Completing bases in four dimensions

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Criteria and constructive methods for the completion of an incomplete basis of, or context in, a four-dimensional Hilbert space by (in)decomposable vectors are given.

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I. COMPLETION OF INCOMPLETE CONTEXTS

We shall find and analyze orthogonal vectors spanning two-dimensional subspaces of four-dimensional real or complex Hilbert space that are orthogonal to a given two-dimensional subspace. In particular, we are interested whether those vectors are indecomposable—a property of pure state vectors signifying entanglement of multipartite quantized systems.

In physics, this question is pertinent to a variety of tasks: First, any orthonormal basis can be, by dyadic or tensor products, rewritten as a system of mutually perpendicular orthogonal projection operators. This system can be extended to maximal hermitian operators associated with maximal quantum observables in terms of their spectral sums containing mutually distinct eigenvalues. Often these maximal operators are denoted as, and identified with, quantum mechanical contexts.

Quantum contexts serve as the basic building blocks of quantum logical and probabilistic certification of quantization. Boole-Bell type arguments consider three or more isolated contexts and compare classical with quantum predictions of expectation functions. Hardy-type arguments involve multiple intertwining contexts with two endpoints, such that classical predictions relate the truth values of these endpoints. Intertwining contexts with “scarce” two-valued states—featuring classical nonseparability of elementary propositions or nonunital sets of two-valued states interpretable as classical (truth) value assignments—yield logics that cannot be homomorphically embedded into “larger” Boolean algebras. And Kochen-Specker-type arguments demonstrate the total absence of any classical interpretation in terms of the aforementioned two-valued states.

All of the above tactics to certify quantization need quantum representations of contexts in terms of (intertwining) orthonormal bases. Often the algebraic structures are formulated and depicted in terms of (hyper)graphs [1]. These hypergraphs, to be realizable in terms of quantum observables, need to allow a faithful orthogonal representation [2, 3], essentially a vertex labeling by vectors, such that adjacent vertices correspond to orthogonal vectors. Although in principle the equations resulting from such relations may be solvable, their direct solution turns out to be unattainable. Therefore one is left with heuristic methods of parametrization [4] that yield incomplete orthonormal systems; and therefore the necessity to complete those findings by supplementing missing base vectors.

In four dimensions, concerning indecomposability—or, in physical terms, entanglement—this task is straightforward for three given mutually orthogonal unit vectors—the one-dimensional subspace spanned by the missing vector is uniquely defined, and there is no choice. However, a completion with (in)decomposable vectors is not straightforward for two given unit vectors. As we shall see there are rather subtle criteria of (in)decomposability if the four-dimensional Hilbert space is interpreted as a tensor product of two two-dimensional spaces.

Such analysis is pertinent to the aforementioned task of completing one or more bases or contexts of a (hyper)graph: find a complete faithful orthogonal representation (aka coordinatization) of a hypergraph when only a coordinatization of the intertwining observables is known. For instance, for Hardy type arguments, it is significant whether the resulting completion of the context may comprise (in)decomposable vectors [5].

We shall, in particular, consider a four-dimensional real or complex Hilbert space $\mathscr{H}$, where $\mathscr{H}$ is either the column space $\mathbb{R}^4$ or $\mathbb{C}^4$. Suppose further that two unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are known which are orthogonal, such that $\langle \mathbf{e}_1 | \mathbf{e}_2 \rangle = 0$. An orthogonal basis can be formed with these two known vectors, as well as with two “missing” vectors $\mathbf{a}$ and $\mathbf{b}$. Those latter missing vectors ought to have additional properties we are interested in; in particular, for Hilbert spaces which can be considered as tensor products of smaller-dimensional spaces, (in)decomposability.

One uniform way of finding the general form of the missing vectors is by arranging a to-be-completed orthonormal basis (aka context) $B = \left\{ \mathbf{e}_1, \mathbf{e}_2, \mathbf{a}, \mathbf{b} \right\}$ into a unitary matrix

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$\mathbf{U} = \left(\mathbf{e}_1, \mathbf{e}_2, \mathbf{a}, \mathbf{b}\right)$ and solving [6, Theorem 2]

$$|\det(\mathbf{U})| = \frac{1}{4}\mathrm{Tr}\left(\mathbf{U}\mathbf{U}^\dagger\right) = 1, \tag{1}$$

where "†" stands for transposition and complex conjugation (which, in the real case, reduces to transposition "⊺").

The subspace $\mathscr{M}^\perp$ of the Hilbert space $\mathscr{H}$ orthogonal to the subspace $\mathscr{M}$ spanned by both $\mathbf{e}_1$ and $\mathbf{e}_2$ will be two-dimensional and spanned by $\mathbf{a}$ and $\mathbf{b}$. This leaves a continuity of freedom in choosing those latter vectors.

Continuity of choices aside; whether or not the missing vectors can be selected to be (in)decomposable is not merely a question of choice but depends on the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ one started with. For instance, if $\mathbf{e}_1 = \left(1,0,0,0\right)^\intercal$ and $\mathbf{e}_2 = \left(0,1,0,0\right)^\intercal$ an elementary calculation shows that there is no option for both $\mathbf{a}$ as well as $\mathbf{b}$ not to be decomposable: Suppose $\mathbf{a} \propto \left(a_1,a_2,a_3,a_4\right)^\intercal$; then $\langle\mathbf{a}|\mathbf{e}_1\rangle = \langle\mathbf{a}|\mathbf{e}_2\rangle = 0$ implies $a_1 = a_2 = 0$. Therefore, $\mathbf{a}$ must be of the form $\left(0,0,a_3,a_4\right)^\intercal$. The same argument holds for $\mathbf{b}$. By the criterion of decomposability for vectors derived later, by which the scalar product of the two "outer" components of the vector must be equal to the two "inner" components of the vector, both $\mathbf{a}$ as well as $\mathbf{b}$ are decomposable.

For the sake of an example in which $\mathbf{a}$ as well as $\mathbf{b}$ may either be chosen decomposable or indecomposable, consider the instance $\mathbf{e}_1 = \left(1,0,0,0\right)^\intercal$ and $\mathbf{e}_2 = \left(0,0,0,1\right)^\intercal$ which allows either decomposable completions such as $\mathbf{a} = \left(0,1,0,0\right)^\intercal$ and $\mathbf{b} = \left(0,0,1,0\right)^\intercal$ or indecomposable completions such as $\mathbf{a} = \frac{1}{\sqrt{2}}\left(0,1,1,0\right)^\intercal$ and $\mathbf{b} = \frac{1}{\sqrt{2}}\left(0,1,-1,0\right)^\intercal$.

The general question therefore remains: given two orthogonal unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, when is it possible for those missing vectors $\mathbf{a}$ and $\mathbf{b}$ of a "completed" orthogonal basis (aka context) $B = \left\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{a}, \mathbf{b}\right\}$ to be (in)decomposable?

II. NOMENCLATURE

Let $\mathscr{H}_2$ and $\mathscr{H}$ either denote the real vector spaces $\mathbb{R}^2$ and $\mathbb{R}^4$ or the complex vector spaces $\mathbb{C}^2$ and $\mathbb{C}^4$. The standard inner product $\langle\cdot|\cdot\rangle$ makes $\mathscr{H}_2$ and $\mathscr{H}$ into a Hilbert space. We identify the outer or tensor product $\mathscr{H}_2 \otimes \mathscr{H}_2$ with $\mathscr{H}$ as follows. Given vectors $\mathbf{u} = \left(u_1,u_2\right)^\intercal$ and $\mathbf{v} = \left(v_1,v_2\right)^\intercal$ in $\mathscr{H}$ we let $\mathbf{u}\otimes\mathbf{v} = \left(u_1v_1, u_1v_2, u_2v_1, u_2v_2\right)^\intercal \in \mathscr{H}$, which is a form of "vectorization" (that is, a flattening) of this tensor product. This product can be compared to the general form of a vector in four dimensions $\mathbf{z} = \left(z_1,z_2,z_3,z_4\right)^\intercal$. Therefore, for $\mathbf{z}$ to be decomposable $z_1 = x_1y_1$, $z_2 = x_1y_2$, $z_3 = x_2y_1$, and $z_4 = x_2y_2$, from which, because of commutativity of scalar multiplication, follows that

$$z_1z_4 = x_1y_1x_2y_2 = x_1x_2y_1y_2 = x_1y_2x_2y_1 = z_2z_3. \tag{2}$$

That is, the product of the "outer components" z_1z_4 of $\mathbf{z}$ must be equal to the product of its "inner components" z_2z_3, or equivalently, $z_1z_4 - z_2z_3 = 0$ [7, p. 18]. This condition is also sufficient, as it renders three equations for the four unknowns x_1, x_2, y_1 and y_2.

Criterion (2) for decomposability can be rewritten in terms of a symmetric bilinear form as follows. The mapping $\mathbf{z} = \left(z_1,z_2,z_3,z_4\right)^\intercal \mapsto 2(z_1z_4 - z_2z_3)$ is a quadratic form which has an associated bilinear form (not to be confused with the scalar or inner product denoted by $\langle\mathbf{a}|\mathbf{b}\rangle$)

$$\begin{aligned}(\mathbf{a}|\mathbf{b}) &= (a_1b_4 - a_2b_3 - a_3b_2 + a_4b_1)\\ &= \left(a_1,a_2,a_3,a_4\right)\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}b_1\\b_2\\b_3\\b_4\end{pmatrix}\\ &= \mathbf{a}^\intercal\cdot\mathbf{A}\cdot\mathbf{b},\\ \text{with } \mathbf{A} &:= \begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix}.\end{aligned} \tag{3}$$

Therefore,

$$(\mathbf{z}|\mathbf{z}) = 2(z_1z_4 - z_2z_3) = 0 \tag{4}$$

characterises $\mathbf{z}$ as decomposable.

