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POLYNOMIALS RELATED TO HARMONIC NUMBERS AND EVALUATION OF HARMONIC NUMBER SERIES II

Ayhan Dil, Veli Kurt

In this paper we focus on r-geometric polynomials, r-exponential polynomials and their harmonic versions. We show that harmonic versions of these polynomials and their generalizations are useful for obtaining closed forms of some series related to harmonic numbers.

1. INTRODUCTION

In [**15**] the concept of harmonic-geometric polynomials and harmonic-exponential polynomials are introduced and hyperharmonic generalizations of these polynomials and numbers are obtained. Furthermore, it was shown that these polynomials are quite useful for obtaining closed forms of some series related to harmonic numbers. In this paper, we extend this analysis to r-versions of these polynomials and numbers.

Boyadzhiev [**6**] has presented and discussed the following transformation formula:

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} f(n) x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^{n} \left\{ n \atop k \right\} x^k g^{(k)}(x) \tag{1}$$

where f, g are appropriate functions and $\left\{ n \atop k \right\}$ are Stirling numbers of the second kind.

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One of the principal objectives of the present paper is to give closed forms of some series related to harmonic numbers as well. To this end, we give a useful generalization of (1) which contains r-Stirling numbers of the second kind:

$$\sum_{n=r}^{\infty} \frac{g^{(n)}(0)}{n!} \binom{n}{r} \frac{r!}{n^r} f_r(n) x^n = \sum_{n=r}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r x^k g^{(k)}(x), \tag{2}$$

where $f_r(x)$ denotes the Maclaurin series of $f(x)$ without the first r terms.

Based on formula (2) we introduce the concept of r-geometric and r-exponential polynomials and numbers. We obtain explicit relations between the r-versions and the classical versions of these polynomials and numbers. Besides, we present harmonic (and hyperharmonic) versions of r-geometric and r-exponential polynomials and numbers as well.

On the other hand, formula (2) and harmonic r-geometric polynomials enable us to obtain closed forms of the following series

$$\sum_{n=r}^{\infty} \binom{n}{r} r! n^{m-r} H_n x^n,$$

where m and r are integers such that $m \geq r$ and H_n is the n-th partial sum of the harmonic series.

In the rest of this section we discuss some important notions.

Stirling numbers of the first and second kind

Stirling numbers of the first kind $\left[{n \atop k} \right]$ and Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ are quite important in combinatorics [**4, 5, 11, 21**]. For integers $n \geq k \geq 0$; $\left[{n \atop k} \right]$ represents the number of permutations of n elements with exactly k cycles and $\left\{ {n \atop k} \right\}$ represents the number of ways to partition a set with n elements into k disjoint, nonempty subsets [**11**].

We note that for $n \geq k \geq 1$, the following identity holds for Stirling numbers of the second kind.

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}. \tag{3}$$

There is a certain generalization of these numbers, namely r-Stirling numbers [**8**], which are similar to the weighted Stirling numbers [**9, 10**]. Representations and combinatorial meanings of these numbers are as follows [**8**]:

r-Stirling numbers of the first kind;

$\left[{n \atop k} \right]_r$ = The number of permutations of the set $\{1, 2, \ldots, n\}$ with

k cycles, such that the numbers $1, 2, \ldots, r$ are in separate cycles,

r-Stirling numbers of the second kind;

$$\left\{ {n \atop k} \right\}_r = \text{The number of partitions of the set } \{1, 2, \ldots, n\} \text{ into } k \text{ non-empty disjoint subsets, such that the numbers } 1, 2, \ldots, r \text{ are in separate subsets.}$$

In particular, $r = 0$ gives the classical Stirling numbers.

The r-Stirling numbers of the second kind satisfy the same recurrence relation as (3), except for the initial conditions, i.e. [**8**].

$$\begin{aligned} &\left\{ {n \atop k} \right\}_r = 0, \quad n < r, \qquad \left\{ {n \atop k} \right\}_r = \delta_{k,r}, \quad n = r, \\ &\left\{ {n \atop k} \right\}_r = \left\{ {n-1 \atop k-1} \right\}_r + k \left\{ {n-1 \atop k} \right\}_r, \quad n > r. \end{aligned} \tag{4}$$

Exponential polynomials and numbers

Exponential polynomials (or single variable Bell polynomials) $\phi_n(x)$ are used in [**2, 7, 16, 21**] as follows:

$$\phi_n(x) := \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k.$$

The well known exponential numbers (or Bell numbers) are obtained by setting $x = 1$ in $\phi_n(x)$ i.e., [**3, 11, 12**].

