NO ROUNDING REVERSE FUZZY MORPHOLOGICAL ASSOCIATIVE MEMORIES

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Abstract: The fuzzy morphological associative memories (FMAM) have many attractive advantages, but their recall effects for hetero associative memories are poor. This shortcoming impedes the application of hetero-FMAM. Aiming at the problem, and inspired by the unified framework of morphological associative memories, a new method called no rounding reverse fuzzy morphological associative memories (NR^2FMAM) is presented by the paper. The value of the new method lies in hetero associative memories. Analyses and experiments show that, it has significantly affected hetero associative morphological memories and with a certain noise robustness. In some cases, it can work more effectively with greater correct recall rate than FMAM. The paper analyzes the reason that NR^2FMAM is sometimes better than FMAM, and thinks that no rounding neural computing is one of passable reasons. At the same time, the condition that the recall rate of NR^2FMAM is greater than FMAM is given by the corresponding theorem in this paper. The NR^2FMAM not only enriched the theory of the morphological associative mnemonic framework, but also helps contribute to the solution of the hetero associative mnemonic problem which is incomplete. At the same time, it can serve as a new identification technology in social networks.

Key words: *fuzzy morphological associative memories, framework, hetero association, incomplete recall, no rounding*

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1. Introduction

Associative memories (AM) are one of the functions of the human brain, also the source of thinking and innovation. Using computers to realize the simulations for associative memories is one of our pursuits of goals. In 1982, Hopfield proposed the famous Hopfield neural network [9], and opened a new era of simulating associative memories. However, the Hopfield network requires orthogonal input patterns.

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Meanwhile, the convergence problem with the network exists. Besides,its storage capacity is limited, no more than 15% of the total network neurons [8]. This means that a large number of neurons in the network are wasted!

Hopfield network, radial basis function network, probabilistic neural network [10, 12], and so on are all belonged to traditional artificial neural networks. Although they have many applications in pattern recognition and picture processing [11,13,25], classification [14], prediction, etc., their limitations are also obvious, for example, the "black box", longer training time, and poorer generalization ability.

Morphological neural networks (MNN) turned over a new leaf for the development of artificial associative memories. In 1998, Ritter et al. realized morphological associative memories (MAM) by using morphological neural networks [17]. After that, people presented some of the other theories and methods of morphological associative memories, for example, morphological bidirectional associative memories (MBAM) [16], complex morphological associative memories (CMAM) [1], and so on. Wang et al. treated input vectors and output vectors as fuzzy sets, therefore they presented the fuzzy morphological associative memories (FMAM) [24], enhanced FMAM (EFMAM) based on empirical kernel map [22] and economized EFMAM (E^2FMAM) [23]. Feng et al. proposed the unified framework of morphological associative memories in complex domain ($\mathrm{UFMAM_{CD}}$) [3], and the MAM based on four-dimensional storage (MAM-FDS) [7], and the Logarithmic and exponential MAM (LEMAM) [6].

Morphological associative memories have many advantages. In contrast to traditional associative memories, morphological associative memories converge in one step. Thus, convergence problems do not exist. Morphological analogues to the Hopfield network not only proved to be far more robust in the presence of noise but have also unlimited storage capacity for perfect inputs. Wang et al. also pointed out that morphological auto-associative memories (auto-MAM) and fuzzy morphological auto-associative memories (auto-FMAM) have many attractive advantages such as unlimited storage capacity, one-shot recall speed and good noise-tolerance to single erosive or dilative noise. Morphological associative memories have many applications in pattern recognition [19, 21], image processing [2, 5], classification and prediction [20], psychology research [4], and so on.

However, the researches and applications of morphological associative memories mainly concentrated on the auto-associative memories, but the studies and applications for hetero-associative memories were relatively less. The main reason is that hetero-associative morphological memories (HAMM) are incomplete, namely, they do not guarantee perfect recall memories, even if their inputs are complete. We call this problem"imperfectness of HAMM". This problem does not exist for auto-associative morphological memories (AAMM). In the case of complete inputs, auto-associative morphological memories will ensure complete recall memories. Although the HAMM based on four-dimensional storage can guarantee perfect recall memories for hetero-associative morphological memories in the case of complete inputs, this method obviously increases some overhead of time and space. Under the circumstances, it is not suitable for real-time processing and large-scale problems. Therefore, it is necessary to develop some different methods of MAM in order to improve the performance of HAMM.

In this paper, aiming at the shortcoming of HAMM, we put forward a new method of MAM, called no rounding reverse FMAM (NR^{2}FMAM). It is formed on the basis of UFMAM$_{\mathrm{CD}}$. Firstly, it was inspired by UFMAM$_{\mathrm{CD}}$ and secondly described by the symbol system of UFMAM$_{\mathrm{CD}}$. It should be pointed out that NR^{2}FMAM and FMAM is different. FMAM is a kind of morphological associative memories "from division to multiplication", while NR^{2}FMAM a kind of morphological associative memories "from multiplication to division". This method of NR^{2}FMAM looks simple, but sometimes will bring amazing changes. Experiments show that NR^{2}FMAM is a very useful morphological associative mnemonic method. In many cases, it can get better effects of HAMM than FMAM. At the same time, our analysis shows that it can serve as a new identification technology in social networks.

The rest of this paper is organized as follows. A brief introduction to UFMAM$_{\mathrm{CD}}$ is given in Section 2. Our NR^{2}FMAM is detailed in Section 3. Section 4 is the performance of NR^{2}FMAM and its comparison with FMAM. This part presents the experimental results of our method. Experiments show that, in some cases, the NR^{2}FMAM achieved better effects on hetero-associative memories. Sometimes, its correct recall rate even approached 100%! Why is this? Why can a simple change bring us the good results? And under what conditions, NR^{2}FMAM can obtain better results of hetero-associative memories than FMAM? These problems are discussed and analyzed in Section 5. The conclusions are given in Section 6.

2. UFMAM$_{\mathrm{CD}}$

It can bring us some inspirations to know UFMAM$_{\mathrm{CD}}$ from the following five aspects.