The (non-degenerate) bilinear form (3) can then be used to define a Gramian matrix of two vectors $\mathbf{a}$ and $\mathbf{b}$ by

$$G_{\mathbf{ab}} = \begin{pmatrix}(\mathbf{a}|\mathbf{a}) & (\mathbf{a}|\mathbf{b})\\ (\mathbf{b}|\mathbf{a}) & (\mathbf{b}|\mathbf{b})\end{pmatrix}. \tag{5}$$

This definition of the Gramian matrix for two vectors has a straightforward generalization for an arbitrary finite number of vectors which we shall use later.

Because of symmetry $(\mathbf{a}|\mathbf{b}) = (\mathbf{b}|\mathbf{a})$ the Gram determinant satisfies

$$\begin{vmatrix}(\mathbf{a}|\mathbf{a}) & (\mathbf{a}|\mathbf{b})\\ (\mathbf{b}|\mathbf{a}) & (\mathbf{b}|\mathbf{b})\end{vmatrix} = (\mathbf{a}|\mathbf{a})(\mathbf{b}|\mathbf{b}) - (\mathbf{a}|\mathbf{b})^2. \tag{6}$$

The symmetric matrix $\mathbf{A}$ that is defined in (3) coincides with its inverse $\mathbf{A}^{-1}$. Let $\overline{x} = \Re x - i\Im x$ stand for complex conjugation, so that, for real vector spaces, $\overline{\mathbf{x}} = \mathbf{x}$ for all vectors $\mathbf{x}$. The matrix $\mathbf{A}^{-1}$ defines a bijection

$$\begin{aligned}\mathbf{x} = \left(x_1,x_2,x_3,x_4\right)^\intercal \mapsto \quad & \mathbf{A}^{-1}\cdot\overline{\mathbf{x}} = \left(\overline{x}_4, -\overline{x}_3, -\overline{x}_2, \overline{x}_1\right)^\intercal\\ &= \left(x_4,-x_3,-x_2,x_1\right)^\dagger =: \widetilde{\mathbf{x}},\end{aligned} \tag{7}$$

which is linear in the real case and antilinear in the complex case.

For two vectors $\mathbf{x}$ and $\mathbf{y}$, because of $\mathbf{A}^{-1} = \left(\mathbf{A}^{-1}\right)^\intercal$,

$$\begin{aligned}\langle\mathbf{x}|\mathbf{y}\rangle &= \mathbf{x}^\dagger\cdot\mathbf{y} = \mathbf{x}^\dagger\cdot\left(\mathbf{A}^{-1}\cdot\mathbf{A}\right)\cdot\mathbf{y}\\ &= \overline{\mathbf{x}}^\intercal\cdot\left(\mathbf{A}^{-1}\right)^\intercal\cdot\mathbf{A}\cdot\mathbf{y}\\ &= \left[\left(\mathbf{A}^{-1}\right)\cdot\overline{\mathbf{x}}\right]^\intercal\cdot\mathbf{A}\cdot\mathbf{y} = (\widetilde{\mathbf{x}}|\mathbf{y}),\end{aligned} \tag{8}$$

where $\widetilde{\mathbf{x}} = \left(\mathbf{A}^{-1}\right)\cdot\overline{\mathbf{x}}$. That is, the inner product can be rewritten in terms of the bilinear form (3) which enters the Gram matrix (5). This is a central facility for the following classification of two-dimensional planes in four-dimensional Hilbert space.

As a side note observe that, though a coordinate change in the new coordinates $\mathbf{x} \mapsto \mathbf{x}'$ with $x_1 \mapsto x_1' = (x_1 + x_4)/\sqrt{2}$, $x_2 \mapsto x_2' = (x_2 - x_3)/\sqrt{2}$, $x_3 \mapsto x_3' = (x_2 + x_3)/\sqrt{2}$, and $x_4 \mapsto x_4' = (x_1 - x_4)/\sqrt{2}$ "mixing outer as well as inner components, respectively", this symmetric bilinear form can be rewritten in terms of a diagonal matrix $\mathsf{A}' = \operatorname{diag}(1,1,-1,-1)$, such that $(\mathbf{x}|\mathbf{y}) = (\mathbf{x}')^\intercal \cdot \mathsf{A}' \cdot \mathbf{y}'$; and, in particular, $(\mathbf{z}|\mathbf{z}) = (\mathbf{z}')^\intercal \cdot \mathsf{A}' \cdot \mathbf{z}'$. [For a proof, expand $(\mathbf{x}')^\intercal \cdot \mathsf{A}' \cdot \mathbf{y}'$ in terms of $\mathbf{x}$ and $\mathbf{y}$.] In these new coordinates $\mathbf{z}'$ a necessary and sufficient criterion for $\mathbf{z}$ to be decomposable is $(\mathbf{z}')^\intercal \cdot \mathsf{A}' \cdot \mathbf{z}' = 0$.

Over the complex numbers only, a second coordinate change $\mathbf{x}' \mapsto \mathbf{x}''$ with $x_1' \mapsto x_1'' = x_1'$, $x_2' \mapsto x_2'' = x_2'$, $x_3' \mapsto x_3'' = ix_3'$, and $x_4' \mapsto x_4'' = ix_4''$ yields $\mathsf{A}'' = \operatorname{diag}(1,1,1,1)$ as matrix of this symmetric bilinear form.

We shall make use of the following equality. Let $\mathbf{s} = (s_1, s_2)^\intercal$, $\mathbf{t} = (t_1, t_2)^\intercal$, $\mathbf{u} = (u_1, u_2)^\intercal$ and $\mathbf{v} = (v_1, v_2)^\intercal$ be arbitrary vectors of $\mathscr{H}_2$. Then $\sum_{j,k=1}^2 \overline{(s_j t_k)}(u_j v_k) = \left(\sum_{j=1}^2 \bar{s}_j u_j\right)\left(\sum_{k=1}^2 \bar{t}_k v_k\right)$ implies

$$\langle \mathbf{s} \otimes \mathbf{t} | \mathbf{u} \otimes \mathbf{v} \rangle = \langle \mathbf{s} | \mathbf{u} \rangle \langle \mathbf{t} | \mathbf{v} \rangle. \tag{9}$$

Eq. (9) can be rephrased in the following way. The inner product on $\mathscr{H}$ is the second tensor power of the inner product on $\mathscr{H}_2$; see [7, Appendix A, p. 164] or [8, Sect. 3.4, pp. 47–48].

Likewise, $s_1 t_1 u_2 v_2 - s_1 t_2 u_2 v_1 - s_2 t_1 u_1 v_2 + s_2 t_2 u_1 v_1 = (s_1 u_2 - s_2 u_1)(t_1 v_2 - t_2 v_1)$ results in

$$(\mathbf{s} \otimes \mathbf{t} | \mathbf{u} \otimes \mathbf{v}) = \det(\mathbf{s}, \mathbf{u}) \det(\mathbf{t}, \mathbf{v}), \tag{10}$$

where $(\mathbf{s}, \mathbf{u})$ stands for the matrix whose first and second column are $\mathbf{s}$ and $\mathbf{u}$, respectively. That is, the symmetric bilinear form $(\cdot|\cdot)$ on $\mathscr{H}$ from (3) is the second tensor power of the skew-symmetric bilinear form given by the determinant on $\mathscr{H}_2$ [9, Sect. 1.22, p. 30–31].

The (anti)linear transformation of $\mathscr{H}_2$ sending $\mathbf{u} = (u_1, u_2)^\intercal$ to $\mathbf{u}^\times = (\bar{u}_2, -\bar{u}_1)^\intercal$ satisfies

$$\langle \mathbf{u} | \mathbf{u}^\times \rangle = 0. \tag{11}$$

Furthermore, it allows us to rewrite any inner product in terms of the determinant:

$$\langle \mathbf{u} | \mathbf{v} \rangle = \bar{u}_1 v_1 + \bar{u}_2 v_2 = \begin{vmatrix} \bar{u}_2 & v_1 \\ -\bar{u}_1 & v_2 \end{vmatrix} = \det(\mathbf{u}^\times, \mathbf{v}). \tag{12}$$

We also observe that

$$\langle \mathbf{u}^\times | \mathbf{v}^\times \rangle = u_2 \bar{v}_2 + u_1 \bar{v}_1 = \overline{\langle \mathbf{u} | \mathbf{v} \rangle} = \langle \mathbf{v} | \mathbf{u} \rangle. \tag{13}$$

Furthermore, the second tensor power of the (anti)linear transformation $\mathbf{u} \mapsto \mathbf{u}^\times$ equals the (anti)linear transformation from Eq. (7):

$$\mathbf{u}^\times \otimes \mathbf{v}^\times = \left(\bar{u}_2 \bar{v}_2, -\bar{u}_2 \bar{v}_1, -\bar{u}_1 \bar{v}_2, \bar{u}_1 \bar{v}_1\right)^\intercal = \widetilde{(\mathbf{u} \otimes \mathbf{v})}. \tag{14}$$

III. PLANE TYPES

Let $\mathscr{V}$ be any finite dimensional vector space over the real or complex numbers. Basic results about symmetric bilinear forms on such a vector space can be found, for example, in [10, Theorems 11.21, 23, 24, 25, 26, pp. 283–288]. We briefly recall these results in a form which is tailored to our needs. That is, we consider a k-dimensional subspace $\mathscr{S}$ of $\mathscr{H}$ together with the restriction of $(\cdot|\cdot)$ to $\mathscr{S}$ rather than $\mathscr{V}$ together with an arbitrary symmetric bilinear form on $\mathscr{V}$.