$$\phi_n := \phi_n(1) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}.$$

In [**14**] the authors obtained new proofs of some fundamental properties of the exponential polynomials and numbers using Euler-Seidel method as:

$$\phi_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} \phi_k(x) \text{ and } \phi_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \phi_k. \tag{5}$$

Recently, Mező [**18**] has defined the *r-Bell polynomials and numbers* as:

$$B_{n,r}(x) = \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r x^k \text{ and } B_{n,r} = \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r,$$

respectively. The r-exponential polynomials and numbers which we discuss in the present paper are slightly different than the r-Bell polynomials and numbers in [**18**].

Geometric polynomials and numbers

Geometric polynomials are used in [**6, 22, 23**] as follows:

$$\mathcal{F}_n(x) := \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k! x^k. \tag{6}$$

In particular, $x = 1$ in (6) we get geometric numbers (or ordered Bell numbers) $\mathcal{F}_n$ [**6, 23, 24**] as:

$$\mathcal{F}_n := \mathcal{F}_n(1) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k!.$$

Moreover, these numbers are called Fubini numbers (this explains "$\mathcal{F}$") and preferential arrangements as well.

Boyadzhiev [**6**] introduced the *general geometric polynomials* as

$$\mathcal{F}_{n,r}(x) = \frac{1}{\Gamma(r)} \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \Gamma(k+r) x^k, \tag{7}$$

where $\mathrm{Re}(r) > 0$. In section 5 we deal with the general geometric polynomials.

Exponential and geometric polynomials are connected by the following integral relation [**6**]

$$\mathcal{F}_n(x) = \int_0^{\infty} \phi_n(x\lambda) e^{-\lambda} \mathrm{d}\lambda. \tag{8}$$

In [**14**], the authors also obtained some fundamental properties of the geometric polynomials and numbers using Euler-Seidel method as:

$$\mathcal{F}_{n+1}(x) = x\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathcal{F}_n(x) + x\mathcal{F}_n(x)\right] \text{ and } \mathcal{F}_n = \sum_{k=0}^{n-1} \binom{n}{k} \mathcal{F}_k.$$

By means of r-Stirling numbers, Mező and Nyul [**20**] introduced *r-geometric polynomials and numbers* (*or r-Fubini or ordered r-Bell polynomials and numbers*) are as follows:

$$\mathcal{F}_{n,r}(x) = \sum_{k=0}^{n} (k+r)! \left\{ {n+r \atop k+r} \right\}_r x^k \text{ and } \mathcal{F}_{n,r} = \sum_{k=0}^{n} (k+r)! \left\{ {n+r \atop k+r} \right\}_r,$$

respectively.

In this paper, the r-geometric polynomials come up naturally in an application of the generalized transformation formula as well.

Note that our definition of r-geometric polynomials (23) slightly differs from that of Mező and Nyul [**20**].

Harmonic and Hyperharmonic numbers

The n-th harmonic number is the n-th partial sum of the harmonic series:

$$H_n := \sum_{k=1}^{n} \frac{1}{k},$$

where $H_0 = 0$.

For an integer $\alpha > 1$, let

$$H_n^{(\alpha)} := \sum_{k=1}^{n} H_k^{(\alpha-1)},$$

with $H_n^{(1)} := H_n$ being the n-th hyperharmonic number of order α [**4, 12**].

These numbers can be expressed in terms of binomial coefficients and ordinary harmonic numbers as: [**4, 12, 19**]:

$$H_n^{(\alpha)} = \binom{n+\alpha-1}{\alpha-1}(H_{n+\alpha-1} - H_{\alpha-1}).$$

The well-known generating functions of the harmonic and hyperharmonic numbers are given by

$$\sum_{n=1}^{\infty} H_n x^n = -\frac{\ln(1-x)}{1-x} \text{ and } \sum_{n=1}^{\infty} H_n^{(\alpha)} x^n = -\frac{\ln(1-x)}{(1-x)^{\alpha}},$$

respectively [**13**].

The following relations connect harmonic and hyperharmonic numbers with the Stirling and r-Stirling numbers of the first kind [**4**]:

$$\begin{bmatrix} k+1 \\ 2 \end{bmatrix} = k!H_k, \tag{9}$$

and

$$k!H_k^{(r)} = \begin{bmatrix} n+r \\ r+1 \end{bmatrix}_r. \tag{10}$$

2. GENERALIZATION OF THE TRANSFORMATION FORMULA

In this section we first mention Boyadzhiev's Theorem 4.1 in [**6**] and give a useful generalization of it. As a result of this generalization we introduce r-geometric polynomials and numbers.

