 13. *Let $n \ge 2$ be an integer and P_n be a path graph of order n. Then $r(P_n \times K_2) = -1$.*

Proof. Let S be an arbitrary vertex cut of $P_n \times K_2$ and set $|S| = x$.

If $x \le n$, then $w((P_n \times K_2) - S) \le x$. Therefore, we have $m((P_n \times K_2) - S) \ge \left\lceil \frac{2n-x}{x} \right\rceil$. Hence,

$$\begin{aligned} w((P_n \times K_2) - S) - |S| - m((P_n \times K_2) - S) &\le x - x - \left\lceil \frac{2n-x}{x} \right\rceil \\ &\le -\left\lceil \frac{2n-x}{n} \right\rceil \le -1. \end{aligned}$$

If $x \ge n$, then $w((P_n \times K_2) - S) \le 2n - x$ and $m((P_n \times K_2) - S) \ge 1$. So

$$\begin{aligned} w((P_n \times K_2) - S) - |S| - m((P_n \times K_2) - S) &\le 2n - x - x - 1 \\ &\le 2n - 2n - 1 \le -1. \end{aligned}$$

By the choice of S, we have

$$r(P_n \times K_2) \leq -1. \tag{11}$$

It is easy to see that there is a vertex cut S^* of $P_n \times K_2$ such that $|S^*| = n$, $w((P_n \times K_2) - S^*) = n$ and $m((P_n \times K_2) - S^*) = 1$. Then

$$\begin{aligned} r(P_n \times K_2) &\geq w((P_n \times K_2) - S^*) - |S^*| - m((P_n \times K_2) - S^*) \\ &= n - n - 1 = -1. \end{aligned}$$

Hence,

$$r(P_n \times K_2) \geq -1. \tag{12}$$

The proof is completed by (11) and (12). □

Finally, we give an upper bound for the rupture degree of the Cartesian product of an even cycle graph C_n of order n and a complete graph K_p of order p.

Theorem 14. *Let n be an even integer and p be an integer. For $n \geq 4$ and $p \geq 2$,*

$$r(C_n \times K_p) \leq -2\sqrt{\frac{n}{2}p\ (p-2)} + \frac{p}{2}.$$

Proof. Let S be a vertex cut and set $|S| = x$. If we remove $|S| = x$ vertices, then the remaining graph has at most $\frac{2x}{p}$ components. Then the remaining connected components has at least $m((C_n \times K_p) - S) \geq \frac{p\ (np - x)}{2x}$ vertices. Therefore,

$$\begin{aligned} w((C_n \times K_p) - S) - |S| - m((C_n \times K_p) - S) &\leq \frac{2x}{p} - x - \frac{p\ (np-x)}{2x} \\ &= \frac{4x^2 - 2x^2p - p^3n + p^2x}{2xp}. \end{aligned}$$

Let $f(x) = \frac{4x^2 - 2x^2p - p^3n + p^2x}{2xp}$. f has a maximum value at $x = p\sqrt{\frac{\frac{n}{2}p}{p-2}}$ and so

$$r(C_n \times K_p) \leq -2\sqrt{\frac{n}{2}p\ (p-2)} + \frac{p}{2}.$$

□

6. Conclusion

In a communication network, the vulnerability measures are essential to guide the designer in choosing an appropriate topology. They measure the stability of the network to disruption of operation after the failure of certain stations or communication links. In the graph theory, many parameters measuring the vulnerability of communication networks have been defined. The rupture degree of a graph is a measure of vulnerability that deals with the number of elements that are not functioning, the number of remaining connected subnetworks and the size of a largest remaining group within which mutual communication can still occur in a disrupted network. In this paper, we investigate the rupture degree of complete k-ary trees, some resultant graphs obtained by graph operations such as Cartesian product, composition and power. We also give some bounds for rupture degree that are better than the results given in [10].

References

[1] Bagga K. S, Beineke L. W., Goddard W. D., Lipman M. J., Pipperd R. E.: A Survey of Integrity. Discrete Applied Mathematics, **37/38**, 1992, pp. 13–28.

[2] Barefoot C. A, Entringer R., Swart H.: Vulnerability in Graphs-A Comparative Survey. J. Combin. Math. Combin. Comput., **1**, 1987, pp. 13–22.

[3] Bondy J. A., Murty U. S. R.: Graph Theory with Applications. London, Macmillan, 1976.

[4] Chvatal V.: Tough Graphs and Hamiltonian Circuits. Discrete Math., 1973, pp. 215–228.

[5] Cozzens M. B., Moazzami D., Stueckle S.: The Tenacity of a Graph. Siam Journal on Discrete Math., 1994, pp. 1111–1122.

[6] Jung H. A.: On a Class of Posets and the Corresponding Comparability Graphs. J. Combin. Theory Ser. B, **1124**, 1978, pp. 125–133.

[7] Douglas B. West: Introduction to Graph Theory, Prentice Hall, 2001.

[8] Kırlangıç A.: The Edge-Integrity of Some Graphs. J. Combin. Math. Combin. Comput., **37**, 2001, pp. 139–148.

[9] Newport K. T., Varshney P. K.: Design of survivable communication networks under performance constraints. IEEE Transactions on Reliability, **40** 1991, pp. 433–440.

[10] Li Y., Zhang S., Li X.: Rupture Degree of Graphs. International Journal of Computer Mathematics, **82-7**, 2005, pp. 793–803.

[11] Li F., Li X.: Computing the Rupture Degrees of Graphs. Proceedings of the 7th International Symposium on Parallel Architectures, Algorithms and Networks (ISPAN'04), 2004, IEEE.

[12] Zhang S., Peng S.: Relationships between scattering number and other vulnerability parameters. Int. J. Comput. Math., **81**, 2004, pp. 291–298.