Suppose that an arbitrary basis of $\mathscr{S}$ is given. Then the Gramian matrix of this basis with respect to $(\cdot|\cdot)$, which is defined in analogy to (5), is a symmetric $(k \times k)$-matrix. In a first step, one can switch to a (not necessarily orthogonal) basis $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k\}$ of $\mathscr{S}$ which has a Gramian matrix in diagonal form. Next, by scaling and reordering the vectors $\mathbf{b}_j$, $j = 1, 2\ldots, k$, in an adequate way, one obtains a basis $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k\}$ of $\mathscr{S}$ such that its Gramian matrix with respect to $(\cdot|\cdot)$ takes the form

$$\operatorname{diag}(\underbrace{1, 1, \ldots, 1}_{p \geq 0}, \underbrace{-1, -1, \ldots, -1}_{r-p \geq 0}, \underbrace{0, 0, \ldots, 0}_{k-r \geq 0}) \tag{15}$$

for some p, r in the real case and the form

$$\operatorname{diag}(\underbrace{1, 1, \ldots, 1}_{r \geq 0}, \underbrace{0, 0, \ldots, 0}_{k-r \geq 0}) \tag{16}$$

for some r in the complex case. (The need to distinguish between these two cases stems from the fact that negative real numbers do not admit a real square root.) The numbers r and p appearing in (15) and (16) are thereby uniquely determined by $\mathscr{S}$ and $(\cdot|\cdot)$, that is, they do not depend on the choice of an appropriate basis of $\mathscr{S}$. We also observe that the *radical* of $\mathscr{S}$, which is defined as

$$\operatorname{rad}(\mathscr{S}) = \{\mathbf{x} \in \mathscr{S} \mid (\mathbf{x}|\mathbf{y}) = 0 \text{ for all } \mathbf{y} \in \mathscr{S}\},$$

satisfies

$$\operatorname{rad}(\mathscr{S}) = \operatorname{span}\{\mathbf{c}_{r+1}, \mathbf{c}_{r+2}, \ldots, \mathbf{c}_k\}. \tag{17}$$

In particular, letting $\mathscr{S} = \mathscr{H}$ yields the matrices $\mathsf{A}' = \operatorname{diag}(1,1,-1,-1)$ (real case) and $\mathsf{A}'' = \operatorname{diag}(1,1,1,1)$ (complex case) that we already encountered at the end of Section II.

Let $\mathscr{S} = \mathscr{M}$ be some plane, defined as a two-dimensional subspace of the Hilbert space $\mathscr{H}$. Then, by the above, there exists at least one basis $\{\mathbf{c}_1, \mathbf{c}_2\}$ of $\mathscr{M}$ such that the Gramian matrix $G_{\mathbf{c}_1\mathbf{c}_2}$ defined in (5) with respect to the bilinear form defined in (3) takes on one of the following forms:

(i) real (Hilbert space) case:

- (i.1) $G_{\mathbf{c}_1\mathbf{c}_2} = \operatorname{diag}(0,0)$,
- (i.2) $G_{\mathbf{c}_1\mathbf{c}_2} = \operatorname{diag}(1,0)$,
- (i.3) $G_{\mathbf{c}_1\mathbf{c}_2} = \operatorname{diag}(-1,0)$,
- (i.4) $G_{\mathbf{c}_1\mathbf{c}_2} = \operatorname{diag}(1,1)$,
- (i.5) $G_{\mathbf{c}_1\mathbf{c}_2} = \operatorname{diag}(-1,-1)$,
- (i.6) $G_{\mathbf{c}_1\mathbf{c}_2} = \operatorname{diag}(1,-1)$;

(ii) complex (Hilbert space) case:

- (ii.1) $G_{\mathbf{c}_1\mathbf{c}_2} = \operatorname{diag}(0,0)$, (ii.2) $G_{\mathbf{c}_1\mathbf{c}_2} = \mathrm{diag}(1,0)$,

(ii.3) $G_{\mathbf{c}_1\mathbf{c}_2} = \mathrm{diag}(1,1)$.

If, say, the plane $\mathscr{M}$ is (uniquely) associated with the Gramian matrix of the form $\mathrm{diag}(1,0)$ then we shall denote $\mathscr{M}$ as a *plane of type* $(1,0)$. The other cases are treated accordingly. Our discussion below will establish the *existence* of all these possible plane types.

Let us come back to an earlier question: Suppose that an *arbitrary* basis $\{\mathbf{a},\mathbf{b}\}$ of $\mathscr{M}$ has been found. The question then is: does this two-dimensional subspace allow or support (in)decomposable vectors in four-dimensional space?

The cases (i.1)–(i.6) for real four-dimensional Hilbert space $\mathscr{H} = \mathbb{R}^4$ as well as (ii.1)–(ii.3) for complex four-dimensional Hilbert space $\mathscr{H} = \mathbb{C}^4$ discussed earlier present a means to answer this question. Thereby the Gramian matrix $G_{\mathbf{ab}}$ is used for an identification and characterization of the particular unique plane type of $\mathscr{M}$.

For real four-dimensional Hilbert space $\mathscr{H} = \mathbb{R}^4$ there are six types of planes, corresponding to the cases (i.1) to (i.6) mentioned earlier. In what follows three cases and the respective subcases will be discussed which characterize those six plane types. We thereby apply results that provide, for real vector spaces of any finite dimension, necessary and sufficient conditions for the (semi)definiteness of a quadratic form in terms of principal minors of its Gramian matrix with respect to an arbitrary basis.

In the following analysis, based on the earlier classification (i.1)–(i.6) for real four-dimensional Hilbert space as well as (ii.1)–(ii.3) for complex four-dimensional Hilbert space, the Gram determinant $\det(G_{\mathbf{ab}})$ will be denoted by G, and G_{ij} stands for the element in the ith row and the jth column of the Gramian matrix $G_{\mathbf{ab}}$.

A. Gram determinant $G > 0$, plane of types $(1,1)$ or $(-1,-1)$

$G > 0$ means that $(\mathbf{a}|\mathbf{a})$ as well as $(\mathbf{b}|\mathbf{b})$ have the same sign and are both non-zero.

1. $G_{11} = (\mathbf{a}|\mathbf{a}) > 0$, *plane of type* $(1,1)$

In this subcase $(\mathbf{a}|\mathbf{a})$ is positive, which indicates a plane of type $(1,1)$ [11, Thm. 3, p. 306]. Consequently, all non-zero vectors of $\mathscr{M}$ are indecomposable.

A typical example is the two-dimensional subspace spanned by $\mathbf{a} = \left(0,1,-1,0\right)^\intercal$ and $\mathbf{b} = \left(1,0,0,1\right)^\intercal$. Any element of the span of $\mathbf{a}$ and $\mathbf{b}$ can be written as $\left(x_1,x_2,-x_2,x_1\right)^\intercal$. The associated Gramian is of the form $G_{\mathbf{ab}} = \mathrm{diag}(2,2)$.

2. $G_{11} = (\mathbf{a}|\mathbf{a}) < 0$, *plane of type* $(-1,-1)$

In this subcase $(\mathbf{a}|\mathbf{a})$ is negative, which indicates a plane of type $(-1,-1)$ [11, Thm. 5, p. 308]. Consequently, all non-zero vectors of $\mathscr{M}$ are indecomposable.

A typical example is the two-dimensional subspace spanned by $\mathbf{a} = \left(0,1,1,0\right)^\intercal$ and $\mathbf{b} = \left(1,0,0,-1\right)^\intercal$. Any element of the span of $\mathbf{a}$ and $\mathbf{b}$ can be written as $\left(x_1,x_2,x_2,-x_1\right)^\intercal$. The associated Gramian is of the form $G_{\mathbf{ab}} = \mathrm{diag}(-2,-2)$.

B. Gram determinant $G = 0$, plane of types $(0,0)$, $(1,0)$ or $(-1,0)$

1. $G_{11} = (\mathbf{a}|\mathbf{a}) = G_{22} = (\mathbf{b}|\mathbf{b}) = 0$, *plane of type* $(0,0)$

In this subcase $G = (\mathbf{a}|\mathbf{b})^2 = 0$, so that the Gramian vanishes – that is, $G_{\mathbf{ab}} = \mathrm{diag}(0,0)$. Hence, by definition, $\mathscr{M}$ is a plane of type $(0,0)$. Any plane of this type contains a continuity of decomposable vectors and no indecomposable vector.

A typical example is the two-dimensional subspace spanned by $\mathbf{a} = \left(1,0,0,0\right)^\intercal$ and $\mathbf{b} = \left(0,1,0,0\right)^\intercal$. Any element of the span of $\mathbf{a}$ and $\mathbf{b}$ can be written as $\left(x_1,x_2,0,0\right)^\intercal$.