Suppose we are given an entire function f and a function g, analytic in a region containing the annulus $K = \{x \in \mathbb{C} : r < |x| < R\}$, where $0 < r < R$. Hence these functions have the following series expansions:

$$f(x) = \sum_{n=0}^{\infty} p_n x^n \text{ and } g(x) = \sum_{n=-\infty}^{\infty} q_n x^n.$$

Now we are ready to state Boyadzhiev's theorem.

Theorem 1. [**6**] *Let the functions f and g be described as above. If the series*

$$\sum_{n=-\infty}^{\infty} q_n f(n) x^n$$

converges absolutely on K, then

$$\sum_{n=-\infty}^{\infty} q_n f(n) x^n = \sum_{m=0}^{\infty} p_m \sum_{k=0}^{m} \left\{ {m \atop k} \right\} x^k g^{(k)}(x) \tag{11}$$

holds for all $x \in K$.

2.1. Generalization of the operator (xD)

The operator (xD) is defined as:

$$(xD)f(x) := xf'(x),$$

where f' is the first derivative of the function f.

Stirling numbers of the second kind appear in the formula (11) due to the operator (xD). Our aim is to obtain a more general formula than (11) which contains r-Stirling numbers of the second kind instead of Stirling numbers of the second kind. Accordingly, first we generalize the operator (xD).

For any m-times differentiable function f we have [**6**],

$$(xD)^m f(x) = \sum_{k=0}^{m} \left\{ {m \atop k} \right\} x^k f^{(k)}(x) . \tag{12}$$

This fact can be easily proven with induction on m by the help of (3).

Our first aim is to generalize the operator (xD). Later we use this generalization to obtain r-geometric and r-exponential polynomials and numbers.

Definition 2. *Let f be a function which is at least m-times differentiable and r be a nonnegative integer. We define (xD_r) as*

$$(xD_r)^m f(x) := \begin{cases} 0, & m < r \\ (xD)^{m-r} x^r f^{(r)}(x), & m \geq r. \end{cases}$$

From Definition 2 and the recurrence relation (4), using induction on m, we can prove that

$$(xD_r)^m f(x) = \sum_{k=0}^{m} \left\{ {m \atop k} \right\}_r x^k f^{(k)}(x), \tag{13}$$

where $m \geq r$.

Equation (13) is a generalization of equation (12) since setting $r = 0$ in (13) gives equation (12).

If n is an integer and $m \geq r$, then it follows from (13)

$$(xD_r)^m x^n = n^{m-r} \binom{n}{r} r! x^n. \tag{14}$$

2.2. Generalization of the transformation formula

We now give our main theorem that is a generalization of Theorem 1.

Theorem 3. *Let $f(x)$ be an entire function and $g(x)$ be an analytic function on the annulus $K = \{x \in \mathbb{C}, s < |x| < S\}$, where $0 \leq s < S$. Suppose that their power series given as*

$$f(x) = \sum_{m=0}^{\infty} p_m x^m \text{ and } g(x) = \sum_{n=-\infty}^{\infty} q_n x^n,$$

and $f_r(x)$ denotes the power series $\sum\limits_{m=r}^{\infty} p_m x^m$. If the series

$$\sum_{n=-\infty}^{\infty} q_n \binom{n}{r} \frac{r!}{n^r} f_r(n) x^n \tag{15}$$

converges absolutely on K where r is a nonnegative integer, then

$$\sum_{n=-\infty}^{\infty} q_n \binom{n}{r} \frac{r!}{n^r} f_r(n) x^n = \sum_{m=r}^{\infty} p_m \sum_{k=0}^{m} \left\{ {m \atop k} \right\}_r x^k g^{(k)}(x)$$

holds for all $x \in K$.

Proof. By considering the power series expansion of $g(x)$ with (13) and (14), we have

$$\sum_{n=-\infty}^{\infty} q_n \binom{n}{r} n^{m-r} r! x^n = \sum_{k=0}^{m} \left\{ {m \atop k} \right\}_r x^k g^{(k)}(x) \tag{16}$$

where m and r are integer such that $m \geq r \geq 0$. If we multiply both sides of equation (16) by p_m and sum on m from r to infinity we get

$$\sum_{n=-\infty}^{\infty} q_n \binom{n}{r} \frac{r!}{n^r} \sum_{m=r}^{\infty} p_m n^m x^n = \sum_{m=r}^{\infty} p_m \sum_{k=0}^{m} \left\{ {m \atop k} \right\}_r x^k g^{(k)}(x),$$

since (15) is converges absolutely on K. This completes the proof.