2.1 Computational basis of UFMAM$_{\mathrm{CD}}$

The basic computation occurring in UFMAM$_{\mathrm{CD}}$ is based on the algebraic lattice structure $(\mathbb{U}, \wedge, \vee, \bigcirc)$, where the symbol $\mathbb{U}$ denotes a set or domain, such as $\mathbb{U} = \mathbb{R}(\mathbb{R} = (-\infty, +\infty)), \mathbb{U} = \mathbb{R}_+(\mathbb{R}_+ = (0, +\infty))$, or $\mathbb{U} = \mathbb{C}(\mathbb{C} = \{c\,|c = a \pm bi, a$ and $b \in \mathbb{R}, i = \sqrt{-1}\})$; the symbols $\wedge$ and $\vee$ denote the binary operations of minimum and maximum, respectively. The symbol $\bigcirc$ represents the closed operation on $\mathbb{U}$. For example, $\bigcirc$ can be arithmetical operator $+$ or $-$, which is closed operation on $\mathbb{R}$. Also, it can be $\cdot$ (multiplication) or / (division), which is closed operation on $\mathbb{R}_+$. $\bigcirc$ can even be beyond operator, such as the logarithmic operator or exponential operator, which is closed operation on $\mathbb{R}_+$ if we constrain that logarithmic base and logarithmic antilogarithm are all greater than 1. We also use the symbol $\ominus$ to denote the inverse operation of $\bigcirc$. Of course, $\bigcirc$ is also the inverse operation of $\ominus$.

2.2 Conditions accepted by UFMAM$_{\mathrm{CD}}$

An object in UFMAM$_{\mathrm{CD}}$ satisfies the following conditions:

(1) Ordered: if $a, b \in \mathbb{U}$, then $a \leqslant b$ or $b \leqslant a$;

(2) Closed: if $a, b \in \mathbb{U}$ and $a \bigcirc b = r$, then $r \in \mathbb{U}$;

(3) Morphological: adopting morphological paradigm and operators in UFMAM_{CD};

(4) Valid: following the right algorithms.

2.3 Reciprocal inverse operations in UFMAM_{CD}

UFMAM_{CD} pointed out that in morphological associative memories, there are two kinds of reciprocal inverse operations, namely,

(1) Reciprocal inverse operations $\wedge$ and $\vee$. They are the minimum operator and the maximum operator, respectively, and also the basic erosion and dilation operations in mathematical morphology, respectively.

(2) Reciprocal inverse operators $\bigcirc$ and $\ominus$. They are two abstract reciprocal inverse operators. To be specific, they can be $+$ and $-$, also can be $\cdot$ and $/$. Of course, if the conditions accepted by UFMAM_{CD} are satisfied, we do not rule out the possibility that other reciprocal inverse operators are used. UFMAM_{CD} pointed out that among various kinds of morphological associative memories, an obvious common characteristic is that if an operator and an operation are used in the process of memories, then the corresponding inverse operator and inverse operation are must be used in the process of associative recall. In the memory phase of FMAM, for example, the memory matrix $\mathbf{A_{XY}}$ for the input pattern matrix $\mathbf{X}$ and the output pattern matrix $\mathbf{Y}$ is computed by using operator $/$ and operation $\wedge$; or the memory matrix $\mathbf{B_{XY}}$ for $\mathbf{X}$ and $\mathbf{Y}$ is computed by using operator $/$ and operation $\vee$. But in the recall phase of FMAM, the $\mathbf{Y}$ is computed by using operator $\cdot$ and operation $\vee$ for $\mathbf{A_{XY}}$ and $\mathbf{X}$; or the $\mathbf{Y}$ is computed by using operator $\cdot$ and operation $\wedge$ for $\mathbf{B_{XY}}$ and $\mathbf{X}$.

2.4 Symbols in UFMAM_{CD}

In UFMAM_{CD}, the abstract operator $\bigcirc$ or $\ominus$, and the operation $\wedge$ or $\vee$ fuse together and transform into an organic whole, thus form the abstract morphological memory operator $\overset{\circ}{\wedge}$ or $\overset{\circ}{\vee}$, and the corresponding abstract morphological recall operator $\overset{\ominus}{\vee}$ or $\overset{\ominus}{\wedge}$. For arithmetic operations, we can construct eight specific morphological operators. They are $\overset{+}{\wedge}, \overset{-}{\wedge}, \overset{\cdot}{\wedge}, \overset{/}{\wedge}, \overset{+}{\vee}, \overset{-}{\vee}, \overset{\cdot}{\vee}$ and $\overset{/}{\vee}$, respectively.

2.5 Memory and recall in UFMAM_{CD}

Let $(\mathbf{x}^1, \mathbf{y}^1), \ldots, (\mathbf{x}^k, \mathbf{y}^k)$ be k vector pairs with$(x_1^\xi, \ldots, x_n^\xi) \in \mathbb{R}^n$ and $(y_1^\xi, \ldots, y_n^\xi) \in \mathbb{R}^m$ for $\xi = 1, \ldots, k$. For a given set of pattern associations $(\mathbf{x}^\xi, \mathbf{y}^\xi) : \xi = 1, \ldots, k$ we define a pair of associated pattern matrices $(\mathbf{X}, \mathbf{Y})$, where $\mathbf{X}{=}(\mathbf{x}^1, \ldots, \mathbf{x}^k)$ and $\mathbf{Y}{=}(\mathbf{y}^1, \ldots, \mathbf{y}^k)$. Thus, $\mathbf{X}$ is of dimension $n \times k$ and $\mathbf{Y}$ is of dimension $m \times k$. With each pair of matrices $(\mathbf{X}, \mathbf{Y})$ we define two natural morphological $m \times n$ memories $\mathbf{W_{XY}}$ and $\mathbf{M_{XY}}$ as follows.

Definition 1. Morphological $\overset{\circ}{\wedge}$-memory $\mathbf{W_{XY}}$ is defined by

$$\mathbf{W_{XY}} = \mathbf{Y} \overset{\circ}{\wedge} \mathbf{X}' = \bigwedge_{\xi=1}^{k} \left[\mathbf{y}^\xi \overset{\circ}{\wedge} (\mathbf{x}^\xi)' \right] = \bigwedge_{\xi=1}^{k} \begin{bmatrix} y_1^\xi \bigcirc x_1^\xi & \ldots & y_1^\xi \bigcirc x_n^\xi \\ \vdots & \ddots & \vdots \\ y_m^\xi \bigcirc x_1^\xi & \ldots & y_m^\xi \bigcirc x_n^\xi \end{bmatrix}. \tag{1}$$

Its arbitrary element w_{ij} is given by

$$w_{ij} = \bigwedge_{\xi=1}^{k} (y_i^\xi \bigcirc x_j^\xi). \tag{2}$$

Definition 2. Morphological $\overset{\bigcirc}{\vee}$-memory $\mathbf{M_{XY}}$ is defined by

$$\mathbf{M_{XY}} = \mathbf{Y} \overset{\bigcirc}{\vee} \mathbf{X}' = \bigvee_{\xi=1}^{k} \left[\mathbf{y}^\xi \overset{\bigcirc}{\vee} (\mathbf{x}^\xi)'\right] = \bigvee_{\xi=1}^{k} \begin{bmatrix} y_1^\xi \bigcirc x_1^\xi & \dots & y_1^\xi \bigcirc x_n^\xi \\ \vdots & \ddots & \vdots \\ y_m^\xi \bigcirc x_1^\xi & \dots & y_m^\xi \bigcirc x_n^\xi \end{bmatrix}. \tag{3}$$