2. $G_{11} = (\mathbf{a}|\mathbf{a}) > 0$ *or* $G_{22} = (\mathbf{b}|\mathbf{b}) > 0$, *plane of type* $(1,0)$

In this subcase one of $(\mathbf{a}|\mathbf{a})$ and $(\mathbf{b}|\mathbf{b})$, say $(\mathbf{a}|\mathbf{a})$, is assumed to be positive. Then the other one, in this case $(\mathbf{b}|\mathbf{b})$, needs to be non-negative, because only then the product $(\mathbf{a}|\mathbf{a})(\mathbf{b}|\mathbf{b})$ is non-negative and therefore may "compensate" the subtraction of the non-negative term $(\mathbf{b}|\mathbf{a})^2$ of the Gram determinant (6). From [11, Thm. 4, p. 307], $\mathscr{M}$ is a plane of type $(1,0)$.

Decomposability (4) requires that, for some ξ, $(\xi\mathbf{a}+\mathbf{b}|\xi\mathbf{a}+\mathbf{b}) = G_{11}\xi^2 + 2G_{12}\xi + G_{22} = 0$, and thus, $\xi = \left(-2G_{12} \pm \sqrt{4G_{12}^2 - 4G_{11}G_{22}}\right)/(2G_{11}) = \Big(-G_{12} \pm \underbrace{\sqrt{-G}}_{=0}\Big)/G_{11} = -G_{11}^{-1}G_{12}$. Note that, in order for the denominator $G_{11} = (\mathbf{a}|\mathbf{a})$ not to vanish, ξ must be multiplied with the indecomposable vector $\mathbf{a}$. Therefore there exists (up to scale factors) only a unique decomposable vector in $\mathscr{M}$, namely $\mathbf{c} = -G_{11}^{-1}G_{12}\mathbf{a} + \mathbf{b}$. All vectors in $\mathscr{M}$ that are not in the span of $\mathbf{c}$ – indeed, a continuity of vectors – are indecomposable.

A typical example is the two-dimensional subspace spanned by $\mathbf{a} = \left(0,1,-1,0\right)^\intercal$ and $\mathbf{b} = \left(1,0,0,0\right)^\intercal$. Any element of the span of $\mathbf{a}$ and $\mathbf{b}$ can be written as $\left(x_1,x_2,-x_2,0\right)^\intercal$. The associated Gramian is of the form $G_{\mathbf{ab}} = \mathrm{diag}(2,0)$.

3. $G_{11} = (\mathbf{a}|\mathbf{a}) < 0$ *or* $G_{22} = (\mathbf{b}|\mathbf{b}) < 0$, *plane of type* $(-1,0)$

In this subcase one of $(\mathbf{a}|\mathbf{a})$ and $(\mathbf{b}|\mathbf{b})$, say $(\mathbf{a}|\mathbf{a})$, is assumed to be negative. Then the other one, in this case $(\mathbf{b}|\mathbf{b})$, needs to be non-positive, because only then the product $(\mathbf{a}|\mathbf{a})(\mathbf{b}|\mathbf{b})$ is non-positive and therefore may "compensate" the subtraction of the non-negative term $(\mathbf{a}|\mathbf{b})^2$ of the Gram determinant (6).

Again, there exists (up to scale factors) only a unique decomposable vector in $\mathscr{M}$, namely $\mathbf{c} = -G_{11}^{-1}G_{12}\mathbf{a} + \mathbf{b}$.

A typical example is the two-dimensional subspace spanned by $\mathbf{a} = \left(0,1,1,0\right)^\intercal$ and $\mathbf{b} = \left(1,0,0,0\right)^\intercal$. Any element of the span of $\mathbf{a}$ and $\mathbf{b}$ can be written as $\left(x_1, x_2, x_2, 0\right)^\intercal$. The associated Gramian is of the form $G_{\mathbf{ab}} = \mathrm{diag}(-2, 0)$.

C. Gram determinant $G < 0$, plane of type $(1,-1)$

In this case $(\mathbf{a}|\mathbf{a})$ as well as $(\mathbf{b}|\mathbf{b})$ can be anything (positive, negative, zero). By the characterizations in [11, Thm. 3, 4, 5, 6, pp. 306–308], the plane $\mathscr{M}$ has to be of the only remaining type, that is, of type $(1,-1)$.

There exists (up to scale factors) only two unique distinct decomposable vectors $\mathbf{c}_\pm$, in accordance with the construction given next. All other vectors – indeed, a continuity of vectors in the plane spanned by $\mathbf{a}$ and $\mathbf{b}$ – are indecomposable. This can again be shown by assuming the case $(\mathbf{a}|\mathbf{a}) \neq 0$, and by noting that decomposability (4) requires that, for some ξ, $(\xi\mathbf{a}+\mathbf{b}|\xi\mathbf{a}+\mathbf{b}) = G_{11}\xi^2 + 2G_{12}\xi + G_{22} = 0$, such that $\xi_\pm = \left(-2G_{12} \pm \sqrt{4G_{12}^2 - 4G_{11}G_{22}}\right) / (2G_{11}) = \Big(-G_{12} \pm \underbrace{\sqrt{-G}}_{\neq 0}\Big)/G_{11}$. Note that these two solutions $\mathbf{c}_\pm = \left[\left(-G_{12} \pm \sqrt{-G}\right)/G_{11}\right]\mathbf{a} + \mathbf{b}$ need not be mutually orthogonal. In the second case one supposes that, instead of $(\mathbf{a}|\mathbf{a}) \neq 0$, now $(\mathbf{b}|\mathbf{b}) \neq 0$, and carries through an analogous calculation. In the third case $(\mathbf{a}|\mathbf{a}) = (\mathbf{b}|\mathbf{b}) = 0$ both vectors $\mathbf{a}$ as well as $\mathbf{b}$ are already decomposable. Note that, in order for the denominator not to vanish, ξ must be multiplied with the respective indecomposable vector.

A typical example is the two-dimensional subspace spanned by $\mathbf{a} = \left(1,0,0,0\right)^\intercal$ and $\mathbf{b} = \left(0,0,0,1\right)^\intercal$. Any element of the span of $\mathbf{a}$ and $\mathbf{b}$ can be written as $\left(x_1, 0, 0, x_2\right)^\intercal$. The associated Gramian is of the form $G_{\mathbf{ab}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The main results of those considerations, as it concerns the question of (in)decomposability, is that, with the exception of type $(0,0)$ planes which contain only decomposable vectors, all other five plane types contain (a continuity of) orthogonal bases spanning them whose basis vectors are both indecomposable: planes of type $(1,-1)$ contain (up to scale factors) a single orthogonal basis whose elements are decomposable; and planes of types $(1,0)$ and $(-1,0)$ contain (up to scale factors and permutations) a single orthogonal basis with one decomposable and one indecomposable element. Planes of plane of types $(1,1)$ and $(-1,-1)$ contain no decomposable non-zero vectors. This completes the characterisation of the real case.

For complex four-dimensional Hilbert space $\mathscr{H} = \mathbb{C}^4$, according to the cases (ii.1) to (ii.3) mentioned earlier, the rank of $G_{\mathbf{ab}}$ determines the type of $\mathscr{M}$ [10, Theorem 11.24, p. 287]:

(i) If $\mathrm{rank}(G_{\mathbf{ab}}) = 0$ then $\mathscr{M}$ is of type $(0,0)$. Earlier remarks concerning properties of a plane of real type $(0,0)$ pertain.

(ii) If $\mathrm{rank}(G_{\mathbf{ab}}) = 1$ then $\mathscr{M}$ is of type $(1,0)$. This situation parallels that of a plane of real type $(1,0)$. In particular, the calculation from there, yielding a unique decomposable vector $\mathbf{c} \in \mathscr{M}$, carries over provided that $G_{11} = (\mathbf{a}|\mathbf{a}) \neq 0$. Moreover, there is continuity of indecomposable vectors in $\mathscr{M}$ which are not in the span of $\mathbf{c}$.

(iii) If $\mathrm{rank}(G_{\mathbf{ab}}) = 2$ then $\mathscr{M}$ is of type $(1,1)$. There is a neat analogy to the case of a plane of real type $(1,-1)$. Note that any quadratic equation over the complex numbers with non-vanishing discriminant has precisely two distinct solutions. Therefore, the calculation of the decomposable vectors $\mathbf{c}_\pm \in \mathscr{M}$ carries over, provided that $G_{11} = (\mathbf{a}|\mathbf{a}) \neq 0$.

All three plane types actually occur. This follows immediately from a reinterpretation of our various examples in the real case.

The various types of planes admit a geometric interpretation in terms of the *projective space* $\mathbb{P}(\mathscr{H})$. We recall that the *points* of $\mathbb{P}(\mathscr{H})$ are the one-dimensional subspaces of $\mathscr{H}$. A set of points is called a *projective line* (*projective plane*) of $\mathbb{P}(\mathscr{H})$ if it comprises all one-dimensional subspaces of $\mathscr{H}$ that are contained in some fixed two-dimensional (three-dimensional) subspace of $\mathscr{H}$; see, for example, [12, p. 122]. All points of $\mathbb{P}(\mathscr{H})$ that are spanned by decomposable vectors constitute a *ruled quadric* Φ, say, with equation $z_1z_4 - z_2z_3 = 0$ [12, pp. 143–144]. The *type* of a projective line is understood to be the type of the associated subspace of $\mathscr{H}$.

