Is There a Sequence on Four Symbols in Which No Two Adjacent Segments Are Permutations of One Another?

T. C. Brown

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It has long been known (see [1–3, 5, 6, 10–12, 15, 16]) that there exist sequences on 3 symbols which contain no 2 identically equal consecutive segments, and sequences on 2 symbols which contain no 3 identically equal consecutive segments. Indeed, Axel Thue obtained these results around 1906. See [6] for a brief account of the contexts of the various independent rediscoveries of these results, and see [7, 14] for an account of other properties of these sequences.

Let X be a set and let $s = x_1x_2x_3\cdots$ be a sequence on X. Then for $i+1 \leq k$, $s[i+1,k] = x_{i+1}x_{i+2}\cdots x_k$ is a *segment* of s, and the segments $s[i+1,j], s[j+1,k]$ are *consecutive*. The segments $s[i+1,j]$ and $s[p+1,q]$ are *identically equal* if $k-i = q-p$ and $x_{i+1} = x_{p+1}, x_{i+2} = x_{p+2}, \ldots, x_k = x_q$ or, in other words, if $s[i+1,k] = s[p+1,q]$ in X^*, the free semigroup generated by the set X.

An interesting situation arises when we allow the symbols *within a segment* to commute with each other. It will be convenient to use the following terminology.

Given a set X and a sequence s on X, we regard segments of s as elements of X^*. (Thus the results mentioned above say that there exist sequences on 3 symbols without 2nd powers as segments, and sequences on 2 symbols without 3rd powers.) Now let X^+ denote the free commutative semigroup generated by X, and let $\alpha : X^* \mapsto X^+$ be the natural homomorphism ($\alpha(x) = x$ for $x \in X$). If s has k consecutive segments $f_1, \ldots, f_k$ such that $\alpha(f_1) = \cdots = \alpha(f_k)$, then we say that s has a kth power mod α. In this language, the question of the title is: Does there exist a sequence on four symbols without 2nd powers mod α?

It is an easy matter to verify that every sequence on 3 symbols contains 2nd powers mod α, and that every sequence on 2 symbols has 3rd powers mod α. For example, if $X = \{x, y\}$, one can show by examining all cases that the longest elements of X^* which do not contain a 3rd power mod α are $xxyyxyyxx, xxyyxyyxy$, and a few others. Evdomikov [4] constructed a sequence on 25 symbols without 2nd powers mod α, and conjectured that perhaps 5 symbols would suffice. Justin [8], with a remarkable half-page proof, constructed a sequence on 2 symbols without 5th powers mod α. This sequence is obtained by successive iterations of the transformation $x \mapsto xxxxy$ and $y \mapsto xyyyy$, starting with x. Thus the first few iterations give $x, xxxxy, (xxxxy)^4xyyyy, [(xxxxy)^4xyyyy]^4xxxxy(xyyyy)^4$. Then in 1970 a paper appeared [13] in which P. A. B. Pleasants gave a construction of a sequence on 5 symbols without 2nd powers mod α. Pleasants' sequence, which extends to infinity in both directions, is constructed by successive iterations of the transformation

$$\begin{aligned} a &\mapsto bacaeacadaeadab \\ b &\mapsto cbdbabdbebabebc \\ c &\mapsto dcecbcecacbcacd \\ d &\mapsto edadcdadbdcdbde \\ e &\mapsto aebedebecedecea \end{aligned}$$

The reader has perhaps notice the "duality" between the number of symbols and the power, according to which the "dual" of each known result is also known. Indeed, it is conceivable that by inserting a third symbol into suitable places in Justin's sequence one could break up all the 4th powers mod α and so obtain a sequence on 3 symbols without 4th powers mod α. Doing this twice more would give another sequence on 5 symbols without 2nd powers mod α, and would tend to clarify the existence of the "duality" in the first place.

Little seems to be known about the $(2,4)$ case, that is, the existence of sequences on 2 symbols without 4th powers mod α and on 4 symbols without 2nd powers mod α, beyond Justin's statement [9] that one can construct on 4 symbols segments of length 7500 without 2nd powers mod α. Pleasants [13] states "it seems certain" that there is a sequence on 4 symbols without 2nd powers mod α, and gives several hints as to how one might manage to construct such a sequence. Even less appears to be known about the self-dual (3,3) case.

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