Its arbitrary element m_{ij} is given by

$$m_{ij} = \bigvee_{\xi=1}^{k} (y_i^\xi \bigcirc x_j^\xi). \tag{4}$$

Obviously, when $k = 1$ and $(\mathbf{X}, \mathbf{Y}) = (\mathbf{x}^\xi, \mathbf{y}^\xi)$, we have

$$\mathbf{W_{XY}} = \mathbf{M_{XY}} = \mathbf{Y} \overset{\bigcirc}{\wedge} \mathbf{X}' = \mathbf{Y} \overset{\bigcirc}{\vee} \mathbf{X}' = \begin{bmatrix} y_1^\xi \bigcirc x_1^\xi & \dots & y_1^\xi \bigcirc x_n^\xi \\ \vdots & \ddots & \vdots \\ y_m^\xi \bigcirc x_1^\xi & \dots & y_m^\xi \bigcirc x_n^\xi \end{bmatrix}. \tag{5}$$

Under the stimulus of input pattern $\mathbf{x}^\gamma$, the morphological associative memory network generates association and recall. UFMAM$_{\mathrm{CD}}$ also defines two abstract morphology recall paradigms with the recall pattern $\mathbf{y}$.

Definition 3. Morphological $\overset{\ominus}{\vee}$-recall paradigm is defined by

$$\mathbf{y} = \mathbf{W_{XY}} \overset{\ominus}{\vee} \mathbf{x}^\gamma = \begin{bmatrix} \bigvee_{i=1}^{n} (w_{1i} \ominus x_i^\gamma) \\ \vdots \\ \bigvee_{i=1}^{n} (w_{mi} \ominus x_i^\gamma) \end{bmatrix}. \tag{6}$$

Definition 4. Morphological $\overset{\ominus}{\wedge}$-recall paradigm is defined by

$$\mathbf{y} = \mathbf{M_{XY}} \overset{\ominus}{\wedge} \mathbf{x}^\gamma = \begin{bmatrix} \bigwedge_{i=1}^{n} (m_{1i} \ominus x_i^\gamma) \\ \vdots \\ \bigwedge_{i=1}^{n} (m_{mi} \ominus x_i^\gamma) \end{bmatrix}. \tag{7}$$

Distinctly, if $(\mathbf{X}, \mathbf{Y}) = (\mathbf{x}^\xi, \mathbf{y}^\xi)$, namely $(\mathbf{X}, \mathbf{Y})$ has only a pair of vectors,then

$$\mathbf{y} = \mathbf{W_{XY}} \overset{\ominus}{\vee} \mathbf{x}^\xi = \mathbf{y}^\xi \overset{\bigcirc}{\wedge} (\mathbf{x}^\xi)' \overset{\ominus}{\vee} \mathbf{x}^\xi = \begin{bmatrix} \bigvee_{i=1}^{n} (y_1^\xi \bigcirc x_i^\xi \ominus x_i^\xi) \\ \vdots \\ \bigvee_{i=1}^{n} (y_m^\xi \bigcirc x_i^\xi \ominus x_i^\xi) \end{bmatrix} = \mathbf{y}^\xi \tag{8}$$

or

$$\mathbf{y} = \mathbf{M_{XY}} \overset{\ominus}{\wedge} \mathbf{x}^\xi = \mathbf{y}^\xi \overset{\bigcirc}{\vee} (\mathbf{x}^\xi)' \overset{\ominus}{\wedge} \mathbf{x}^\xi = \begin{bmatrix} \bigwedge_{i=1}^{n} (y_1^\xi \bigcirc x_i^\xi \ominus x_i^\xi) \\ \vdots \\ \bigwedge_{i=1}^{n} (y_m^\xi \bigcirc x_i^\xi \ominus x_i^\xi) \end{bmatrix} = \mathbf{y}^\xi. \tag{9}$$

3. No rounding reverse FMAM

We now introduce our no rounding reverse FMAM (NR^{2}FMAM).

3.1 Computational basis of NR^{2}FMAM

In the unified framework of morphological associative memories, for the abstract algebraic lattice structure $(\mathbb{U}, \wedge, \vee, \bigcirc)$, if we take $\mathbb{U} = \mathbb{R}_+, \bigcirc = \cdot$ (accordingly, $\ominus = /$), then the computational basis of NR^{2}FMAM is just established.

3.2 Memory of NR^{2}FMAM

Definition 5. In NR^{2}FMAM, the $\dot{\wedge}$ Memory denoted by $\mathbf{C_{XY}}$ is defined by

$$\mathbf{C_{XY}} = \mathbf{Y} \dot{\wedge} \mathbf{X}' = \bigwedge_{\xi=1}^{k} [\mathbf{y}^{\xi} \dot{\wedge} (\mathbf{x}^{\xi})']. \tag{10}$$

Its arbitrary element c_{ij} is given by

$$c_{ij} = \bigwedge_{\xi=1}^{k} (y_i^{\xi} \cdot x_j^{\xi}). \tag{11}$$

Definition 6. In NR^{2}FMAM, the $\dot{\vee}$_Memory denoted by $\mathbf{D_{XY}}$ is defined by

$$\mathbf{D_{XY}} = \mathbf{Y} \dot{\vee} \mathbf{X}' = \bigvee_{\xi=1}^{k} [\mathbf{y}^{\xi} \dot{\vee} (\mathbf{x}^{\xi})']. \tag{12}$$

Its arbitrary element d_{ij} is given by

$$d_{ij} = \bigvee_{\xi=1}^{k} (y_i^{\xi} \cdot x_{\mathrm{j}}^{\xi}). \tag{13}$$

3.3 Recall of NR^{2}FMAM

NR^{2}FMAM takes into the association and recall process with the stimulation of input pattern $\mathbf{x}^{\gamma}$. The recall or output pattern can be obtained by replacing $\mathbf{W_{XY}}$ with $\mathbf{C_{XY}}$ and replacing $\ominus$ with / in the formula (6), or by replacing $\mathbf{M_{XY}}$ with $\mathbf{D_{XY}}$ and replacing $\ominus$ with / in the formula (7). The two types of recall are called $\overset{/}{\vee}$-recall and $\overset{/}{\wedge}$-recall by us, respectively. Their definitions are as follows.

