Chapter 24
Roots and (Re)sources of Value (In)definiteness *Versus* Contextuality

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Abstract In Itamar Pitowsky's reading of the Gleason and the Kochen-Specker theorems, in particular, his Logical Indeterminacy Principle, the emphasis is on the value indefiniteness of observables which are not within the preparation context. This is in stark contrast to the prevalent term contextuality used by many researchers in informal, heuristic yet omni-realistic and potentially misleading ways. This paper discusses both concepts and argues in favor of value indefiniteness in all but a continuum of contexts intertwining in the vector representing a single pure (prepared) state. Even more restrictively, and inspired by operationalism but not justified by Pitowsky's Logical Indeterminacy Principle or similar, one could identify with a "quantum state" a single quantum context – aka the respective maximal observable, or, in terms of its spectral decomposition, the associated orthonormal basis – from the continuum of intertwining context, as per the associated maximal observable actually or implicitly prepared.

Keywords Logical indeterminacy principle · Contextuality · Conditions of possible experience · Quantum clouds · Value indefiniteness · Partition logic

24.1 Introduction

An upfront *caveat* seems in order: The following is a rather subjective narrative of my reading of Itamar Pitowsky's thoughts about classical value indeterminacy on quantum logical structures of observables, amalgamated with my current thinking on related issues. I have never discussed these matters with Itamar Pitovsky explicitly; therefore the term "my reading" should be taken rather literally; namely

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as taken from his publications. In what follows classical value indefiniteness on collections of (intertwined) quantum observables will be considered a consequence, or even a synonym, of what he called indeterminacy. Whether or not this identification is justified is certainly negotiable; but in what follows this is taken for granted.

The term value indefiniteness has been stimulated by recursion theory (Rogers, Jr. 1967; Odifreddi 1989; Smullyan 1993), and in particular by *partial functions* (Kleene 1936) – indeed the notion of partiality has not diffused into physical theory formation, and might even appear alien to the very notion of functional value assignments – and yet it appears to be necessary (Abbott et al. 2012, 2014, 2015) if one insists (somewhat superficially) on classical interpretations of quantized systems.

Value indefiniteness/indeterminacy will be contrasted with some related interpretations and approaches, in particular, with contextuality. Indeed, I believe that contextuality was rather foreign to Itamar Pitowsky's thinking: the term *"contextuality"* appears marginally – as in "a different context" – in his book *Quantum Probability – Quantum Logic* (Pitowsky 1989b), nowhere in his reviews on Boole-Bell type inequalities (Pitowsky 1989a, 1994), and mostly with reference to *contextual quantum probabilities* in his late writings (Pitowsky 2006). The emphasis on value indefiniteness/indeterminacy was, I believe, independently shared by Asher Peres as well as Ernst Specker.

I met Itamar Pitowsky (Bub and Demopoulos 2010) personally rather late; after he gave a lecture entitled *"All Bell Inequalities"* in Vienna (ESI – The Erwin Schrödinger International Institute for Mathematical Physics 2001) on September 6th, 2000. Subsequent discussions resulted in a joint paper (Pitowsky and Svozil 2001) (stimulating further research (Sliwa 2003; Colins and Gisin 2004)). It presents an application of his correlation polytope method (Pitowsky 1986, 1989a,b, 1991, 1994) to more general configurations than had been studied before. Thereby semi-automated symbolic as well as numeric computations have been used.

Nevertheless, the violations of what Boole called (Boole 1862, p. 229) *"conditions of possible experience,"* obtained through solving the hull problem of classical correlation polytopes, was just one route to quantum indeterminacy pursued by Itamar Pitowsky. One could identify at least two more passages he contributed to: One approach (Pitowsky 2003, 2006) compares differences of classical with quantum predictions through conditions and constraints imposed by certain intertwined configurations of observables which I like to call *quantum clouds* (Svozil 2017b). And another approach (Pitowsky 1998; Hrushovski and Pitowsky 2004) pushes these predictions to the limit of logical inconsistency; such that any attempt of a classical description fails relative to the assumptions. In what follows we shall follow all three pursuits and relate them to new findings.

24.2 Stochastic Value Indefiniteness/Indeterminacy by Boole-Bell Type Conditions of Possible Experience

The basic idea to obtain all classical predictions – including classical probabilities, expectations as well as consistency constraints thereof – associated with (mostly complementary; that is, non-simultaneously measurable) collections of observables is quite straightforward: Figure out all "extreme" cases or states which would be classically allowed. Then construct all classically conceivable situations by forming suitable combinations of the former.

Formally this amounts to performing the following steps (Pitowsky 1986, 1989a,b, 1991, 1994):

- Contemplate about some concrete structure of observables and their interconnections in intertwining observables – the quantum cloud.
- Find all two-valued states of that quantum cloud. (In the case of "contextual inequalities" (Cabello 2008) include all variations of true/1 and false/0, irrespective of exclusivity; thereby often violating the Kolmogorovian axioms of probability theory even within a single context.)
- Depending on one's preferences, form all (joint) probabilities and expectations.
- For each of these two-valued states, evaluate the joint probabilities and expectations as products of the single particle probabilities and expectations they are formed of (this reflects statistical independence of the constituent observables).
- For each of the two-valued states, form a tuple containing these relevant (joint) probabilities and expectations.
- Interpret this tuple as a vector.
- Consider the set of all such vectors – there are as many as there are two-valued states, and their dimension depends on the number of (joint) probabilities and expectations considered – and interpret them as vertices forming a convex polytope.
- The convex combination of all conceivable two-valued states yields the surface of this polytope; such that every point inside its convex hull corresponds to a classical probability distribution.
- Determine the conditions of possible experience by solving the hull problem – that is, by computing the hyperplanes which determine the inside–versus–outside criteria for that polytope. These then can serve as necessary criteria for all classical probabilities and expectations considered.