In the real case the points off the quadric Φ fall into two classes, namely the sets of points $\mathrm{span}\{\mathbf{z}\}$, $\mathbf{z} \in \mathscr{H}$, with $(\mathbf{z}|\mathbf{z}) > 0$ and $(\mathbf{z}|\mathbf{z}) < 0$, respectively. We call these two classes the *positive* and the *negative side* of Φ, respectively. (From a geometric point of view, the attributes "positive" and "negative" are immaterial. Indeed, multiplying the equation of Φ by some negative real number will change the labelling of the two sides but not the quadric Φ.) A projective line is of type $(0,0)$ precisely when it is contained in Φ. A projective line is of type $(1,0)$ [of type $(-1,0)$] if and only if it meets Φ at a unique point whereas all its other points are on the positive [negative] side of Φ. The projective lines of type $(1,1)$ [of type $(-1,-1)$] are those which are contained in the positive [negative] side of Φ. Finally, a projective line is of type $(1,-1)$ precisely when it meets Φ at exactly two distinct points. (Any such line contains points from either side.)

In the complex case, a projective line is of type $(0,0)$, $(1,0)$ or $(1,1)$ precisely when it is contained in Φ, it meets Φ at a unique point or it meets Φ at exactly two distinct points.

In order to visualize this situation in the real case, we consider the *affine space* on $\mathbb{R}^3$; its *points* are the vectors of $\mathbb{R}^3$, an *affine line* (*affine plane*) is a translate of a one-dimensional (two-dimensional) subspace of $\mathbb{R}^3$. There is a one-one correspondence between the set of points of $\mathbb{P}(\mathscr{H})$ that are not contained in the projective plane $z_4 = 0$ and the set of points of the affine space on $\mathbb{R}^3$ as follows:

$$\mathrm{span}\{(z_1,z_2,z_3,z_4)^\intercal\} \mapsto (w_1,w_2,w_3)^\intercal = \left(\frac{z_1}{z_4}, \frac{z_2}{z_4}, \frac{z_3}{z_4}\right)^\intercal.$$

Under this correspondence a projective line (plane) corresponds to an affine line (plane) unless it is contained in the projective plane $z_4 = 0$ [12, p. 124].

The points (off the plane $z_4 = 0$) of the ruled quadric Φ correspond to the points of a *hyperbolic paraboloid* with equation $w_1 = w_2 w_3$, which is depicted in Fig. 1. The figure also shows several affine lines together with the type of their associated projective lines. All affine lines on the paraboloid, among which are the w_2-axis and the w_3-axis are of type $(0,0)$. The points "above" ("below") the paraboloid illustrate the positive (negative) side. The w_1-axis thereby is understood to be "tending upwards". Take notice that this picture lacks all points of $\mathbb{P}(\mathscr{H})$ in the projective plane $z_4 = 0$. Therefore, in some cases, it provides an incomplete illustration. For example, the w_1-axis has just one point in common with the paraboloid, namely $(0,0,0)^\mathsf{T}$ even though it corresponds to a projective line of type $(1,-1)$, which is spanned by the decomposable vectors $(1,0,0,0)^\mathsf{T}$ and $(0,0,0,1)^\mathsf{T}$.

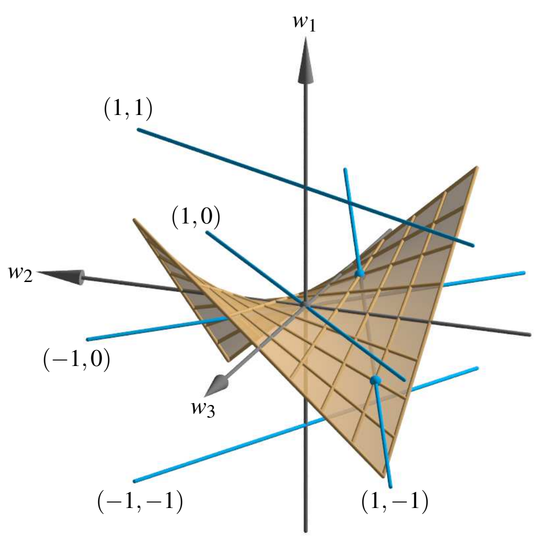

FIG. 1. Schematic drawing of various plane types.

The paraboloid from Fig. 1 is one way to visualize the ruled quadric Φ comprising all points that are spanned by decomposable vectors; this can also be found in Refs. [13, Fig. 6, p. 4687] and [14, Fig. 16.1, p. 438]. For an alternative point of view, from which Φ appears as a hyperboloid of one sheet, see Refs. [12, Fig. 2.17, p. 35] and [14, Fig. 4.3, p. 113].

IV. IDENTIFICATION AND CHARACTERIZATION OF (ORTHOGONAL) PLANES

We are now in a position to solve the problem mentioned earlier: suppose we are given two orthogonal unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ spanning a plane (aka the two-dimensional subspace) $\mathscr{M}$ of a four-dimensional Hilbert space $\mathscr{H}$. One intermediate task – a straightforward [e.g. via the system of non-linear equations (1)] exercise – is to find an orthogonal basis $\{\mathbf{a},\mathbf{b}\}$ of the plane $\mathscr{M}^\perp$ orthogonal to $\mathscr{M}$. Here we are not concerned with the explicit realization of two remaining vectors $\mathbf{a}$ and $\mathbf{b}$. We focus instead on the identification and analysis of the plane $\mathscr{M}^\perp$, which allows us to decide whether or not $\mathbf{a}$ and $\mathbf{b}$ can be chosen indecomposable.

We shall solve this latter problem directly by substitution of the inner product $\langle\cdot|\cdot\rangle$ by the bilinear form $(\cdot|\cdot)$ from (3), as exposed in Eq. (8).

We start by introducing the plane which is defined as the image of $\mathscr{M}$ under the (anti)linear transformation (7):

$$\widetilde{\mathscr{M}} = \left\{ \left(\mathsf{A}^{-1}\right) \cdot \overline{\mathbf{x}} \mid \mathbf{x} \in \mathscr{M} \right\} = \left\{ \widetilde{\mathbf{x}} \mid \mathbf{x} \in \mathscr{M} \right\}. \qquad (18)$$

The first essential point is as follows. *The planes $\mathscr{M}$ and $\widetilde{\mathscr{M}}$ are of the same type regarding* $(\cdot|\cdot)$. In order to prove the assertion, we note that, by a straightforward calculation,

$$(\widetilde{\mathbf{x}}|\widetilde{\mathbf{y}}) = \overline{(\mathbf{x}|\mathbf{y})} \text{ for all } \mathbf{x},\mathbf{y} \in \mathscr{H}. \qquad (19)$$

Now let $\{\mathbf{c}_1,\mathbf{c}_2\}$ be a basis of $\mathscr{M}$ such that the Gramian matrix $G_{\mathbf{c}_1\mathbf{c}_2}$ has the distinguished form as described in (15) for the real case or as in (16) for the complex case. Then $\{\widetilde{\mathbf{c}}_1,\widetilde{\mathbf{c}}_2\}$ is a basis of $\widetilde{\mathscr{M}}$ and (19) gives

$$G_{\widetilde{\mathbf{c}}_1\widetilde{\mathbf{c}}_2} = \overline{G_{\mathbf{c}_1\mathbf{c}_2}} = G_{\mathbf{c}_1\mathbf{c}_2}, \qquad (20)$$

which establishes the result.

The second essential point is that the *type of $\mathscr{M}^\perp$ regarding $(\cdot|\cdot)$ is co-determined by the type of $\widetilde{\mathscr{M}}$ regarding* $(\cdot|\cdot)$. Notice, however, that, as earlier, the real and complex cases have to be treated separately: whereas in the complex (Hilbert space) case $\mathscr{M}^\perp$ and $\widetilde{\mathscr{M}}$ are of the same type, in the real (Hilbert space) case

(i) $\mathscr{M}^\perp$ is of type $(0,0) \Leftrightarrow \widetilde{\mathscr{M}}$ is of type $(0,0)$,

(ii) $\mathscr{M}^\perp$ is of type $(1,-1) \Leftrightarrow \widetilde{\mathscr{M}}$ is of type $(1,-1)$,

(iii) $\mathscr{M}^\perp$ is of type $(\pm1,\pm1) \Leftrightarrow \widetilde{\mathscr{M}}$ is of type $(\mp1,\mp1)$,

(iv) $\mathscr{M}^\perp$ is of type $(\pm1,0) \Leftrightarrow \widetilde{\mathscr{M}}$ is of type $(\mp1,0)$.

Our proof is based on the following alternative description of $\widetilde{\mathscr{M}}$, which makes use of Eq. (18), the identity $\mathscr{M} = (\mathscr{M}^\perp)^\perp$, Eq. (3) and the bijectivity of the transformation (7):

$$\begin{aligned}
\widetilde{\mathscr{M}} &= \{\widetilde{\mathbf{x}} \in \mathscr{H} \mid \mathbf{x} \in \mathscr{M}\} \\
&= \{\widetilde{\mathbf{x}} \in \mathscr{H} \mid \mathbf{x} \in (\mathscr{M}^\perp)^\perp\} \\
&= \{\widetilde{\mathbf{x}} \in \mathscr{H} \mid \langle\mathbf{x}|\mathbf{y}\rangle = 0 \text{ for all } \mathbf{y} \in \mathscr{M}^\perp\} \\
&= \{\widetilde{\mathbf{x}} \in \mathscr{H} \mid (\widetilde{\mathbf{x}}|\mathbf{y}) = 0 \text{ for all } \mathbf{y} \in \mathscr{M}^\perp\} \\
&= \{\mathbf{z} \in \mathscr{H} \mid (\mathbf{z}|\mathbf{y}) = 0 \text{ for all } \mathbf{y} \in \mathscr{M}^\perp\}.
\end{aligned} \qquad (21)$$