Definition 7. In NR^{2}FMAM, the $\overset{/}{\vee}$-recall is defined by

$$\mathbf{C_{XY}} \overset{/}{\vee} \mathbf{x}^{\gamma} = \begin{bmatrix} \bigvee_{i=1}^{n} (c_{1i}/x_i^{\gamma}) \\ \vdots \\ \bigvee_{i=1}^{n} (c_{mi}/x_i^{\gamma}) \end{bmatrix} \tag{14}$$

Definition 8. In NR^{2}FMAM, the $\acute{\wedge}$-recall is defined by

$$\mathbf{D_{XY}}\acute{\wedge}\mathbf{x}^\gamma = \begin{bmatrix} \bigwedge_{i=1}^n (d_{1i}/x_i^\gamma) \\ \vdots \\ \bigwedge_{i=1}^n (d_{mi}/x_i^\gamma) \end{bmatrix} \tag{15}$$

4. Performance of NR^{2}FMAM and its comparison with FMAM

Theorem 1. $\mathbf{C_{XY}}$ *is a* $\dot{\wedge}$*-perfect memory for* $(\mathbf{X}, \mathbf{Y})$*, if and only if for each* $\xi = 1, \ldots, k$*, each row of the matrix* $[\mathbf{y}^\xi \,\dot{\wedge}\, (\mathbf{x}^\xi)'] - \mathbf{C_{XY}}$ *contains a zero entry.*

Proof. $\mathbf{C_{XY}}$ is a $\dot{\wedge}$-perfect memory for $(\mathbf{X}, \mathbf{Y}), \forall \xi = 1, \ldots, k$ and $\forall i = 1, \ldots, m$

$$\begin{aligned} \Leftrightarrow (\mathbf{C_{XY}}\acute{\vee}\mathbf{x}^\xi)_i &= y_i^\xi \Leftrightarrow y_i^\xi - (\mathbf{C_{XY}}\acute{\vee}\mathbf{x}^\xi)_i = 0 \\ \Leftrightarrow y_i^\xi - \bigvee_{j=1}^n (c_{ij}/x_j^\xi) &= 0 \Leftrightarrow \bigwedge_{j=1}^n \left[y_i^\xi - (c_{ij}/x_j^\xi)\right] = 0 \\ \Leftrightarrow \bigwedge_{j=1}^n (y_i^\xi \cdot x_j^\xi - c_{ij}) &= 0 \Leftrightarrow \bigwedge_{j=1}^n \left[\left(\mathbf{y}^\xi \dot{\wedge} (\mathbf{x}^\xi)'\right) - \mathbf{C_{XY}}\right]_{ij} = 0. \end{aligned} \tag{16}$$

This last set of equations is true if and only if for each $\xi = 1, \ldots, k$, and each integer $i = 1, \ldots, m$, each column entry of the i_{th} row of the matrix $[\mathbf{y}^\xi \,\dot{\wedge}\, (\mathbf{x}^\xi)'] - \mathbf{C_{XY}}$ contains at least one zero entry. □

Theorem 2. $\mathbf{D_{XY}}$ *is a* $\dot{\bigvee}$*-perfect memory for* $(\mathbf{X}, \mathbf{Y})$*, if and only if for each* $\xi = 1, \ldots, k$*,each row of the matrix* $\mathbf{D_{XY}} - [\mathbf{y}^\xi \dot{\bigvee} (\mathbf{x}^\xi)']$ *contains a zero entry.*

Proof. $\mathbf{D_{XY}}$ is a $\dot{\bigvee}$-perfect memory for $(\mathbf{X}, \mathbf{Y})$,$\forall \xi = 1, \ldots, k$ and $\forall i = 1, \ldots, m$

$$\begin{aligned} \Leftrightarrow (\mathbf{D_{XY}}\acute{\wedge}\mathbf{x}^\xi)_i &= y_i^\xi \Leftrightarrow (\mathbf{D_{XY}}\acute{\wedge}\mathbf{x}^\xi)_i - y_i^\xi = 0 \\ \Leftrightarrow \bigwedge_{j=1}^n (d_{ij}/x_j^\xi) - y_i^\xi &= 0 \Leftrightarrow \bigwedge_{j=1}^n (d_{ij}/x_j^\xi - y_i^\xi) = 0 \\ \Leftrightarrow \bigwedge_{j=1}^n (d_{ij} - y_i^\xi \cdot x_j^\xi) &= 0 \Leftrightarrow \bigwedge_{j=1}^n \left[\mathbf{D_{XY}} - \left(\mathbf{y}^\xi \,\dot{\vee}\, (\mathbf{x}^\xi)'\right)\right]_{ij} = 0. \end{aligned} \tag{17}$$

This last set of equations is true if and only if for each $\xi = 1, \ldots, k$, and each integer $i = 1, \ldots,$ m, each column entry of the i_{th} row of the matrix $\mathbf{D_{XY}} - [\mathbf{y}^\xi \,\dot{\vee}\, (\mathbf{x}^\xi)']$ contains at least one zero entry. □

The two theorems show that sometimes $\mathbf{C_{XY}}$ or $\mathbf{D_{XY}}$ of NR^{2}FMAM can realize perfect associative memories for $(\mathbf{X}, \mathbf{Y})$ when the condition in Theorem 1 or in Theorem 2 is satisfied. Therefore, we can utilize them in order to obtain much better effectiveness for hetero associative memories.

Example 1. Assume that $\mathbb{U} = \mathbb{R}_+, \bigcirc = \cdot, \ominus = /$, and

$$\mathbf{X} = \begin{bmatrix} 1 & 2 & 2 \\ 4 & 2 & 2 \\ 4 & 2 & 4 \end{bmatrix}, \qquad \mathbf{Y} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}. \tag{18}$$

According to the method of FMAM,the memories $\mathbf{A_{XY}}$ and $\mathbf{B_{XY}}$ are given by

$$\mathbf{A_{XY}} = \mathbf{Y} \overset{/}{\wedge} \mathbf{X}' = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 1 & 0.5 & 0.5 \\ 0.5 & 0.25 & 0.25 \end{bmatrix}, \mathbf{B_{XY}} = \mathbf{Y} \overset{/}{\vee} \mathbf{X}' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 0.5 & 0.5 \end{bmatrix}. \tag{19}$$

Accordingly, their recall memories are

$$\mathbf{A_{XY}} \dot{\vee} \mathbf{X} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \neq \mathbf{Y}, \mathbf{B_{XY}} \overset{/}{\wedge} \mathbf{X} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \neq \mathbf{Y}. \tag{20}$$

Both $\mathbf{A_{XY}}$ and $\mathbf{B_{XY}}$ can not do the perfect recall memories for $(\mathbf{X}, \mathbf{Y})$.