The systematic application of this method yields necessary criteria for classical probabilities and expectations which are violated by the quantum probabilities and expectations. Since I have reviewed this subject exhaustively (Svozil 2018c, Sect. 12.9) (see also Svozil 2017a) I have just sketched it to obtain a taste for its relevance for quantum indeterminacy. As is often the case in mathematical physics the method seems to have been envisioned independently a couple of times. From its (to the best of my knowledge) inception by Boole (1862) it has been discussed in the measure theoretic context by Chochet theory (Bishop and Leeuw 1959) and by Vorob'ev (1962). Froissart (Froissart 1981; Cirel'son (=Tsirel'son) 1993) might have been the first explicitly proposing it as a method to generalized Bell-type inequalities. I suggested its usefulness for non-Boolean cases (Svozil 2001) with "enough" two-valued states; preferable sufficiently many to allow a proper distinction/separation of all observables (cf. Kochen and Specker's Theorem 0 (Kochen and Specker 1967, p. 67)). Consideration of the pentagon/pentagram logic – that is, five cyclically intertwined contexts/blocks/Boolean subalgebras/cliques/orthonormal bases popularized the subject and also rendered new predictions which could be used to differentiate classical from quantized systems (Klyachko 2002; Klyachko et al. 2008; Bub and Stairs 2009, 2010; Badzięg et al. 2011).

A *caveat:* the obtained criteria involve multiple mutually complementary summands which are not all simultaneously measurable. Therefore, different terms, when evaluated experimentaly, correspond to different, complementary measurement configurations. They are obtained at different times and on different particles and samples.

Explicit, worked examples can, for instance, be found in Pitowsky's book (Pitowsky 1989b, Section 2.1), or papers (Pitowsky 1994) (see also Froissart's example (Froissart 1981)). Empirical findings are too numerous to even attempt a just appreciation of all the efforts that went into testing classicality. There is overwhelming evidence that the quantum predictions are correct; and that they violate Boole's conditions of possible classical experience (Clauser 2002) relative to the assumptions (basically non-contextual realism and locality).

So, if Boole's conditions of possible experience are violated, then they can no longer be considered appropriate for any reasonable ontology forcing "reality" upon them. This includes the realistic (Stace 1934) existence of hypothetical counterfactual observables: "unperformed experiments seem to have no consistent outcomes" (Peres 1978). The inconsistency of counterfactuals (in Specker's scholastic terminology *infuturabilities* (Specker 1960, 2009)) provides a connection to value indefiniteness/indeterminacy – at least, and let me again repeat earlier provisos, relative to the assumptions. More of this, piled higher and deeper, has been supplied by Itamar Pitowsky, as will be discussed later.

24.3 Interlude: Quantum Probabilities from Pythagorean "Views on Vectors"

Quantum probabilities are vector based. At the same time those probabilities mimic "classical" ones whenever they must be classical; that is, among mutually commuting observables which can be measured simultaneously/concurrently on the same particle(s) or samples – in particular, whenever those observables correspond to projection operators which are either orthogonal (exclusive) or identical (inclusive).

At the same time, quantum probabilities appear "contextual" (I assume he had succumbed to the prevalent nomenclature at that late time) according to Itamar Pitowsky's late writings (Pitowsky 2006) if they need not be classical: namely among non-commuting observables. (The term "needs not" derives its justification from the finding that there exist situations (Moore 1956; Wright 1990) involving complementary observables with a classical probability interpretation (Svozil 2005)).

Thereby, classical probability theory is maintained for simultaneously co-measurable (that is, non-complementary) observables. This essentially amounts to the validity of the Kolmogorov axioms of probability theory of such observables within a given context/block/Boolean subalgebra/clique/orthonormal basis, whereby the probability of an event associated with an observable

- is a non-negative real number between 0 and 1;
- is 1 for an event associated with an observable occurring with certainty (in particular, by considering any observable or its complement); as well as
- additivity of probabilities for events associated with mutually exclusive observables.

Sufficiency is assured by an elementary geometric argument (Gleason 1957) which is based upon the Pythagorean theorem; and which can be used to explicitly construct vector-based probabilities satisfying the aforementioned Kolmogorov axioms within contexts: Suppose a pure state of a quantized system is formalized by the unit state vector $|\psi\rangle$. Consider some orthonormal basis $\mathscr{B} = \{|\mathbf{e}_1\rangle, \dots, |\mathbf{e}_n\rangle\}$ of $\mathscr{V}$. Then the square $P_\psi(\mathbf{e}_i) = |\langle\psi|\mathbf{e}_i\rangle|^2$ of the length/norm $\sqrt{\langle\psi|\mathbf{e}_i\rangle\langle\mathbf{e}_i|\psi\rangle}$ of the orthogonal projection $(\langle\psi|\mathbf{e}_i\rangle)\,|\mathbf{e}_i\rangle$ of that unit vector $|\psi\rangle$ along the basis element $|\mathbf{e}_i\rangle$ can be interpreted as the probability of the event associated with the $0-1$-observable (proposition) associated with the basis vector $|\mathbf{e}_i\rangle$ (or rather the orthogonal projector $\mathbf{E}_i = |\mathbf{e}_i\rangle\langle\mathbf{e}_i|$ associated with the dyadic product of the basis vector $|\mathbf{e}_i\rangle$); given a quantized physical system which has been prepared to be in a pure state $|\psi\rangle$. Evidently, $1 \leq P_\psi(\mathbf{e}_i) \leq 1$, and $\sum_{i=1}^{n} P_\psi(\mathbf{e}_i) = 1$. In that Pythagorean way, every context, formalized by an orthonormal basis $\mathscr{B}$, "grants a (probabilistic) view" on the pure state $|\psi\rangle$.