Corollary 4. *Let g be an analytic function on the disk $D = \{x \in \mathbb{C}, 0 \leq |x| < S\}$ then*

$$\sum_{n=r}^{\infty} \frac{g^{(n)}(0)}{n!} \binom{n}{r} \frac{r!}{n^r} f_r(n)\, x^n = \sum_{n=r}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r x^k g^{(k)}(x). \tag{17}$$

Most of the results in the subsequent sections depend on Corollary 4.

Remark 5. The particular case $r = 0$ in the Theorem 3 refers to the Theorem 4.1 of Boyadzhiev [**6**]. Therefore from now on we consider the case $r \geq 1$.

3. *r*-EXPONENTIAL POLYNOMIALS AND NUMBERS, *r*-GEOMETRIC POLYNOMIALS AND NUMBERS

Stirling numbers of the first and second kind are notable in many branches of mathematics, especially in combinatorics, computational mathematics and computer science [**1, 5, 11, 12, 17**]. The importance of the exponential polynomials and numbers are substantial because of their direct connection with Stirling numbers. r-Stirling numbers [**8**] are one of the reputable generalizations of Stirling numbers. Therefore, it is meaningful to introduce the concepts of the r-exponential and r-geometric polynomials and numbers as follows.

3.1. *r*-exponential polynomials and numbers

First, we consider $g(x) = e^x$ in equation (17). Hence we get

$$\sum_{n=r}^{\infty} \binom{n}{r} \frac{r!}{n^r} f_r(n) \frac{x^n}{n!} = e^x \sum_{n=r}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r x^k. \tag{18}$$

The finite sum on the RHS is a generalization of exponential polynomials. We call these polynomials the *r-exponential polynomials* in notation ${}_r\phi_n(x)$:

$${}_r\phi_n(x) := \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r x^k. \tag{19}$$

As in the classical case, *r-exponential numbers* can be defined by setting $x = 1$ in (19), i.e.,

$${}_r\phi_n := \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r .$$

Now we give an explicit formula which connects r-exponential polynomials with the classical exponential polynomials. This formula also allows us to calculate ${}_r\phi_n(x)$ easily.

Proposition 6. *We have*

$${}_r\phi_{n+r}(x) = x^r \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \phi_k(x), \tag{20}$$

where n and r are nonnegative integers.

Proof. Let m be an integer such that $m \geq r \geq 0$ and we put $f(x) = x^m$ in (18). Then we get the following equality:

$$ {}_r\phi_m(x)e^x = \sum_{n=r}^{\infty} \binom{n}{r} \frac{r!}{n!} n^{m-r} x^n. \tag{21} $$

The right hand side of (21) can be written as

$$ x^r \sum_{n=0}^{\infty} (n+r)^{m-r} \frac{x^n}{n!} = x^r \sum_{k=0}^{m-r} \binom{m-r}{k} r^{m-r-k} \sum_{n=0}^{\infty} n^k \frac{x^n}{n!}. $$

Using the definition of the operator (xD) we obtain

$$ x^r \sum_{k=0}^{m-r} \binom{m-r}{k} r^{m-r-k} (xD)^k e^x. $$

From (12), we have

$$ x^r e^x \sum_{k=0}^{m-r} \binom{m-r}{k} r^{m-r-k} \phi_k(x). $$

Comparision of the LHS and the RHS completes the proof. □

It is easy to see that equation (20) is a generalization of equation (5). As analogue of (20) appears in the paper of Mező [**18**].

Remark 7. As a corollary of Proposition 6, a similar relation can be given between classical exponential numbers and r-exponential numbers as:

$$ {}_r\phi_{n+r} = \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \phi_k. $$

3.2. r-geometric polynomials and numbers

By considering $g(x) = \dfrac{1}{1-x}$ in equation (17) we get

$$ \sum_{n=r}^{\infty} \binom{n}{r} \frac{r!}{n^r} f_r(n) x^n = \frac{1}{1-x} \sum_{n=r}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r k! \left(\frac{x}{1-x} \right)^k. \tag{22} $$

We call the finite sum of the RHS as *r-geometric polynomials* and indicate them with ${}_r\mathcal{F}_n(x)$. Hence

$$ {}_r\mathcal{F}_n(x) := \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r k! x^k. \tag{23} $$

We define *r-geometric numbers* by specializing $x = 1$ in (23) as

$$ {}_r\mathcal{F}_n := \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r k!. $$

The following proposition gives an explicit relation between r-geometric polynomials and generalized geometric polynomials which were given by equation (7).