According to the method of MAM,the memories $\mathbf{W_{XY}}$ and $\mathbf{M_{XY}}$ of MAM are given by

$$\mathbf{W_{XY}} = \mathbf{Y} \overset{-}{\wedge} \mathbf{X}' = \begin{bmatrix} -1 & -3 & -3 \\ 0 & -2 & -2 \\ -1 & -3 & -3 \end{bmatrix}, \mathbf{M_{XY}} = \mathbf{Y} \overset{-}{\vee} \mathbf{X}' = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}. \tag{21}$$

Accordingly, their recall memories are

$$\mathbf{W_{XY}} \overset{+}{\vee} \mathbf{X} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \neq \mathbf{Y}, \mathbf{M_{XY}} \overset{+}{\wedge} \mathbf{X} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \neq \mathbf{Y}. \tag{22}$$

Both $\mathbf{W_{XY}}$ and $\mathbf{M_{XY}}$ can't do the perfect recall memories for $(\mathbf{X}, \mathbf{Y})$.

But according to NR^{2}FMAM method, the memories $\mathbf{C_{XY}}$ and $\mathbf{D_{XY}}$, as well as the according recall outputs are respectively given by

$$\mathbf{C_{XY}} = \mathbf{Y} \dot{\wedge} \mathbf{X}' = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 4 \\ 1 & 2 & 2 \end{bmatrix}, \mathbf{C_{XY}} \overset{/}{\vee} \mathbf{X} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \mathbf{Y}. \tag{23}$$

$$\mathbf{D_{XY}} = \mathbf{Y} \dot{\vee} \mathbf{X}' = \begin{bmatrix} 4 & 4 & 4 \\ 4 & 8 & 8 \\ 2 & 4 & 4 \end{bmatrix}, \mathbf{D_{XY}} \overset{/}{\wedge} \mathbf{X} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \mathbf{Y}. \tag{24}$$

The result shows that both $\mathbf{C_{XY}}$ and $\mathbf{D_{XY}}$ are perfect recall memories for $(\mathbf{X}, \mathbf{Y})$.

Our NR^{2}FMAM also has a certain ability to resist noise. $\mathbf{C_{XY}}$ resists dilative noise, and $\mathbf{D_{XY}}$ erosive noise. The following theorem shows this ability of NR^{2}FMAM, and simultaneously gives the boundary conditions of noise.

Theorem 3. *Let* $\tilde{\mathbf{x}}^\gamma$ *denote the distorted version of the pattern* $\mathbf{x}^\gamma$, $\gamma \in \{1, \ldots, k\}$. *Then* $\mathbf{C_{XY}} \overset{/}{\vee} \tilde{\mathbf{x}}^\gamma = \mathbf{y}^\gamma$, *if and only if*

$$\tilde{x}_j^\gamma \geq x_j^\gamma \wedge \bigvee_{i=1}^{m} \left[\bigwedge_{\xi \neq \gamma} (y_i^\xi / y_i^\gamma \cdot x_j^\xi) \right] \forall j = 1, \ldots, n \tag{25}$$

and for each row index $i \in \{1, \ldots, m\}$ *there exists a column index* $j_i \in \{1, \ldots, n\}$ *such that*

$$\tilde{x}_{j_i}^\gamma = x_{j_i}^\gamma \wedge \left[\bigwedge_{\xi \neq \gamma} (y_i^\xi / y_i^\gamma \cdot x_{j_i}^\xi) \right] \forall j = 1, \ldots, n. \tag{26}$$

Proof. 1) Suppose that $\tilde{\mathbf{x}}^\gamma$ denotes a distorted version of $\mathbf{x}^\gamma$ and that for $\gamma = 1, \ldots, k, \mathbf{C_{XY}} \overset{/}{\vee} \tilde{\mathbf{x}}^\gamma = \mathbf{y}^\gamma$. Then

$$y_i^\gamma = \left(\mathbf{C_{XY}} \overset{/}{\vee} \tilde{\mathbf{x}}^\gamma \right)_i = \bigvee_{l=1}^{n} (c_{il} / \tilde{x}_l^\gamma) \geq c_{ij} / \tilde{x}_j^\gamma \ \forall i = 1, \ldots, m \text{ and } \forall j = 1, \ldots, n. \tag{27}$$

Therefore

$$\begin{aligned}
& \tilde{x}_j^\gamma \geq c_{ij} / y_i^\gamma \ \forall i = 1, \ldots, m \text{ and } \forall j = 1, \ldots, n \\
\Leftrightarrow\ & \tilde{x}_j^\gamma \geq \bigvee_{i=1}^{m} (c_{ij} / y_i^\gamma) \quad \forall j = 1, \ldots, n \\
\Leftrightarrow\ & \tilde{x}_j^\gamma \geq \bigvee_{i=1}^{m} \left[\bigwedge_{\xi=1}^{k} (y_i^\xi \cdot x_j^\xi) / y_i^\gamma \right] \quad \forall j = 1, \ldots, n \\
\Leftrightarrow\ & \tilde{x}_j^\gamma \geq \bigvee_{i=1}^{m} \left[\bigwedge_{\xi \neq \gamma} (y_i^\xi \cdot x_j^\xi / y_i^\gamma) \wedge x_j^\gamma \right] \quad \forall j = 1, \ldots, n \\
\Leftrightarrow\ & \tilde{x}_j^\gamma \geq x_j^\gamma \wedge \bigvee_{i=1}^{m} \left[\bigwedge_{\xi \neq \gamma} (y_i^\xi / y_i^\gamma \cdot x_j^\xi) \right] \quad \forall j = 1, \ldots, n.
\end{aligned} \tag{28}$$

This shows that the inequalities given by (25) are satisfied. It also follows that

$$\tilde{x}_j^\gamma \geq x_j^\gamma \wedge \left[\bigwedge_{\xi \neq \gamma} (y_i^\xi / y_i^\gamma \cdot x_j^\xi) \right] \forall j = 1, \ldots, n \text{ and } \forall i = 1, \ldots, m. \tag{29}$$

Suppose that the set of inequalities given by (29) does not contain an equality for $i = 1, \ldots, m$, namely, assume that there exists a row index $i \in \{1, \ldots, m\}$, such that

$$\tilde{x}_j^\gamma > x_j^\gamma \wedge \left[\bigwedge_{\xi \neq \gamma} (y_i^\xi / y_i^\gamma \cdot x_j^\xi) \right] \forall j = 1, \ldots, n. \tag{30}$$