It can be expected that these Pythagorean-style probabilities are different from classical probabilities almost everywhere – that is, for almost all relative measurement positions. Indeed, for instance, whereas classical two-partite correlations are linear in the relative measurement angles, their respective quantum correlations follow trigonometric functions – in particular, the cosine for "singlets" (Peres 1993). These differences, or rather the vector-based Pythagorean-style quantum probabilities, are the "root cause" for violations of Boole's aforementioned conditions of possible experience in quantum setups.

Because of the convex combinations from which they are derived, all of these conditions of possible experience contain only *linear* constraints (Boole 1854, 1862; Fréchet 1935; Hailperin 1965, 1986; Ursic 1984, 1986, 1988; Beltrametti and Maçzyński 1991, 1993, 1994, 1995; Pykacz and Santos 1991; Sylvia and Majernik 1992; Dvurečenskij and Länger 1994; Beltrametti et al. 1995; Del Noce

1995; Länger and Mączyński 1995; Dvurečenskij and Länger 1995a,b; Beltrametti and Bugajski 1996; Pulmannová 2002). And because linear combinations of linear operators remain linear, one can identify the terms occurring in conditions of possible experience with linear self-adjoint operators, whose sum yields a self-adjoint operator, which stands for the "quantum version" of the respective conditions of possible experience. This operator has a spectral decomposition whose min-max eigenvalues correspond to the quantum bounds (Filipp and Svozil 2004a,b), which thereby generalize the Tsirelson bound (Cirel'son (=Tsirel'son) 1980). In that way, every condition of possible experience which is violated by the quantum probabilities provides a direct criterium for non-classicality.

24.4 Classical Value Indefiniteness/Indeterminacy by Direct Observation

In addition to the "fragmented, explosion view" criteria allowing "nonlocality" *via* Einstein separability (Weihs et al. 1998) among its parts, classical predictions from quantum clouds – essentially intertwined (therefore the Hilbert space dimensionality has to be greater than two) arrangements of contexts – can be used as a criterium for quantum advantage over (or rather "otherness" or "distinctiveness" from) classical predictions. Thereby it is sufficient to observe of a single outcome of a quantized system which directly contradicts the classical predictions.

One example of such a configuration of quantum observables forcing a "one-zero rule" (Svozil 2009b) because of a true-implies-false set of two-valued classical states (TIFS) (Cabello et al. 2018) is the "Specker bug" logic (Kochen and Specker 1965, Fig. 1, p. 182) called "cat's cradle" (Pitowsky 2003, 2006) by Itamar Pitowsky (see also Belinfante (1973, Fig. B.l. p. 64), Stairs (1983, p. 588–589), Clifton (1993, Sects. IV, Fig. 2) and Pták and Pulmannová (1991, p. 39, Fig. 2.4.6) for early discussions), as depicted in Fig. 24.1.

For such configurations, it is often convenient to represent both its labels as well as the classical probability distributions in terms of a partition logic (Svozil 2005) of the set of two-valued states – in this case, there are 14 such classical states. Every maximal observable is characterized by a context. The atoms of this context are labeled according to the indices of the two-valued measure with the value 1 on this atom. The axioms of probability theory require that, for each two-valued state, and within each context, there is exactly one such atom. As a result, as long as the set of two-valued states is separating (Kochen and Specker 1967, Theorem 0), one obtains a set of partitions of the set of two-valued states; each partition corresponding to a context.

Classically, if one prepares the system to be in the state $\{1, 2, 3\}$ – standing for any one of the classical two-valued states 1, 2 or 3 or their convex combinations – then there is no chance that the "remote" target state $\{7, 10, 13\}$ can be observed. A direct observation of quantum advantages (or rather superiority in terms of the

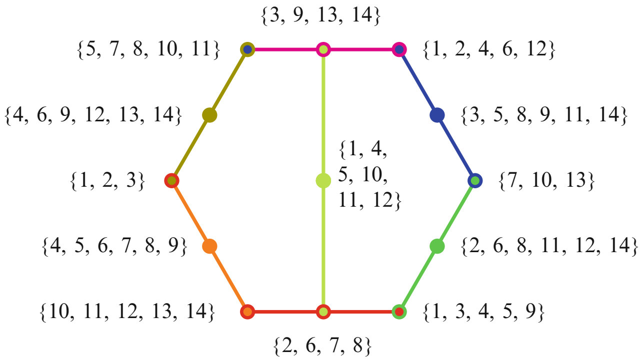

Fig. 24.1 The convex structure of classical probabilities in this (Greechie) orthogonality diagram representation of the Specker bug quantum or partition logic is reflected in its partition logic, obtained through indexing all 14 two-valued measures, and adding an index $1 \le i \le 14$ if the ith two-valued measure is 1 on the respective atom. Concentrate on the outermost left and right observables, depicted by squares: Positivity and convexity requires that $0 \le \lambda_i \le 1$ and $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_7 + \lambda_{10} + \lambda_{13} \le \sum_{i=1}^{14} \lambda_i = 1$. Therefore, if a classical system is prepared (a generalized urn model/automaton logic is "loaded") such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$, then $\lambda_7 + \lambda_{10} + \lambda_{13} = 0$, which results in a TIFS: the classical prediction is that the latter outcome never occurs if the former preparation is certain