Proposition 8. *For any nonnegative integers n and r we have*

$$ {}_r\mathcal{F}_{n+r}(x) = x^r r! \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \mathcal{F}_{k,r+1}(x) \tag{24}$$

Proof. Let m be a nonnegative integer such that $m \geq r$. By setting $f(x) = x^m$ in (22) we get the following curious formula:

$$ \frac{1}{1-x}\, {}_r\mathcal{F}_m\left(\frac{x}{1-x}\right) = \sum_{n=r}^{\infty} \binom{n}{r} r! n^{m-r} x^n. \tag{25}$$

Rearranging RHS of (25) gives

$$x^r r! \sum_{k=0}^{m-r} \binom{m-r}{k} r^{m-r-k} \sum_{n=0}^{\infty} \binom{n+r}{r} n^k x^n.$$

We can write this by means of (xD) operator as

$$x^r r! \sum_{k=0}^{m-r} \binom{m-r}{k} r^{m-r-k} (xD)^k \frac{1}{(1-x)^{r+1}}.$$

Considering the fact that (equation (3.26) in [**6**])

$$(xD)^k \frac{1}{(1-x)^{r+1}} = \frac{1}{(1-x)^{r+1}} \mathcal{F}_{k,r+1}\left(\frac{x}{1-x}\right)$$

completes the proof.

Remark 9. Letting $x = 1/2$ in (25), we obtain another expression for ${}_r\mathcal{F}_n$:

$$ {}_r\mathcal{F}_m = r! \sum_{n=r}^{\infty} \binom{n}{r} \frac{n^{m-r}}{2^{n-1}} \tag{26}$$

Remark 10. A similar result between numbers is as follows.

$$ {}_r\mathcal{F}_{n+r} = r! \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \mathcal{F}_{k,r+1} \tag{27}$$

From (24) and (27), we have the following relations for classical geometric polynomials and numbers as a corollary:

$$\mathcal{F}_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} \mathcal{F}_{k,2}(x) \text{ and } \mathcal{F}_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \mathcal{F}_{k,2}.$$

4. HARMONIC r-GEOMETRIC POLYNOMIALS AND NUMBERS, HARMONIC r-EXPONENTIAL POLYNOMIALS AND NUMBERS

We introduced concepts of harmonic-geometric and harmonic-exponential polynomials and numbers in [**15**]. In this section we follow similar approach as in [**15**] to investigate harmonic r-geometric and harmonic r-exponential polynomials and numbers.

4.1. Harmonic r-geometric polynomials and numbers

We consider the generating function of harmonic numbers as the function g in the transformation formula (17). From [**15**] we have

$$g^{(k)}(x) = \frac{k!\big(H_k - \ln(1-x)\big)}{(1-x)^{k+1}} \quad \text{and} \quad g^{(k)}(0) = k!H_k. \tag{28}$$

Using Theorem 3 we state the following transformation formula for harmonic numbers.

Proposition 11. *Let r be a nonnegative integer and let f be an entire function. Then we have*

$$\sum_{n=r}^{\infty} \binom{n}{r} H_n \frac{r!}{n^r} f_r(n) x^n = \frac{1}{1-x} \sum_{n=r}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r k! H_k \left(\frac{x}{1-x} \right)^k - \frac{\ln(1-x)}{1-x} \sum_{n=r}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r k! \left(\frac{x}{1-x} \right)^k. \tag{29}$$

Proof. Employing (28) in (17) gives the statement. □

Second part of the RHS of equation (29) contains r-geometric polynomials which are familiar to us from the previous section. But the first part contains a new family of polynomials which is a generalization of harmonic-geometric polynomials [**15**]. We call them the *harmonic r-geometric polynomials* and denote them as ${}_r\mathcal{F}_n^h(x)$. Thus

$${}_r\mathcal{F}_n^h(x) := \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r k! H_k x^k. \tag{30}$$

Harmonic r-geometric numbers can be defined by setting $x = 1$ in (30), i.e.,

$${}_r\mathcal{F}_n^h := \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r k! H_k.$$

Hence with this notation we state the formula (29) simply as

$$\begin{aligned}\sum_{n=r}^{\infty}\binom{n}{r}H_n\frac{r!}{n^r}f_r\left(n\right)x^n \\ =\frac{1}{1-x}\sum_{n=r}^{\infty}\frac{f^{(n)}(0)}{n!}\left\{{}_r\mathcal{F}_n^h\left(\frac{x}{1-x}\right)-{}_r\,\mathcal{F}_n\left(\frac{x}{1-x}\right)\ln(1-x)\right\}.\end{aligned} \tag{31}$$

In the following corollary we obtain closed forms of some series related to harmonic numbers and binomial coefficients.