Then

$$\begin{aligned}
\left(\mathbf{C_{XY}} \overset{/}{\vee} \tilde{\mathbf{x}}^{\gamma}\right)_i &= \bigvee_{j=1}^{n} \left(c_{ij}/\tilde{x}_j^{\gamma}\right) < \bigvee_{j=1}^{n} \left[c_{ij}/\left(x_j^{\gamma} \wedge \bigwedge_{\xi\neq\gamma} \left(y_i^{\xi} \cdot x_j^{\xi}/y_i^{\gamma}\right)\right)\right] = \\
&= \bigvee_{j=1}^{n} \left[c_{ij}/\left(\bigwedge_{\xi=1}^{k} \left(y_i^{\xi} \cdot x_j^{\xi}/y_i^{\gamma}\right)\right)\right] = \bigvee_{j=1}^{n} \left[c_{ij} \cdot y_i^{\gamma}/\bigwedge_{\xi=1}^{k} \left(y_i^{\xi} \cdot x_j^{\xi}\right)\right] = \\
&= \bigvee_{j=1}^{n} \left[c_{ij} \cdot y_i^{\gamma}/c_{ij}\right] = y_i^{\gamma}. \qquad (31)
\end{aligned}$$

Therefore, $\mathbf{C_{XY}} \overset{/}{\vee} \tilde{\mathbf{x}}^{\gamma} < \mathbf{y}^{\gamma}$ which contradicts the hypothesis that $\mathbf{C_{XY}} \overset{/}{\vee} \tilde{\mathbf{x}}^{\gamma} = \mathbf{y}^{\gamma}$. It follows that for each row index i there must exist a column index j_i satisfying (26).

2) Suppose that

$$\tilde{x}_j^{\gamma} \geq x_j^{\gamma} \wedge \bigvee_{i=1}^{m} \left[\bigwedge_{\xi\neq\gamma} (y_i^{\xi}/y_i^{\gamma} \cdot x_j^{\xi})\right] \forall j = 1,\ldots,n \qquad (32)$$

By the first part of our proof, this inequality is true if and only if

$$\tilde{x}_j^{\gamma} \geq c_{ij}/y_i^{\gamma} \ \forall i = 1,\ldots,m \text{ and } \forall j = 1,\ldots,n \qquad (33)$$

or, equivalently, if and only if

$$\begin{aligned}
& c_{ij}/\tilde{x}_j^{\gamma} \leq y_i^{\gamma} \quad \forall i = 1,\ldots,m \text{ and } \forall j = 1,\ldots,n \\
\Leftrightarrow & \bigvee_{j=1}^{n} \left(c_{ij}/\tilde{x}_j^{\gamma}\right) \leq y_i^{\gamma} \forall i = 1,\ldots,m \qquad (34) \\
\Leftrightarrow & (\mathbf{C_{XY}} \overset{/}{\vee} \tilde{\mathbf{x}}^{\gamma})_i \leq y_i^{\gamma} \forall i = 1,\ldots,m
\end{aligned}$$

which implies that $\mathbf{C_{XY}} \overset{/}{\vee} \tilde{\mathbf{x}}^{\gamma} \leq \mathbf{y}^{\gamma}$, $\forall \gamma = 1,\ldots,k$. Thus, if we can show that $\mathbf{C_{XY}} \overset{/}{\vee} \tilde{\mathbf{x}}^{\gamma} \geq \mathbf{y}^{\gamma}$, $\forall \gamma = 1,\ldots,k$, then we must improve that $\mathbf{C_{XY}} \overset{/}{\vee} \tilde{\mathbf{x}}^{\gamma} = \mathbf{y}^{\gamma}$, $\forall \gamma = 1,\ldots,k$. Let $\gamma \in \{1,\ldots,k\}$ and $i \in \{1,\ldots,m\}$ be arbitrarily chosen. Then

$$\begin{aligned}
\left(\mathbf{C_{XY}} \overset{/}{\vee} \tilde{\mathbf{x}}^{\gamma}\right)_i &= \bigvee_{j=1}^{n} (c_{ij}/\tilde{x}_j^{\gamma}) \geq c_{ij_i}/\tilde{x}_{j_i}^{\gamma} = c_{ij_i}/\left[x_{j_i}^{\gamma} \wedge \bigwedge_{\xi\neq\gamma} \left(y_i^{\xi}/y_i^{\gamma} \cdot x_{j_i}^{\xi}\right)\right] \\
&= c_{ij_i}/\left[\bigwedge_{\xi=1}^{k} \left(y_i^{\xi}/y_i^{\gamma} \cdot x_{j_i}^{\xi}\right)\right] = c_{ij_i} \cdot y_i^{\gamma}/\left[\bigwedge_{\xi=1}^{k} \left(y_i^{\xi} \cdot x_{j_i}^{\xi}\right)\right] \\
&= c_{ij_i} \cdot y_i^{\gamma}/c_{ij_i} = y_i^{\gamma}. \qquad (35)
\end{aligned}$$

This shows that $\mathbf{C_{XY}} \overset{/}{\vee} \tilde{\mathbf{x}}^{\gamma} \geq \mathbf{y}^{\gamma}$. □

Theorem 4. *Let* $\tilde{\mathbf{x}}^{\gamma}$ *denote the distorted version of the pattern* $\mathbf{x}^{\gamma}$*. Then* $\mathbf{D}_{\mathbf{XY}} \overset{/}{\wedge} \tilde{\mathbf{x}}^{\gamma} = \mathbf{y}^{\gamma}$*,if and only if*

$$\tilde{x}_j^{\gamma} \leq x_j^{\gamma} \vee \bigwedge_{i=1}^{m} \left[\bigvee_{\xi \neq \gamma} (y_i^{\xi} / y_i^{\gamma} \cdot x_j^{\xi}) \right] \forall j = 1, \ldots, n \tag{36}$$

and for each row index $i \in \{1, \ldots, m\}$ *there exists a column index* $j_i \in \{1, \ldots, n\}$ *such that*

$$\tilde{x}_{j_i}^{\gamma} = x_{j_i}^{\gamma} \vee \left[\bigvee_{\xi \neq \gamma} (y_i^{\xi} / y_i^{\gamma} \cdot x_{j_i}^{\xi}) \right]. \tag{37}$$

Proof. Because the proof of Theorem 4 is similar to the proof of Theorem 3, so it is omitted here. □