frequencies predicted with respect to classical frequencies) is then suggested by some faithful orthogonal representation (FOR) (Lovász et al. 1989; Parsons and Pisanski 1989; Cabello et al. 2010; Solís-Encina and Portillo 2015) of this graph. In the particular Specker bug/cats cradle configuration, an elementary geometric argument (Cabello 1994, 1996) forces the relative angle between the quantum states $|\{1, 2, 3\}\rangle$ and $|\{7, 10, 13\}\rangle$ in three dimensions to be not smaller than $\arctan\left(2\sqrt{2}\right)$, so that the quantum prediction of the occurrence of the event associated with state $|\{7, 10, 13\}\rangle$, if the system was prepared in state $|\{1, 2, 3\}\rangle$ is that the probability can be at most $|\langle\{1, 2, 3\}|\{7, 10, 13\}\rangle|^2 = \cos^2\left[\arctan\left(2\sqrt{2}\right)\right] = \frac{1}{9}$. That is, on the average, if the system was prepared in state $|\{1, 2, 3\}\rangle$ at most one of 9 outcomes indicates that the system has the property associated with the observable $|\{7, 10, 13\}\rangle\langle|\{7, 10, 13\}|$. The occurrence of a single such event indicates quantum advantages over the classical prediction of non-occurrence.

This limitation is only true for the particular quantum cloud involved. Similar arguments with different quantum clouds resulting in TIFS can be extended to arbitrary small relative angles between preparation and measurement states, so that the relative quantum advantage can be made arbitrarily high (Abbott et al. 2015; Ramanathan et al. 2018). Classical value indefiniteness/indeterminacy comes naturally: because – at least relative to the assumptions regarding non-contextual value definiteness of truth assignments, in particular, of intertwining, observables – the existence of such definite values would enforce non-occurrence of outcomes which are nevertheless observed in quantized systems.

Very similar arguments against classical value definiteness can be inferred from quantum clouds with true-implies-true sets of two-valued states (TITS) (Stairs 1983; Clifton 1993; Johansen 1994; Vermaas 1994; Belinfante 1973; Pitowsky 1982; Hardy 1992, 1993; Boschi et al. 1997; Cabello and García-Alcaine 1995; Cabello et al. 1996, 2013, 2018; Cabello 1997; Badziąg et al. 2011; Chen et al. 2013). There the quantum advantage is in the non-occurrence of outcomes which classical predictions mandate to occur.

24.5 Classical Value Indefiniteness/Indeterminacy Piled Higher and Deeper: The Logical Indeterminacy Principle

For the next and final stage of classical value indefiniteness/indeterminacy on quantum clouds (relative to the assumptions) one can combine two logics with simultaneous classical TIFS and TITS properties at the same terminals. That is, suppose one is preparing the same "initial" state, and measuring the same "target" observable; nevertheless, contemplating the simultaneous counterfactual existence of two different quantum clouds of intertwined contexts interconnecting those fixated "initial" state and measured "target" observable. Whenever one cloud has the TIFS and another cloud the TITS property (at the same terminals), those quantum clouds induce contradicting classical predictions. In such a setup the only consistent choice (relative to the assumptions; in particular, omni-existence and context independence) is to abandon classical value definiteness/determinacy. Because the assumption of classical value definiteness/determinacy for any such logic, therefore, yields a complete contradiction, thereby eliminating prospects for hidden variable models (Abbott et al. 2012, 2015; Svozil 2017b) satisfying the assumptions.

Indeed, suppose that a quantized system is prepared in some pure quantum state. Then Itamar Pitowsky's (Pitowsky 1998; Hrushovski and Pitowsky 2004) *indeterminacy principle* states that – relative to the assumptions; in particular, global classical value definiteness for all observables involved, as well as context-independence of observables in which contexts intertwine – any other distinct (non-collinear) observable which is not orthogonal can neither occur nor not occur. This can be seen as an extension of both Gleason's theorem (Gleason 1957; Zierler and Schlessinger 1965) as well as the Kochen-Specker theorem (Kochen and Specker 1967) implying and utilizing the non-existence of any two-valued global truth assignments on even finite quantum clouds.