Corollary 12.

$$\sum_{n=r}^{\infty}\binom{n}{r}r!n^{m-r}H_nx^n=\frac{1}{1-x}\left\{{}_r\mathcal{F}_m^h\left(\frac{x}{1-x}\right)-{}_r\,\mathcal{F}_m\left(\frac{x}{1-x}\right)\ln(1-x)\right\}, \tag{32}$$

where m and r are integers such that $m\geq r\geq 0$.

Proof. It follows by setting $f\left(x\right)=x^m$ in equation (31).

Remark 13. Formula (32) allows us to calculate closed forms of several harmonic number series. The cases $r=0$ and $r=1$ in (32) has been analyzed in [**15**] already.

The case $r=2$ gives

$$\begin{aligned}\sum_{n=2}^{\infty}n^{m-1}(n-1)H_nx^n \\ =\frac{1}{1-x}\left\{{}_2\mathcal{F}_m^h\left(\frac{x}{1-x}\right)-{}_2\,\mathcal{F}_m\left(\frac{x}{1-x}\right)\ln(1-x)\right\}.\end{aligned} \tag{33}$$

Hence some series and their closed forms that we get from (33) are as follows.

For $m=2$ we have

$$\sum_{n=2}^{\infty}n(n-1)H_nx^n=\frac{x^2\{3-2\ln(1-x)\}}{(1-x)^3}.$$

For $m=3$ we have

$$\sum_{n=2}^{\infty}n^2(n-1)H_nx^n=\frac{x^2\{6+5x-(4+2x)\ln(1-x)\}}{(1-x)^4},$$

and so on.

The case $r=3$ gives

$$\begin{aligned}\sum_{n=3}^{\infty}n^{m-2}(n-1)(n-2)H_nx^n \\ =\frac{1}{1-x}\left\{{}_3\mathcal{F}_m^h\left(\frac{x}{1-x}\right)-{}_3\,\mathcal{F}_m\left(\frac{x}{1-x}\right)\ln(1-x)\right\}.\end{aligned} \tag{34}$$

Hence some series and their closed forms that we get from (34) are as follows:

For $m = 3$ we have

$$\sum_{n=3}^{\infty} n(n-1)(n-2)H_n x^n = \frac{x^3\{11 - 6\ln(1-x)\}}{(1-x)^4}.$$

For $m = 4$ we have

$$\sum_{n=3}^{\infty} n^2(n-1)(n-2)H_n x^n = \frac{x^3\{33 + 17x - (18+6x)\ln(1-x)\}}{(1-x)^5},$$

and so on.

REMARK 14. In the following proposition and from now on we use the notation

$$\sum_{0\le k_1\le k_2\le\cdots\le k_{s+1}\le n}$$

to indicate the following multiple type sums:

$$\sum_{k_{s+1}=0}^{n}\sum_{k_s=0}^{k_{s+1}}\cdots\sum_{k_1=0}^{k_2}.$$

Now we give a summation formula for the multiple series.

Proposition 15.

$$\begin{aligned}
&\sum_{n=r}^{\infty}\left(\sum_{k=0}^{n-r}\binom{k}{r}\binom{n+s-k}{s} r!k^{m-r}H_k\right)x^n \\
&= \sum_{n=r}^{\infty}\left(\sum_{0\le k_1\le k_2\le\cdots\le k_{s+1}\le n}\binom{k_1}{r} r!k_1^{m-r}H_{k_1}\right)x^n \qquad (35)\\
&= \frac{1}{(1-x)^{s+2}}\left\{{}_r\mathcal{F}_m^h\left(\frac{x}{1-x}\right) - {}_r\mathcal{F}_m\left(\frac{x}{1-x}\right)\ln(1-x)\right\}
\end{aligned}$$

where m, r and s are nonnegative integers such that $m \ge r$.