Likewise, we also have

$$\begin{aligned}
\mathscr{M}^\perp &= \{\mathbf{y} \in \mathscr{H} \mid \langle\mathbf{x}|\mathbf{y}\rangle = 0 \text{ for all } \mathbf{x} \in \mathscr{M}\} \\
&= \{\mathbf{y} \in \mathscr{H} \mid (\widetilde{\mathbf{x}}|\mathbf{y}) = 0 \text{ for all } \mathbf{x} \in \mathscr{M}\} \\
&= \{\mathbf{y} \in \mathscr{H} \mid (\mathbf{z}|\mathbf{y}) = 0 \text{ for all } \mathbf{z} \in \widetilde{\mathscr{M}}\}.
\end{aligned} \qquad (22)$$

Eqs. (21) and (22) imply that

$$\operatorname{rad}(\widetilde{\mathscr{M}}) = \widetilde{\mathscr{M}} \cap \mathscr{M}^\perp = \operatorname{rad}(\mathscr{M}^\perp). \qquad (23)$$

There exist bases $\{\mathbf{d}_1,\mathbf{d}_2\}$ of $\widetilde{\mathscr{M}}$ and $\{\mathbf{d}_3,\mathbf{d}_4\}$ of $\mathscr{M}^\perp$ such that their Gramian matrices have the distinguished form as described in (15) for the real case or as in (16) for the complex case. Let m denote the dimension of the subspace appearing in Eq. (23). Then Eq. (17), applied to $\widetilde{\mathscr{M}}$ and its basis $\{\mathbf{d}_1,\mathbf{d}_2\}$, together with one of Eqs. (15) and (16) shows that the leading $2-m$ diagonal entries of the Gramian matrix $G_{\mathbf{d}_1\mathbf{d}_2}$ are non-zero, whereas the remaining m diagonal entries are zero. The same result holds, mutatis mutandis, for the Gramian matrix $G_{\mathbf{d}_3\mathbf{d}_4}$. There are three cases.

In the first case, $\mathscr{H}$ is a complex Hilbert space or $m=2$. Then, by the above, $G_{\mathbf{d}_1\mathbf{d}_2}=G_{\mathbf{d}_3\mathbf{d}_4}$ so that $\widetilde{\mathscr{M}}$ and $\mathscr{M}^\perp$ are of the same type. In particular, for $m=2$ both planes are of type $(0,0)$. This establishes the result for a complex space as well as (i) for a real space.

In the second case, $\mathscr{H}$ is a real Hilbert space and $m=0$. The planes $\widetilde{\mathscr{M}}$ and $\mathscr{M}^\perp$ are of types $(\varepsilon_1,\varepsilon_2)$ and $(\varepsilon_3,\varepsilon_4)$, respectively, where $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\in\{1,-1\}$. Since $m=\dim(\widetilde{\mathscr{M}}\cap\mathscr{M}^\perp)=0$, the four vectors $\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3,\mathbf{d}_4$ constitute a basis of $\widetilde{\mathscr{M}}\oplus\mathscr{M}^\perp=\mathscr{H}$ and $G_{\mathbf{d}_1\mathbf{d}_2\mathbf{d}_3\mathbf{d}_4}=\operatorname{diag}(\varepsilon_1,\varepsilon_3,\varepsilon_3,\varepsilon_4)$. By Sylvester's law of inertia [11, Thm. 1, p. 297], this matrix coincides – up to a permutation of its diagonal entries – with the matrix $\mathbf{A}'=\operatorname{diag}(1,1,-1,-1)$ from Section II. This establishes (ii) and (iii).

In the third case, $\mathscr{H}$ is a real Hilbert space and $m=1$. The planes $\widetilde{\mathscr{M}}$ and $\mathscr{M}^\perp$ are of types $(\varepsilon_1,0)$ and $(\varepsilon_3,0)$, respectively, where $\varepsilon_1,\varepsilon_3\in\{1,-1\}$. In order to verify (iv), it remains to show that ε_1 and ε_3 have different signs. Assume to the contrary that, for example, ε_1 and ε_3 are both positive. From Eq. (17), applied to $\widetilde{\mathscr{M}}$ and its basis $\{\mathbf{d}_1,\mathbf{d}_2\}$, we obtain $\mathbf{d}_1\notin\operatorname{rad}(\widetilde{\mathscr{M}})$ and $\mathbf{d}_2\in\operatorname{rad}(\widetilde{\mathscr{M}})$. The same kind of reasoning for $\mathscr{M}^\perp$ and $\{\mathbf{d}_3,\mathbf{d}_4\}$ yields $\mathbf{d}_3\notin\operatorname{rad}(\mathscr{M}^\perp)$ and $\mathbf{d}_4\in\operatorname{rad}(\mathscr{M}^\perp)$. Thus, using Eq. (23), $\mathbf{d}_1,\mathbf{d}_3\notin\widetilde{\mathscr{M}}\cap\mathscr{M}^\perp$ whereas $\mathbf{d}_2,\mathbf{d}_4\in\widetilde{\mathscr{M}}\cap\mathscr{M}^\perp$. The three vectors $\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3$ therefore constitute a basis of $\widetilde{\mathscr{M}}+\mathscr{M}^\perp$ and its Gramian matrix has the form $G_{\mathbf{d}_1\mathbf{d}_2\mathbf{d}_3}=\operatorname{diag}(\varepsilon_1,0,\varepsilon_3)=\operatorname{diag}(1,0,1)$. Therefore $(\mathbf{x}|\mathbf{x})\geq 0$ for all $\mathbf{x}\in\widetilde{\mathscr{M}}+\mathscr{M}^\perp$. On the other hand, there exists a plane $\mathscr{N}$ of type $(-1,-1)$, whence $(\mathbf{x}|\mathbf{x})<0$ for all non-zero vectors $\mathbf{x}\in\mathscr{N}$. Due to $\dim\mathscr{H}=4$, the plane $\mathscr{N}$ has a non-zero intersection with the three-dimensional subspace $\widetilde{\mathscr{M}}+\mathscr{M}^\perp$, that is, there exists a vector $\mathbf{n}\in\widetilde{\mathscr{M}}+\mathscr{M}^\perp$ with $(\mathbf{n},\mathbf{n})<0$, a contradiction.

Summing up, the plane type of $\mathscr{M}^\perp$ can be directly obtained by analyzing the Gramian matrix $G_{\mathbf{e}_1\mathbf{e}_2}$ of the two given "input" vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, and, in the complex case, determining its rank.

V. ORTHOGONALITY OF DECOMPOSABLE VECTORS

Throughout this section, $\mathscr{M}$ denotes a plane of type $(1,-1)$ (real case) or of type $(1,1)$ (complex case). Then there exist vectors $\mathbf{s},\mathbf{t},\mathbf{u},\mathbf{v}$ in $\mathscr{H}_2$ such that

$$\{\mathbf{s}\otimes\mathbf{t},\mathbf{u}\otimes\mathbf{v}\},\tag{24}$$

is a basis of $\mathscr{M}$. Since $\mathscr{M}$ is not of type $(0,0)$, we must have, by virtue of Eq. (10),

$$(\mathbf{s}\otimes\mathbf{t}|\mathbf{u}\otimes\mathbf{v})=\det(\mathbf{s},\mathbf{u})\det(\mathbf{t},\mathbf{v})\neq 0.\tag{25}$$

This in turn shows that $\{\mathbf{s},\mathbf{u}\}$ and $\{\mathbf{t},\mathbf{v}\}$ are bases of $\mathscr{H}_2$. Now (25) implies $\det(\mathbf{s}^\times,\mathbf{u}^\times)=\overline{\det(\mathbf{s},\mathbf{u})}\neq 0$ and $\det(\mathbf{t}^\times,\mathbf{v}^\times)=\overline{\det(\mathbf{t},\mathbf{v})}\neq 0$. Therefore each of the sets $\{\mathbf{s}^\times,\mathbf{u}^\times\}$ and $\{\mathbf{t}^\times,\mathbf{v}^\times\}$ is a basis of $\mathscr{H}_2$. Consequently, we obtain

$$\{\mathbf{s}^\times\otimes\mathbf{t}^\times,\ \underbrace{\mathbf{s}^\times\otimes\mathbf{v}^\times}_{=:\mathbf{b}},\ \underbrace{\mathbf{u}^\times\otimes\mathbf{t}^\times}_{=:\mathbf{a}},\ \mathbf{u}^\times\otimes\mathbf{v}^\times\}\tag{26}$$

as a basis of $\mathscr{H}$. Furthermore, it is immediate from Eqs. (9) and (11) that each of the linearly independent vectors $\mathbf{a}$ and $\mathbf{b}$ is orthogonal to the vectors $\mathbf{s}\otimes\mathbf{t}$ and $\mathbf{u}\otimes\mathbf{v}$, that is, $\{\mathbf{a},\mathbf{b}\}$ is a basis of the plane $\mathscr{M}^\perp$.