For the sake of a concrete example consider the two TIFS and TITS clouds – that is, logics with 35 intertwined binary observables (propositions) in 24 contexts – depicted in Fig. 24.2 (Svozil 2018b). They represent quantum clouds with the same terminal points $\{1\} \equiv \{1'\}$ and $\{2, 3, 4, 5, 6, 7\} \equiv \{1', 2', 3', 4', 5'\}$, forcing

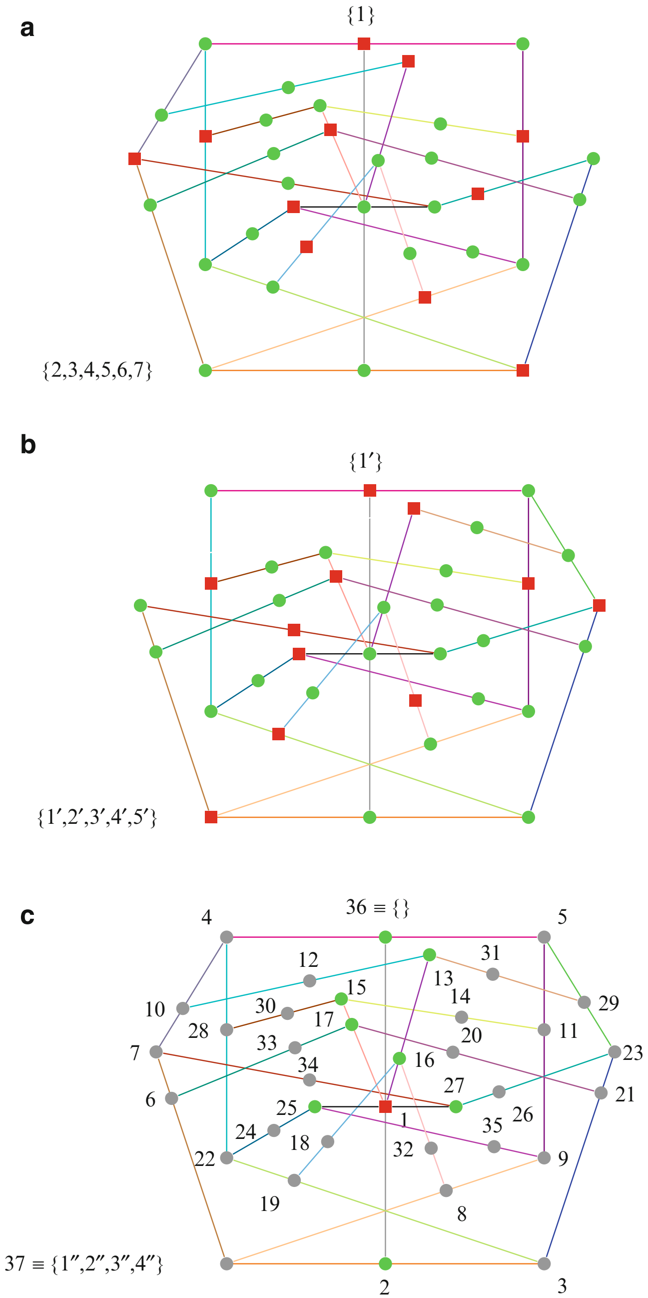

Fig. 24.2 (**a**) TIFS cloud, and (**b**) TITS cloud with only a single overlaid classical value assignment if the system is prepared in state $|1\rangle$ (Svozil 2018b). (**c**) The combined cloud from (**a**) and (**b**) has no value assignment allowing $36 = \{\}$ to be true/1; but still allows 8 classical value assignments enumerated by Table 24.1, with overlaid partial coverage common to all of them. A faithful orthogonal realization is enumerated in Abbott et al. (2015, Table. 1, p. 102201–7)

the latter ones (that is, $\{2, 3, 4, 5, 6, 7\}$ and $\{1', 2', 3', 4', 5'\}$) to be false/0 and true/1, respectively, if the former ones (that is, $\{1\} \equiv \{1'\}$) are true/1.

Formally, the only two-valued states on the logics depicted in Fig. 24.2a and b which allow $v(\{1\}) = v'(\{1'\}) = 1$ requires that $v(\{2, 3, 4, 5, 6, 7\}) = 0$ but $v'(\{1', 2', 3', 4', 5'\}) = 1 - v(\{2, 3, 4, 5, 6, 7\})$, respectively. However, both these logics have a faithful orthogonal representation (Abbott et al. 2015, Table. 1, p. 102201–7) in terms of vectors which coincide in $|\{1\}\rangle = |\{1'\}\rangle$, as well as in $|\{2, 3, 4, 5, 6, 7\}\rangle = |\{1', 2', 3', 4', 5'\}\rangle$, and even in all of the other adjacent observables.

The combined logic, which features 37 binary observables (propositions) in 26 contexts has no longer a classical interpretation in terms of a partition logic, as the 8 two-valued states enumerated in Table 24.1 cannot mutually separate (Kochen and Specker 1967, Theorem 0) the observables 2, 13, 15, 16, 17, 25, 27 and 36, respectively.

It might be amusing to keep in mind that, because of non-separability (Kochen and Specker 1967, Theorem 0) of some of the binary observables (propositions), there does not exist a proper partition logic. However, there exist generalized urn (Wright 1978, 1990) and finite automata (Moore 1956; Schaller and Svozil 1995, 1996) model realisations thereof: just consider urns "loaded" with balls which have no colored symbols on them; or no such balls at all, for the binary observables (propositions) 2, 13, 15, 16, 17, 25, 27 and 36. In such cases it is no more possible to empirically reconstruct the underlying logic; yet if an underlying logic is assumed then – at least as long as there still are truth assignments/two-valued states on the logic – "reduced" probability distributions can be defined, urns can be loaded, and automata prepared, which conform to the classical predictions from a convex combination of these truth assignments/two-valued states – thereby giving rise to "reduced" conditions of experience *via* hull computations.