Example 2. Let

$$\mathbf{X} = \begin{bmatrix} 1 & 2 & 2 \\ 4 & 2 & 2 \\ 4 & 2 & 4 \end{bmatrix}, \ \mathbf{Y} = \begin{bmatrix} 2 & 4 & 2 \\ 4 & 4 & 4 \\ 2 & 2 & 2 \end{bmatrix}, \ \bar{\mathbf{X}} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 4 & 2 & 4 \end{bmatrix}, \tag{38}$$

$$\widetilde{\mathbf{X}} = \begin{bmatrix} 0.1 & 2 & 1 \\ 0.001 & 0.001 & 0.002 \\ 4 & 0.002 & 4 \end{bmatrix}, \widehat{\mathbf{X}} = \begin{bmatrix} 100 & 200 & 100 \\ 2 & 2000 & 2 \\ 4000 & 2 & 40 \end{bmatrix}, \tag{39}$$

where $\bar{\mathbf{X}}$, $\widetilde{\mathbf{X}}$ and $\widehat{\mathbf{X}}$ denote a distorted version of $\mathbf{X}$, respectively. The recall results of $\mathbf{W}_{\mathbf{XY}}$ and $\mathbf{M}_{\mathbf{XY}}$ of MAM, $\mathbf{A}_{\mathbf{XY}}$ and $\mathbf{B}_{\mathbf{XY}}$ of FMAM, and $\mathbf{C}_{\mathbf{XY}}$ and $\mathbf{D}_{\mathbf{XY}}$ of NR^{2}FMAM in the Example 2 are given in the Tab. I. From the table we can see that the recall result of MAM is the worst. Its correct recall rate in the Example 2 is zero. The best recall result belongs to NR^{2}FMAM. At the same time, the $\mathbf{C}_{\mathbf{XY}}$ of NR^{2}FMAM has a good ability to deal with dilative noise when the dilative input is presented to NR^{2}FMAM; and the $\mathbf{D}_{\mathbf{XY}}$ has a good ability to deal with erosive noise when the erosive input is presented to NR^{2}FMAM.

Input	$\mathbf{X}$	$\bar{\mathbf{X}}$	$\widetilde{\mathbf{X}}$	$\widehat{\mathbf{X}}$
Recall of $\mathbf{W}_{\mathbf{XY}}$	$\neq \mathbf{Y}$	$\neq \mathbf{Y}$	$\neq \mathbf{Y}$	$\neq \mathbf{Y}$
Recall of $\mathbf{M}_{\mathbf{XY}}$	$\neq \mathbf{Y}$	$\neq \mathbf{Y}$	$\neq \mathbf{Y}$	$\neq \mathbf{Y}$
Recall of $\mathbf{A}_{\mathbf{XY}}$	$\neq \mathbf{Y}$	$\neq \mathbf{Y}$	$\neq \mathbf{Y}$	$\neq \mathbf{Y}$
Recall of $\mathbf{B}_{\mathbf{XY}}$	$\neq \mathbf{Y}$	$= \mathbf{Y}$	$\neq \mathbf{Y}$	$\neq \mathbf{Y}$
Recall of $\mathbf{C}_{\mathbf{XY}}$	$= \mathbf{Y}$	$= \mathbf{Y}$	$\neq \mathbf{Y}$	$= \mathbf{Y}$
Recall of $\mathbf{D}_{\mathbf{XY}}$	$= \mathbf{Y}$	$= \mathbf{Y}$	$= \mathbf{Y}$	$\neq \mathbf{Y}$

Tab. I *The recall results of MAM, FMAM and NR^{2}FMAM in the Example 2.*

Example 3. In the experiment, $\mathbf{X} = \{\text{B, C, F, H, S, W}\}$, $\mathbf{Y} = \{\text{AI, Child, Dog, Flower, Illusion, Sheep}\}$. Each character in $\mathbf{X}$ is a 35×35 Boolean image, and each image in $\mathbf{Y}$ is a 50×50 gray image (as shown in Fig. 1, where the top row is the input patterns and the bottom row the target patterns). Because of the domain $\mathbb{U} > 0$ for FMAM and NR^{2}FMAM, in order to avoid zero value elements, we can make an appropriate processing, such as each zero element is set to a small positive number.

Fig. 1 *Original associative memory patterns.*

Experiment shows that for FMAM, both $\mathbf{A_{XY}}$ and $\mathbf{B_{XY}}$ cannot perfectly recall memories with the worst outcome, i.e. zero recall rate (see Fig. 2,where the top row shows the recall patterns of the $\mathbf{A_{XY}}$ and the bottom row the recall patterns of the $\mathbf{B_{XY}}$). The result comes from strict procedure analysis, rather than only by visual inspection. The analysis standard is that a recall image is incomplete as long as a pixel is different from the pixel of original image.

Fig. 2 *Experimental result of FMAM*

The experimental result of MAM (or RMAM) is shown in Fig. 3, where the top row shows the recall patterns of the $\mathbf{W_{XY}}$ and the bottom row the recall patterns of the $\mathbf{M_{XY}}$. It shows that the correct recall rate of $\mathbf{W_{XY}}$ was 17% (only Child can be completely recalled), and the correct recall rate of $\mathbf{M_{XY}}$ was 83% (only the AI cannot be completely recalled).

Fig. 3 *Experimental result of MAM.*

The experimental result of NR^{2}FMAM is shown in Fig. 4: the top row is the recall patterns of the $\mathbf{C_{XY}}$ and the bottom row the recall patterns of the $\mathbf{D_{XY}}$. It shows that the correct recall rate of $\mathbf{D_{XY}}$ was 17% (only Child can be completely recalled).

But amazingly, $\mathbf{C_{XY}}$ can achieve perfect recall memories with correct rate of 100%! Experimental result is encouraging! Our experiments demonstrate that in quite a number of cases NR^{2}FMAM has better effect than FMAM and MAM on hetero associative memories. Therefore, the method is a good complement to FMAM and MAM.

Fig. 4 *Experimental result of NR^{2}FMAM.*

5. Discussions

Why can NR^{2}FMAM in many cases get better results than FMAM? What is its application in the social network? We discussed these questions from the following three aspects.

5.1 No rounding

We think that the different operation order and no rounding in memory and recall may be one of possible reasons. In theory, $a \times b/b = a$, and $a/b \times b = a$, i.e. $a \times b/b = a/b \times b$, but in practice and for computers, it may not necessarily turn out that way. In the case of no overflow and operating from left to right, $a \times b/b = a$, which operation result is no problem. This operation is from multiplication to division (FMTD) like in NR^{2}FMAM. But that computing result of $a/b \times b$ is uncertain. We know that computing accuracy of a computer is limited by its word length. In the case of indivisible, the operation of the computer may bring us the error, such that $a/b \times b \neq a$ or $a/b \times b \approx a$ because of the round off. This operation is from division to multiplication (FDTM) with round off. The operation in FMAM just is of FDTM. Therefore, for hetero associative morphological memories, we more inclined to adopt NR^{2}FMAM method. Of course, this method does not preclude FMAM method. In general, they should be complementary.