If the basis vectors of $\mathscr{M}$ appearing in (24) are orthogonal, may we then suspect that there exists a "completed" orthogonal basis of the four-dimensional real or complex Hilbert space $\mathscr{H}$ which (includes these two vectors and) consists solely of decomposable vectors? Stated pointedly, does the orthogonality of decomposable vectors spanning the given plane $\mathscr{M}$ imply that the corresponding two decomposable vectors in the orthogonal subspace $\mathscr{M}^\perp$ [which is again of the same type] are also orthogonal, and *vice versa?* In what follows we shall prove that this is indeed the case; that is, the orthogonality of the two decomposable vectors from (24) is "inherited" by the two decomposable vectors $\mathbf{a}$ and $\mathbf{b}$ defined in (26). Using Eqs. (9) and (13) we obtain:

$$\langle\mathbf{s}\otimes\mathbf{t}|\mathbf{u}\otimes\mathbf{v}\rangle=\langle\mathbf{s}|\mathbf{u}\rangle\langle\mathbf{t}|\mathbf{v}\rangle\tag{27}$$

and

$$\langle\mathbf{u}^\times\otimes\mathbf{t}^\times|\mathbf{s}^\times\otimes\mathbf{v}^\times\rangle=\langle\mathbf{u}^\times|\mathbf{s}^\times\rangle\langle\mathbf{t}^\times|\mathbf{v}^\times\rangle=\langle\mathbf{s}|\mathbf{u}\rangle\langle\mathbf{v}|\mathbf{t}\rangle.\tag{28}$$

Therefore $\mathbf{s}\otimes\mathbf{t}$ and $\mathbf{u}\otimes\mathbf{v}$ are orthogonal if and only if at least one of the inner products $\langle\mathbf{s}|\mathbf{u}\rangle$ and $\langle\mathbf{t}|\mathbf{v}\rangle$ vanishes. This in turn is equivalent to $\mathbf{u}^\times\otimes\mathbf{t}^\times$ and $\mathbf{s}^\times\otimes\mathbf{v}^\times$ being orthogonal.

Our considerations from above do not involve the auxiliary plane $\widetilde{\mathscr{M}}$ that we used before. We add, for the sake of completeness, that a basis of $\widetilde{\mathscr{M}}$ is given by

$$\{\mathbf{s}^\times\otimes\mathbf{t}^\times,\ \mathbf{u}^\times\otimes\mathbf{v}^\times\}.\tag{29}$$

This follows from Eq. (14) applied to the basis vectors of $\mathscr{M}$ from (24). Note that Eq. (27) and the analogue of (28) (obtained by interchanging $\mathbf{s}^\times$ and $\mathbf{u}^\times$) establishes that the orthogonality of the decomposable basis vectors of $\widetilde{\mathscr{M}}$ from (29) holds precisely when the decomposable basis vectors of $\mathscr{M}$ from (24) are orthogonal.

All things considered, we see that the four decomposable basis vectors from (26) give rise to six planes. Two of them are $\mathscr{M}^\perp$ and $\widetilde{\mathscr{M}}$. The remaining four planes are of type $(0,0)$: Take, for example, the plane spanned by $\mathbf{s}^\times\otimes\mathbf{t}^\times$ and $\mathbf{s}^\times\otimes\mathbf{v}^\times$. An arbitrary linear combination of these two vectors reads $\xi_1(\mathbf{s}^\times\otimes\mathbf{t}^\times)+\xi_2(\mathbf{s}^\times\otimes\mathbf{v}^\times)=\mathbf{s}^\times\otimes(\xi_1\mathbf{t}^\times+\xi_2\mathbf{v}^\times)$ and therefore is decomposable.

In a geometric language, the four vectors from (26) generate the vertices of a *tetrahedron* with three specific properties in the projective space $\mathbb{P}(\mathscr{H})$. First, the vertices of the tetrahedron are on the ruled quadric Φ, whose points are given by all non-zero decomposable vectors. Second, two edges of the tetrahedron meet Φ at exactly two distinct points. Third, the remaining four edges lie completely on the quadric Φ. One tetrahedron of this kind is depicted in Fig. 2, where we adopted the same affine viewpoint as in Fig. 1.

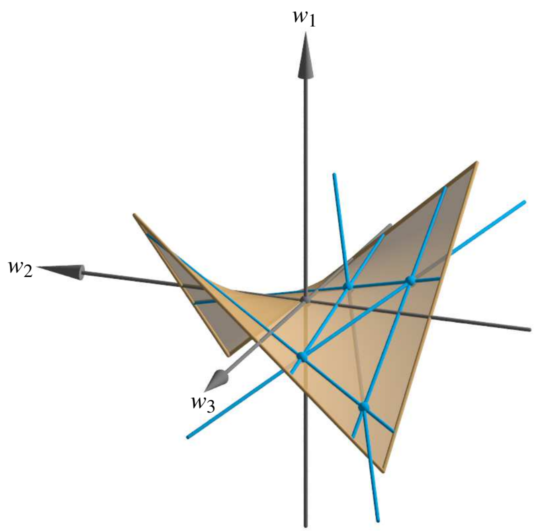

FIG. 2. Tetrahedron arising from the basis (26).

It is a straightforward task to obtain *all* planes of type $(1,-1)$ (real case) and of type $(1,1)$ (complex case) by a reverse approach. Given any two bases $\{\mathbf{s}',\mathbf{u}'\}$ and $\{\mathbf{t}',\mathbf{v}'\}$ of $\mathscr{H}_2$ the analogue of (25) holds. This shows that the plane spanned by $\mathbf{s}'\otimes\mathbf{t}'$ and $\mathbf{u}'\otimes\mathbf{v}'$ has the required type. Furthermore, by an appropriate choice of the initial bases, one can assure that $\mathbf{s}'\otimes\mathbf{t}'$ and $\mathbf{u}'\otimes\mathbf{v}'$ are (non-)orthogonal.

VI. CONSEQUENCES FOR COMPLETION OF CONTEXTS

One of the main results of this categorization exercise is that, as long as the two given vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ do not span a plane of type $(0,0)$ – that is, as long as their Gramian does not vanish such that $G_{\mathbf{e}_1\mathbf{e}_2} = \mathrm{diag}(0,0)$ – the vectors completing the context (four-dimensional orthogonal basis) can always be chosen to be indecomposable, and therefore correspond to entangled states. In the case the Gramian $G_{\mathbf{e}_1\mathbf{e}_2}$ vanishes the entire context consists of decomposable vectors associated with non-entangled states. Moreover, if there exist two orthogonal decomposable vectors spanning a plane it is always possible to “complete” the respective orthogonal basis by adding two orthogonal decomposable vectors spanning the orthogonal subspace.

For the sake of a concrete example consider the faithful orthogonal representation (aka coordinatization) of a hypergraph of the Hardy type, as quoted from the last row of Table I of Ref. [5], as depicted in Fig. 3.

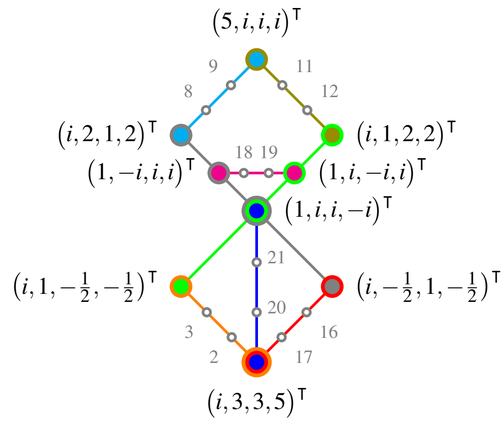

FIG. 3. “Incomplete” faithful orthogonal representation (aka coordinatization) of the orthogonality hypergraph of the Hardy gadget, as quoted from Figure 1 and the last row of Table I of Ref. [5].

It comprises a pasting of two complete (aka orthogonal bases, maximal operators [15, § 84, Theorem 1], Boolean subalgebras or blocks [16], maximal cliques) as well as six incomplete intertwining contexts

$$\begin{aligned}
&\left\{\left(i,1,-\tfrac{1}{2},-\tfrac{1}{2}\right)^\intercal,\left(i,3,3,5\right)^\intercal,2,3\right\},\\
&\left\{\left(5,i,i,i\right)^\intercal,\left(i,2,1,2\right)^\intercal,8,9\right\},\\
&\left\{\left(5,i,i,i\right)^\intercal,\left(i,1,2,2\right)^\intercal,11,12\right\},\\
&\left\{\left(i,-\tfrac{1}{2},1,-\tfrac{1}{2}\right)^\intercal,\left(i,3,3,5\right)^\intercal,16,17\right\},\\
&\left\{\left(1,-i,i,i\right)^\intercal,\left(1,i,-i,i\right)^\intercal,18,19\right\},\\
&\left\{\left(1,i,i,-i\right)^\intercal,\left(i,3,3,5\right)^\intercal,20,21\right\},
\end{aligned} \tag{30}$$

arranged in and 21 atoms or vectors, $2\times 6 = 12$ thereof undefined, namely (partitions indicate same contexts) $\{\{2,3\},\{8,9\},\{11,12\},\{16,17\},\{18,19\},\{20,21\}\}$.

By now it should be clear that all of these undefined vectors can be made to be indecomposable: by a parity argument using their even numbers of imaginary units all of the defined vectors are indecomposable; hence there is no way that these could span a (transformed) type $(0,0)$ plane. But in what plane types exactly are those undefined vectors? All we need to know is the type of the planes spanned by the transformed known vector pairs, which reduces to the task of computing the rank of their Gramian matrices.

For the sake of an explicit computation, take the context defined by $\left\{\left(i,1,-\frac{1}{2},-\frac{1}{2}\right)^\intercal,\left(i,3,3,5\right)^\intercal,2,3\right\}$, and identify $\mathbf{e}_1 = \left(i,1,-\frac{1}{2},-\frac{1}{2}\right)^\intercal$, $\mathbf{e}_2 = \left(i,3,3,5\right)^\intercal$, $\mathbf{a}=2$, $\mathbf{b}=3$, respectively. Then the associated Gramian matrix is $G_{\mathbf{e}_1\mathbf{e}_2} = \frac{1}{4}\begin{pmatrix} 2-2i & -3+9i \\ -3+9i & -36+20i\end{pmatrix}$. The rank of this matrix is two; therefore the type of plane spanned by the vectors $\mathbf{a}=2$ and $\mathbf{b}=3$ is $(1,1)$. Analogous computations show that all planes spanned by the “missing” vectors are of type $(1,1)$.