Proof. Multiplying both sides of equation (32) with the Newton binomial series and considering that

$$\sum_{k=0}^{n-r}\binom{k}{r}\binom{n+s-k}{s} r!k^{m-r}H_k = \sum_{0\le k_1\le k_2\le\cdots\le k_{s+1}\le n}\binom{k_1}{r} r!k_1^{m-r}H_{k_1}$$

we get the statement. □

By setting $r = 2$ and $s = 0$ in formula (35) we can give the following applications:

For $m = 2$ we have

$$\sum_{n=2}^{\infty}\left(\sum_{k=2}^{n} k(k-1)H_k\right)x^n = \frac{x^2\{3 - 2\ln(1-x)\}}{(1-x)^4}.$$

For $m = 3$ we have

$$\sum_{n=2}^{\infty}\left(\sum_{k=2}^{n} k^2(k-1)H_k\right)x^n = \frac{x^2\{6 + 5x - (4+2x)\ln(1-x)\}}{(1-x)^5},$$

and so on.

Remark 16. Using (9) we can state ${}_r\mathcal{F}_n^h(x)$ and ${}_r\mathcal{F}_n^h$ in terms of $r-$Stirling numbers of the second kind and Stirling numbers of the first kind as

$${}_r\mathcal{F}_n^h(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r \left[{k+1 \atop 2} \right] x^k \text{ and } {}_r\mathcal{F}_n^h = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r \left[{k+1 \atop 2} \right].$$

4.2. Harmonic r-exponential polynomials and numbers

Bearing in mind the similarity of exponential and geometric polynomials and using the definition of harmonic exponential polynomials and numbers, we give the following definition.

Definition 17. *For nonnegative integers n and r, harmonic r-exponential polynomials and harmonic r-exponential numbers are defined respectively as*

$${}_r\phi_n^h(x) := \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r H_k x^k \text{ and } {}_r\phi_n^h := \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r H_k.$$

Remark 18. Definition 17 enables us to extend the relation (8) as

$$ {}_r\mathcal{F}_n^h(x) = \int_0^{\infty} {}_r\phi_n^h(x\lambda)\, e^{-\lambda}\mathrm{d}\lambda. \tag{36}$$

5. HYPERHARMONIC r-GEOMETRIC POLYNOMIALS AND NUMBERS, HYPERHARMONIC r-EXPONENTIAL POLYNOMIALS AND NUMBERS

We now consider hyperharmonic numbers and their transformations. In this way we can generalize almost all results of [**15**] and in the previous sections of the present paper.

5.1. Hyperharmonic r-geometric polynomials and numbers

Similar to the previous section, let us consider the function g in the transformation formula (17) as the generating function of the hyperharmonic numbers. From [**15**] we have

$$g^{(k)}(x) = \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)} \cdot \frac{1}{(1-x)^{\alpha+k}} \left(H_{k+\alpha-1} - H_{\alpha-1} - \ln(1-x)\right) \tag{37}$$

and

$$g^{(k)}(0) = k! H_k^{(\alpha)}. \tag{38}$$

Now we give a transformation formula for hyperharmonic numbers.

Proposition 19. *For integers $r \geq 0$ and $\alpha \geq 1$ we have*

$$\begin{aligned} &\sum_{n=r}^{\infty} \binom{n}{r} H_n^{(\alpha)} \frac{r!}{n^r} f_r(n) x^n \\ &= \frac{1}{(1-x)^{\alpha}} \sum_{n=r}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r k! H_k^{(\alpha)} \left(\frac{x}{1-x} \right)^k \\ &\quad - \frac{\ln(1-x)}{(1-x)^{\alpha}} \sum_{n=r}^{\infty} \frac{f^{(n)}(0)}{n!} \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r \Gamma(k+\alpha) \left(\frac{x}{1-x} \right)^k . \end{aligned} \tag{39}$$

Proof. Consideration (37) and (38) in (17) give the statement. □

The first part of the RHS is a generalization of harmonic r-geometric polynomials which contains hyperharmonic numbers instead of harmonic numbers. We call these polynomials the *hyperharmonic r-geometric polynomials* and denote them as ${}_r\mathcal{F}^h_{n,\alpha}(x)$. Thus

$${}_r\mathcal{F}^h_{n,\alpha}(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r k! H_k^{(\alpha)} x^k. \tag{40}$$

The second part of the RHS of (39) contains a generalization of the, *general geometric polynomials*. We call these polynomials the *general r-geometric polynomials* and denote them as ${}_r\mathcal{F}_{n,\alpha}(x)$. Hence

$${}_r\mathcal{F}_{n,\alpha}(x) = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r \Gamma(k+\alpha) x^k. \tag{41}$$