For global/total truth assignments (Pitowsky 1998; Hrushovski and Pitowsky 2004) as well as for local admissibility rules allowing partial (as opposed to total, global) truth assignments (Abbott et al. 2012, 2015), such arguments can be extended to cover all terminal states which are neither collinear nor orthogonal. One could point out that, insofar as a fixed state has to be prepared the resulting value indefiniteness/indeterminacy is state dependent. One may indeed hold that the strongest indication for quantum value indefiniteness/indeterminacy is the *total absence/non-existence* of two-valued states, as exposed in the Kochen-Specker theorem (Kochen and Specker 1967). But this is rather a question of nominalistic taste, as both cases have no direct empirical testability; and as has already been pointed out by Clifton in a private conversation in 1995: *"how can you measure a contradiction?"*

Table 24.1 Enumeration of the 8 two-valued states on 37 binary observables (propositions) of the combined quantum clouds/logics depicted in Fig. 24.2a and b. Row vector indicate the state values on the observables, column vectors the values on all states per the respective observable

#	1	2	3	4	...										...									...										34	35	36	37
1	1	0	0	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	0	1	0	0	1	0	1	0	0	1	1	0	1	1	1	1	0	1
2	1	0	0	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0	0	1	1	0	0	0	0	0	1	1	1	1	1	1	0	1
3	1	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	1	0	0	1	0	1	0	1	0	0	1	1	1	1	1	0	1
4	1	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	1	1	0	0	0	1	0	0	0	1	1	1	1	1	1	0	1
5	1	0	1	1	0	1	0	1	0	0	1	1	0	0	0	0	0	1	0	1	0	0	0	1	0	1	0	0	1	1	0	0	0	1	1	0	0
6	1	0	1	1	0	1	0	0	1	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	1	1	0	1	0	1	0	0	0
7	1	0	1	0	1	1	0	1	0	1	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	1	0	0	1	0	0	1	1	0	0
8	1	0	1	0	1	0	1	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	1	0	0	1	0	1	0	1	0	0

24.6 The "Message" of Quantum (In)determinacy

At the peril of becoming, as expressed by Clauser (2002), "evangelical," let me "sort things out" from my own very subjective and private perspective. (Readers adverse to "interpretation" and the semantic, "meaning" aspects of physical theory may consider stop reading at this point.)

Thereby one might be inclined to follow Planck (against Feynman (Clauser 2002; Mermin 1989a,b)) and hold it as being not too unreasonable to take scientific comprehensibility, rationality, and causality as a (Planck 1932, p. 539) (see also Earman 2007, p. 1372) *"heuristic principle, a sign-post … to guide us in the motley confusion of events and to show us the direction in which scientific research must advance in order to attain fruitful results."*

So what does all of this – the Born rule of quantum probabilities and its derivation by Gleason's theorem from the Kolmogorovian axioms applied to mutually comeasurable observables, as well as its consequences, such as the Kochen-Specker theorem, the plethora of violations of Boole's conditions of possible experience, Pitowsky's indeterminacy principle and more recent extensions and variations thereof – "try to tell us?"

First, observe that all of the aforementioned postulates and findings are (based upon) assumptions; and thus consequences of the latter. Stated differently, these findings are true not in the absolute, ontologic but in the epistemic sense: they hold relative to the axioms or assumptions made.

Thus, in maintaining rationality one needs to grant oneself – or rather one is forced to accept – the abandonment of at least some or all assumptions made. Some options are exotic; for instance, Itamar Pitowsky's suggestions to apply paradoxical set decompositions to probability measures (Pitowsky 1983, 1986). Another "exotic escape option" is to allow only unconnected (non-intertwined) contexts whose observables are dense (Godsil and Zaks 1988, 2012; Meyer 1999; Havlicek et al. 2001). Some possibilities to cope with the findings are quite straightforward, and we shall concentrate our further attention to those (Svozil 2009b).

24.6.1 Simultaneous Definiteness of Counterfactual, Complementary Observables, and Abandonment of Context Independence

Suppose one insists on the simultaneous definite omni-existence of mutually complementary, and therefore necessarily counterfactual, observables. One straightforward way to cope with the aforementioned findings is the abandonment of context-independence of intertwining observables.

There is no indication in the quantum formalism which would support such an assumption, as the respective projection operators do not in any way depend on the contexts involved. However, one may hold that the outcomes are context dependent as functions of the initial state and the context measured (Svozil 2009a, 2012; Dzhafarov et al. 2017); and that they actually "are real" and not just "idealistically occur in our imagination;" that is, being *"mental through-and-through"* (Segal and Goldschmidt 2017/2018). Early conceptualizations of context-dependence aka contextuality can be found in Bohr's remark (in his typical Nostradamus-like style) (Bohr 1949) on *"the impossibility of any sharp separation between the behavior of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear."* Bell, referring to Bohr, suggested (Bell 1966), Sec. 5) that *"the result of an observation may reasonably depend not only on the state of the system (including hidden variables) but also on the complete disposition of the apparatus."*

However, the common, prevalent, use of the term "contextuality" is not an explicit context-dependent form, as suggested by the realist Bell in his earlier quote, but rather a situation where the classical predictions of quantum clouds are violated. More concretely, if experiments on quantized systems violate certain Boole-Bell type classical bounds or direct classical predictions, the narratives claim to have thereby "proven contextuality" (e.g., see Hasegawa et al. (2006), Cabello et al. (2008), Cabello (2008), Bartosik et al. (2009), Amselem et al. (2009), Bub and Stairs (2010) and Cabello et al. (2013) for a "direct proof of quantum contextuality").