5.2 The conditions which NR^{2}FMAM is better than FMAM

Let $(\mathbf{x}^1, \mathbf{y}^1), \ldots, (\mathbf{x}^k, \mathbf{y}^k)$ be k pairs of vector patterns for associative memories, $\mathbf{X} = (\mathbf{x}^1, \ldots, \mathbf{x}^k)$ the input pattern matrix, and $\mathbf{Y} = (\mathbf{y}^1, \ldots, \mathbf{y}^k)$ the output pattern matrix. Assume that R_R is the correct recall rate of NR^{2}FMAM, and R_F the correct recall rate of FMAM. Then, what are the conditions of that $R_R > R_F$? The next theorem answers this question.

Theorem 5. *$R_R > R_F$ if and only if the following conditions are satisfied:*

1) In k pairs of vector patterns, there exist m pairs of vector patterns which satisfy that for each $\xi_i \in \{1, \ldots, k\}$ and $i = 1, \ldots, m$, each row of the matrix $[\mathbf{y}^{\xi_i} \dot{\wedge} (\mathbf{x}^{\xi_i})'] - \mathbf{C_{XY}}$ contains a zero entry, or each row of the matrix $\mathbf{D_{XY}} - [\mathbf{y}^{\xi_i} \dot{\vee} (\mathbf{x}^{\xi_i})']$ contains a zero entry.

2) In k pairs of vector patterns, there exist n pairs of vector patterns which satisfy that for each $\xi_j \in \{1, \ldots, k\}$ and $j = 1, \ldots, n$, each row of the matrix $[\mathbf{y}^{\xi_j} \overset{/}{\wedge} (\mathbf{x}^{\xi_j})'] - \mathbf{A_{XY}}$ contains a zero entry, or each row of the matrix $\mathbf{B_{XY}} - [\mathbf{y}^{\xi_j} \overset{/}{\vee} (\mathbf{x}^{\xi_j})']$ contains a zero entry.

3) $m > n$.

Proof. 1) Assume that $m > n$. According to Theorem 1, we know that m pairs of vectors in k pairs of vectors satisfy that for each $\xi_i \in \{1, \ldots, k\}$ and $i = 1, \ldots, m$, each row of the matrix $\mathbf{D_{XY}} - [\mathbf{y}^{\xi_i} \dot{\vee} (\mathbf{x}^{\xi_i})']$ contains a zero entry or each row of the matrix $[\mathbf{y}^{\xi_i} \dot{\wedge} (\mathbf{x}^{\xi_i})'] - \mathbf{C_{XY}}$ contains a zero entry means that m pairs of vectors can be correctly associated and recalled, and the recall rate $R_R = m/k$; in addition, according to FMAM, we know that n pairs of vectors in k pairs of vectors satisfy that for each $\xi_j \in \{1, \ldots, k\}$ and $j = 1, \ldots, n$, each row of the matrix $[\mathbf{y}^{\xi_j} \overset{/}{\wedge} (\mathbf{x}^{\xi_j})'] - \mathbf{A_{XY}}$ contains a zero entry or each row of the matrix $\mathbf{B_{XY}} - [\mathbf{y}^{\xi_j} \overset{/}{\vee} (\mathbf{x}^{\xi_j})']$ contains a zero entry means that n pairs of vectors can be correctly associated and recalled, and the recall rate $R_F = n/k$. Since $m > n$, we have that $R_R > R_F$.

2) Assume that $R_R > R_F$, might as well set $R_R = m/k, R_F = n/k$, and $m > n$. It means that m pairs of vectors in k pairs of vectors can be correctly associated and recalled by NR^{2}FMAM and n pairs of vectors in k pairs of vectors can be correctly associated and recalled by FMAM. According to the Theorem 1 we know that m pairs of vectors satisfy that for each $\xi_i \in \{1, \ldots, k\}$ and $i = 1, \ldots, m$, each row of the matrix $[\mathbf{y}^{\xi_i} \dot{\wedge} (\mathbf{x}^{\xi_i})'] - \mathbf{C_{XY}}$ contains a zero entry or each row of the matrix $\mathbf{D_{XY}} - [\mathbf{y}^{\xi_i} \dot{\vee} (\mathbf{x}^{\xi_i})']$ contains a zero entry. In addition, according to FMAM, it is certain that n pairs of vectors satisfy that for each $\xi_j \in \{1, \ldots, k\}$ and $j = 1, \ldots, n$, each row of the matrix $[\mathbf{y}^{\xi_j} \overset{/}{\wedge} (\mathbf{x}^{\xi_j})'] - \mathbf{A_{XY}}$ contains a zero entry or each row of the matrix $\mathbf{B_{XY}} - [\mathbf{y}^{\xi_j} \overset{/}{\vee} (\mathbf{x}^{\xi_j})']$ contains a zero entry.

□

5.3 Its application in the social network

We should point out that NR^{2}FMAM has these functions of pattern recognition, clustering, classification and prediction. If $\mathbf{Y} = \mathbf{X}$, NR^{2}FMAM is the auto association from $\mathbf{X}$ to $\mathbf{X}$. Then take advantage of its anti-noise function, we can realize pattern recognition and clustering analysis. Next, we can realize the classification and prediction by using the hetero associative from $\mathbf{X}$ to $\mathbf{Y}$. In a social network [15, 18, 26], the identification of a social group and the group leader's recognition are an important research content, and this recognition depends on the clustering and classification techniques. In the present, improving the identification effect of community organizations and community leaders is still needed [27, 28]. Obviously, NR^{2}FMAM can serve as a new identification technology in social networks, and we should study and use it deeply.

6. Conclusion

In this paper, aiming at the shortcomings of hetero associative morphological memories, we propose the NR^{2}FMAM method. Theoretical analysis shows that NR^{2}FMAM under certain conditions can realize perfect recall memories, at the same time with a certain noise robustness. Experiments have demonstrated that, in many cases, NR^{2}FMAM has better effects of hetero associative morphological memories than FMAM. Why is this? The paper has some discussions for this problem. No rounding is one of those possible reasons that result in higher recall rate than FMAM. The conditions of $R_R > R_F$, namely the conditions of NR^{2}FMAM is better than FMAM are given by Theorem 5. For hetero associative morphological memories, we tend to use NR^{2}FMAM. Of course, it should be pointed out that NR^{2}FMAM and FMAM are complementary, not exclusive. Anyway, NR^{2}FMAM is a beneficial way for HAMM. We believe that combining NR^{2}FMAM with FMAM can significantly improve the effectiveness of HAMM, and solve more problems in practical applications. In the social network, we can find the application of NR^{2}FMAM. Of course, this problem needs to be further studied in the future.

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