Using these notations we can state (39) simply as

$$\begin{aligned}(42)\qquad &\sum_{n=r}^{\infty}\binom{n}{r}H_n^{(\alpha)}\frac{r!}{n^r}f(n)x^n\\ &=\frac{1}{(1-x)^{\alpha}}\sum_{n=r}^{\infty}\frac{f^{(n)}(0)}{n!}\left[{}_r\mathcal{F}_{n,\alpha}^{h}\left(\frac{x}{1-x}\right)-{}_r\,\mathcal{F}_{n,\alpha}\left(\frac{x}{1-x}\right)\ln(1-x)\right].\end{aligned}$$

Remark 20. Putting $x=1$ in (40) we get *hyperharmonic r-geometric numbers* as

$${}_r\mathcal{F}_{n,\alpha}^{h}=\sum_{k=0}^{n}\left\{{n \atop k}\right\}_r k!H_k^{(\alpha)},$$

and putting $x=1$ in (41) gives *general r-geometric numbers* as

$${}_r\mathcal{F}_{n,\alpha}=\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{n}\left\{{n \atop k}\right\}_r\Gamma(k+\alpha).$$

Using the following corollary of Proposition 19 we obtain closed forms of some series related to hyperharmonic numbers and binomial coefficients.

Corollary 21.

$$\begin{aligned}&\sum_{n=r}^{\infty}\binom{n}{r}r!n^{m-r}H_n^{(\alpha)}x^n\\ (43)\qquad &=\frac{1}{(1-x)^{\alpha}}\left[{}_r\mathcal{F}_{m,\alpha}^{h}\left(\frac{x}{1-x}\right)-{}_r\,\mathcal{F}_{m,\alpha}\left(\frac{x}{1-x}\right)\ln(1-x)\right].\end{aligned}$$

Proof. For a positive integers $m\geq r$, setting $f(x)=x^m$ in (42) gives (43).

Remark 22. Putting the values of r, m and α in (43), one can get closed forms of several hyperharmonic numbers series.

Now we extend the formula (35) to hyperharmonic number series.

Proposition 23. *Let m,r,s and α be nonnegative integers such that and $m\geq r$. Then we have*

$$\begin{aligned}(44)\qquad &\sum_{n=r}^{\infty}\left(\sum_{k=0}^{n-r}\binom{k}{r}\binom{n+s-k}{s}r!k^{m-r}H_k^{(\alpha)}\right)x^n\\ &=\sum_{n=r}^{\infty}\left(\sum_{0\leq k_1\leq k_2\leq\cdots\leq k_{s+1}\leq n}\binom{k_1}{r}r!k_1^{m-r}H_{k_1}^{(\alpha)}\right)x^n\\ &=\frac{1}{(1-x)^{\alpha+s+1}}\left\{{}_r\mathcal{F}_m^{h}\left(\frac{x}{1-x}\right)-{}_r\,\mathcal{F}_m\left(\frac{x}{1-x}\right)\ln(1-x)\right\}.\end{aligned}$$

Proof. Multiplying both sides of equation (43) with the Newton binomial series gives the statement.

REMARK 24. Particular values of r, m, s and α in (44) gives closed forms of several mutiplicative hyperharmonic numbers series.

REMARK 25. Using (10) we get an alternative expression of hyperharmonic r-geometric polynomials and numbers as

$$ {}_r\mathcal{F}_{n,\alpha}^h(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r \left[{k+\alpha \atop \alpha+1} \right]_\alpha x^k, \quad {}_r\mathcal{F}_{n,\alpha}^h = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r \left[{k+\alpha \atop \alpha+1} \right]_\alpha . $$

5.2. Hyperharmonic r-exponential polynomials and numbers

Definition 26. *For positive integers α and r, hyperharmonic r-exponential polynomials are defined as*

$$ {}_r\phi_{n,\alpha}^h(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r H_k^{(\alpha)} x^k. $$

Hence *hyperharmonic r-exponential numbers* are defined as

$$ {}_r\phi_{n,\alpha}^h = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r H_k^{(\alpha)}. $$

REMARK 27. We extend the relation (36) as

$$ {}_r\mathcal{F}_{n,\alpha}^h(x) = \int_0^\infty {}_r\phi_{n,\alpha}^h(x\lambda)\, e^{-\lambda} \mathrm{d}\lambda. $$

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Akdeniz University,
Department of Mathematics,
07058 Antalya
Turkey
E-mails: adil@akdeniz.edu.tr,
vkurt@akdeniz.edu.tr

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