What if we take Bell's proposal of a context dependence of valuations – and consequently, "classical" contextual probability theory – seriously? One of the consequences would be the introduction of an *uncountable multiplicity* of counterfactual observables. An example to illustrate this multiplicity – comparable to de Witt's view of Everett's relative state interpretation (Everett III 1973) – is the uncountable set of orthonormal bases of $\mathbb{R}^3$ which are all interconnected at the same single intertwining element. A continuous angular parameter characterizes the angles between the other elements of the bases, located in the plane orthogonal to that common intertwining element. Contextuality suggests that the value assignment of an observable (proposition) corresponding to this common intertwining element needs to be both true/1 and false/0, depending on the context involved, or whenever some quantum cloud (collection of intertwining observables) demands this through consistency requirements.

Indeed, the introduction of multiple quantum clouds would force any context dependence to also implicitly depend on this general perspective – that is, on the respective quantum cloud and its faithful orthogonal realization, which in turn determines the quantum probabilities *via* the Born-Gleason rule: Because there exist various different quantum clouds as "pathways interconnecting" two observables, context dependence needs to vary according to any concrete connection between the prepared and the measured state.

A single context participates in an arbitrary, potentially infinite, multiplicity of quantum clouds. This requires this one context to "behave very differently" when it comes to contextual value assignments. Alas, as quantum clouds are hypothetical constructions of our mind and therefore *"mental through-and-through"* (Segal and Goldschmidt 2017/2018), so appears context dependence: as an idealistic concept, devoid of any empirical evidence, created to rescue the *desideratum* of omni-realistic existence.

Pointedly stated, contextual value assignments appear both utterly ad hoc and abritrary – like a *deus ex machina* “saving” the *desideratum* of a classical omni-value definite reality, whereby it must obey quantum probability theory without grounding it (indeed, in the absence of any additional criterium or principle there is no reason to assume that the likelihood of true/1 and false/0 is other than 50:50); as well as highly discontinuous. In this latter, discontinuity respect, context dependence is similar to the earlier mentioned breakup of the intertwine observables by reducing quantum observables to disconnected contexts (Godsil and Zaks 1988, 2012; Meyer 1999; Havlicek et al. 2001).

It is thereby granted that these considerations apply only to cases in which the assumptions of context independence are valid throughout the entire quantum cloud – that is, uniformly: for all observables in which contexts intertwine. If this were not the case – say, if only a single one observable occurring in intertwining contexts is allowed to be context-dependent (Svozil 2012; Simmons 2020) – the respective clouds taylored to prove Pitowsky’s Logical Indeterminacy Principle and similar, as well as the Kochen-Specker theorems do not apply; and therefore the aforementioned consequences are invalid.

24.6.2 Abandonment of Omni-Value Definiteness of Observables in All But One Context

Nietzsche once speculated (Nietzsche 1887, 2009–,,) that what he has called *“slave morality”* originated from superficially pretending that – in what later Blair (aka Orwell 1949) *“doublespeak”* – weakness means strength. In a rather similar sense the lack of comprehension – Planck’s *“sign-post”* – and even the resulting inconsistencies tended to become reinterpreted as an asset: nowadays consequences of the vector-based quantum probability law are marketed as *“quantum supremacy”* – a *“quantum magic”* or *“hocus-pocus”* (Svozil 2016) of sorts.

Indeed, future centuries may look back at our period, and may even call it a second “renaissance” period of scholasticism (Specker 1960). In years from now historians of science will be amused about our ongoing queer efforts, the calamities and “magic” experienced through our painful incapacity to recognize the obvious – that is, the non-existence and therefore value indefiniteness/indeterminacy of certain counterfactual observables – namely exactly those mentioned in Itamar Pitowsky’s indeterminacy principle.

This principle has a positive interpretation of a quantum state, defined as the maximal knowledge obtainable by simultaneous measurements of a quantized system; or, conversely, as the maximal information content encodable therein. This can be formalized in terms of the *value definiteness* of a single (Zeilinger 1999; Svozil 2002, 2004, 2018b; Grangier 2002) context – or, in a more broader (nonoperational) perspective, the continuum of contexts intertwined by some prepared pure quantum state (formalized as vector or the corresponding one-dimensional orthogonal projection operator). In terms of Hilbert space quantum mechanics this amounts to the claim that the only value definite entity can be a single orthonormal basis/maximal operator; or a continuum of maximal operators whose spectral sum contain proper “true intertwines.” All other “observables” grant an, albeit necessarily stochastic, value indefinite/indeterministic, view on this state.

If more than one context is involved we might postulate that all admissable probabilities should at least satisfy the following criterium: they should be classical Kolmogorov-style *within* any single particular context (Gleason 1957). It has been suggested (Auffféves and Grangier 2017, 2018) that this can be extended and formalized in a quantum multi-context environment by a double stochastic matrix whose entries $P(\mathbf{e}_i, \mathbf{f}_j)$, with $1 \le i, j \le n$ (n is the number of distinct “atoms” or exclusive outcomes in each context) are identified by the conditional probabilities of one atom $\mathbf{f}_j$ in the second context, relative to a given one atom $\mathbf{e}_i$ in the first context. The general multi-context case yields row stochastic matrices (Svozil 2018a). Various types of decompositions of those matrices exist for particular cases:

- By the Birkhoff-von Neumann theorem double stochastic matrices can be represented by the Birkhoff polytope spanned by the convex hull of the set of permutation matrices: let $\lambda_1, \ldots, \lambda_k \ge 0$ such that $\sum_{l=1}^{k} \lambda_l = 1$, then $P(\mathbf{e}_i, \mathbf{f}_j) = \left[\sum_{l=1}^{k} \lambda_l \Pi_l\right]_{ij}$. Since there exist $n!$ permutations of n elements, k will be bounded from above by $k \le n!$. Note that this type of decomposition may not be unique, as the space spanned the permutation matrices is $\left[(n-1)^2+1\right]$-dimensional; with $n! > (n-1)^2 + 1$ for $n > 2$. Therefore, the bound from above can be improved such that decompositions with $k \le (n-1)^2 + 1 = n^2 - 2(n+1)$ exist (Marcus and Ree 1959). Formally, a permutation matrix has a quasi-vectorial (Mermin 2007) decomposition in terms of the canonical (Cartesian) basis, such that, $\Pi_i = \sum_{j=1}^{n} |\mathbf{e}_j\rangle\langle\mathbf{e}_{\pi_i(j)}|$, where $|\mathbf{e}_j\rangle$ represents the n-tuple associated with the jth basis vector of the canonical (Cartesian) basis, and $\pi_i(j)$ stands for the ith permutation of j.
- Vector based probabilities allow the following decomposition (Auffféves and Grangier 2017, 2018): $P(\mathbf{e}_i, \mathbf{f}_j) = \text{Trace}\left(\mathbf{E}_i \mathbf{R} \mathbf{F}_j \mathbf{R}\right)$, where $\mathbf{E}_i$ and $\mathbf{F}_i$ are elements of contexts, formalized by two sets of mutually orthogonal projection operators, and $\mathbf{R}$ is a real positive diagonal matrix such that the trace of $\mathbf{R}^2$ equals the dimension n, and $\text{Trace}\left(\mathbf{E}_i \mathbf{R}^2\right) = 1$. The quantum mechanical Born rule is recovered by identifying $\mathbf{R} = \mathbb{I}_n$ with the identity matrix, so that $P(\mathbf{e}_i, \mathbf{f}_j) = \text{Trace}\left(\mathbf{E}_i \mathbf{F}_j\right)$.
- There exist more “exotic” probability measures on “reduced” propositional spaces such as Wright’s 2-state dispersion-free measure on the pentagon/pentagram (Wright 1978), or another type of probability measure based on a discontinuous 3(2)-coloring of the set of all unit vectors with rational coefficients (Godsil and Zaks 1988, 2012; Meyer 1999; Havlicek et al. 2001) whose decomposition appear to be ad hoc; at least for the time being.

Where might this aforementioned type of stochasticism arise from? It could well be that it is introduced by interactions with the environment; and through the many uncontrollable and, for all practical purposes (Bell 1990), huge number of degrees of freedom in unknown states.

The finiteness of physical resources needs not prevent the specification of a particular vector or context. Because any other context needs to be operationalized within the physically feasible means available to the respective experiment: it is the measurable coordinate differences which count; not the absolute locatedness relative to a hypothetical, idealistic absolute frame of reference which cannot be accessed operationally.

Finally, as the type of context envisioned to be value definite can be expressed in terms of vector spaces equipped with a scalar product – in particular, by identifying a context with the corresponding orthonormal basis or (the spectral decomposition of) the associated maximal observable(s) – one may ask how one could imagine the origin of such entities? Abstractly vectors and vector spaces could originate from a great variety of very different forms; such as from systems of solutions of ordinary linear differential equations. Any investigation into the origins of the quantum mechanical Hilbert space formalism itself might, if this turns out to be a progressive research program (Lakatos 1978), eventually yield to a theory indicating operational physical capacities beyond quantum mechanics.

24.7 Biographical Notes on Itamar Pitowsky

I am certainly not in the position to present a view of Itamar Pitowsky's thinking. Therefore I shall make a few rather anecdotal observations. First of all, he seemed to me as one of the most original physicists I have ever met – but that might be a triviality, given his *opus*. One thing I realized was that he exhibited a – sometimes maybe even unconscious, but sometimes very outspoken – regret that he was working in a philosophy department. I believe he considered himself rather a mathematician or theoretical physicist. To this I responded that being in a philosophy department might be rather fortunate because there one could "go wild" in every direction; allowing much greater freedom than in other academic realms. But, of course, this had no effect on his uneasiness.

He was astonished that I spent a not so little money (means relative to my investment capacities) in an Israeli internet startup company which later flopped, depriving me of all but a fraction of what I had invested. He told me that, at least at that point, many startups in Israel had been put up intentionally only to attract money from people like me; only to collapse later.

A late project of his concerned quantum bounds in general; maybe in a similar – graph theoretical and at the time undirected to quantum – way as Grötschel, Lovász and Schrijver's theta body (Grötschel et al. 1986; Cabello et al. 2014). The idea was not just deriving absolute (Cirel'son (=Tsirel'son) 1980) or parameterized, continuous (Filipp and Svozil 2004a,b) bounds for existing classical conditions of possible experience obtained by hull computations of polytopes; but rather genuine quantum bounds on, say, Einstein-Podolsky-Rosen type setups.

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