Intuitively speaking there exist “much less” decomposable vectors than indecomposable ones: from all vectors of four-dimensional space only those satisfying condition (4) qualify. Therefore, the task of finding a faithful orthogonal representation *with only decomposable vectors* of a (hyper)graph turns out to be more difficult than, say, by requiring indecomposability of the vectors. For some configurations and (hyper)graphs it is impossible to find faithful orthogonal representations by decomposable vectors; even if there exist "plenty" of such representations containing also indecomposable vectors.

Consider, for the sake of such an example, a "triangle" subgraph of the hypergraph in Fig. 3. Suppose we wish to "dress" this hypergraph with a coordinatization involving only decomposable vectors. In order to show that this task cannot be accomplished we exhibit a faithful orthogonal representation of the hypergraph depicted in Fig. 4(a). Thereby we merely require *decomposability* of the vectors $\mathbf{b}_2, \mathbf{b}_3, \ldots, \mathbf{b}_6$ while allowing *arbitrary* vectors $\mathbf{b}_1, \mathbf{b}_7, \mathbf{b}_8, \mathbf{b}_9$. Then there exist non-zero vectors $\mathbf{s}_j, \mathbf{t}_j \in \mathscr{H}_2$ such that $\mathbf{b}_j = \mathbf{s}_j \otimes \mathbf{t}_j$ for all $j = 2, 3, \ldots, 6$.

The plane $(\operatorname{span}\{\mathbf{b}_1, \mathbf{b}_7\})^\perp$ contains the two decomposable vectors $\mathbf{b}_5$, $\mathbf{b}_6$ as well as a third decomposable vector $\mathbf{b}_4$, all of which are two-by-to linearly independent. This forces not only $(\operatorname{span}\{\mathbf{b}_1, \mathbf{b}_7\})^\perp$ but also $\operatorname{span}\{\mathbf{b}_1, \mathbf{b}_7\}$ to be of type $(0,0)$ (otherwise there would exist at most two such vectors). Consequently, there are non-zero vectors $\mathbf{s}_k, \mathbf{t}_k \in \mathscr{H}_2$ such that $\mathbf{b}_k = \mathbf{s}_k \otimes \mathbf{t}_k$ for $k = 1, 7$. Using Eq. (10) we arrive at

$$(\mathbf{b}_1 | \mathbf{b}_7) = \det(\mathbf{s}_1, \mathbf{s}_7) \det(\mathbf{t}_1, \mathbf{t}_7) = 0. \tag{31}$$

Furthermore, from Eqs. (9) and (12) we obtain

$$\langle \mathbf{b}_1 | \mathbf{b}_7 \rangle = \langle \mathbf{s}_1 | \mathbf{s}_7 \rangle \langle \mathbf{t}_1 | \mathbf{t}_7 \rangle = \det(\mathbf{s}_1^\times, \mathbf{s}_7) \det(\mathbf{t}_1^\times, \mathbf{t}_7) = 0. \tag{32}$$

Since $\mathbf{s}_1$ and $\mathbf{s}_7$ are non-zero, the determinants $\det(\mathbf{s}_1, \mathbf{s}_7)$ and $\det(\mathbf{s}_1^\times, \mathbf{s}_7)$ cannot vanish simultaneously. Likewise, $\det(\mathbf{t}_1, \mathbf{t}_7)$ and $\det(\mathbf{t}_1^\times, \mathbf{t}_7)$ are not both zero. Consequently, there are two cases: (i) either $\det(\mathbf{s}_1, \mathbf{s}_7) = \det(\mathbf{t}_1^\times, \mathbf{t}_7) = 0$ and, at the same time, $\det(\mathbf{t}_1, \mathbf{t}_7) \neq 0 \neq \det(\mathbf{s}_1^\times, \mathbf{s}_7)$, (ii) or, alternatively, $\det(\mathbf{s}_1^\times, \mathbf{s}_7) = \det(\mathbf{t}_1, \mathbf{t}_7) = 0$ and, at the same time, $\det(\mathbf{s}_1, \mathbf{s}_7) \neq 0 \neq \det(\mathbf{t}_1^\times, \mathbf{t}_7)$. Therefore, either $\det(\mathbf{s}_1, \mathbf{s}_7) = 0 \neq \det(\mathbf{t}_1, \mathbf{t}_7)$, or, alternatively, $\det(\mathbf{s}_1^\times, \mathbf{s}_7) = 0 \neq \det(\mathbf{t}_1^\times, \mathbf{t}_7)$. Hence, up to an irrelevant scaling factor, $\mathbf{b}_7 = \mathbf{s}_7 \otimes \mathbf{t}_7$ equals one of the following vectors:

$$\mathbf{s}_1 \otimes \mathbf{t}_1^\times, \quad \mathbf{s}_1^\times \otimes \mathbf{t}_1. \tag{33}$$

Next, we repeat the previous reasoning in view of $\mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_7 \in (\operatorname{span}\{\mathbf{b}_1, \mathbf{b}_4\})^\perp$. In this way, we regain the decomposability of $\mathbf{b}_1$ and $\mathbf{b}_4$ and arrive at precisely the same vectors from (33). So, one of the vectors from (33) must be proportional to $\mathbf{b}_4$ while the other vector needs to be proportional to $\mathbf{b}_7$. The plane $\operatorname{span}\{\mathbf{b}_4, \mathbf{b}_7\}$ is of type $(1,-1)$ in the real and of type $(1,1)$ in the complex case, since $(\mathbf{s}_1 \otimes \mathbf{t}_1^\times | \mathbf{s}_1^\times \otimes \mathbf{t}_1) = \det(\mathbf{s}_1, \mathbf{s}_1^\times) \det(\mathbf{t}_1^\times, \mathbf{t}_1) \neq 0$. We are therefore in a position to substitute $\mathbf{s}$ by $\mathbf{s}_1$, $\mathbf{t}$ by $\mathbf{t}_1^\times$, $\mathbf{u}$ by $\mathbf{s}_1^\times$ and $\mathbf{v}$ by $\mathbf{t}_1$ in (24), so that the vectors $\mathbf{b}$, $\mathbf{a}$ appearing in (26) turn into

$$\mathbf{s}_1^\times \otimes \mathbf{t}_1^\times, \quad (-\mathbf{s}_1) \otimes (-\mathbf{t}_1) = \mathbf{s}_1 \otimes \mathbf{t}_1 = \mathbf{b}_1. \tag{34}$$

The decomposable vectors from (34) constitute a basis of the plane $\operatorname{span}\{\mathbf{b}_8, \mathbf{b}_9\}$. This plane, like its orthogonal plane $\operatorname{span}\{\mathbf{b}_4, \mathbf{b}_7\}$, is of type $(1,-1)$ in the real and of type $(1,1)$ in the complex case. Thus, up to scaling factors, the vectors appearing in (34) are the only decomposable vectors of $\operatorname{span}\{\mathbf{b}_8, \mathbf{b}_9\}$. Also, we established in Section V that the orthogonality of $\mathbf{b}_4$ and $\mathbf{b}_7$ forces the vectors from (34) to be orthogonal. Now, since our orthogonal representation is faithful, it turns out that $\mathbf{b}_8$ is not proportional to $\mathbf{s}_1 \otimes \mathbf{t}_1 = \mathbf{b}_1$, which in turn establishes that $\mathbf{b}_9$ is not a multiple of $\mathbf{s}_1^\times \otimes \mathbf{t}_1^\times$. The previous statement remains true when interchanging $\mathbf{b}_8$ and $\mathbf{b}_9$. Our final conclusion therefore is that $\mathbf{b}_8$ *and* $\mathbf{b}_9$ *have to be indecomposable*.

Fig. 4(b) displays an explicit example of a non-faithful orthogonal representation of a "triangle" in terms of decomposable vectors with just one multiplicity. Thereby, it has to be assumed that $\{\mathbf{s}, \mathbf{u}\}$, $\{\mathbf{s}^\times, \mathbf{u}\}$, $\{\mathbf{t}, \mathbf{v}\}$ and $\{\mathbf{t}^\times, \mathbf{v}\}$ are bases of $\mathscr{H}_2$ in order to avoid any further multiplicities. Of course the plane $\operatorname{span}\{\mathbf{s}^\times \otimes \mathbf{t}^\times, \mathbf{s} \otimes \mathbf{t}\}$ admits a continuum of orthogonal bases containing only indecomposable vectors. Replacement of the given basis $\{\mathbf{s}^\times \otimes \mathbf{t}^\times, \mathbf{s} \otimes \mathbf{t}\}$ with any such basis yields a faithful orthogonal representation.

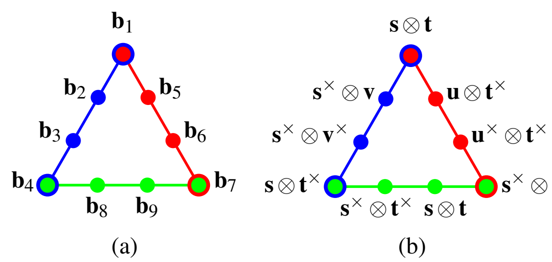

FIG. 4. (a) Subgraph of the triangle hypergraph depicted in Fig. 3 with a faithful orthogonal representation by decomposable vectors $\mathbf{b}_2, \mathbf{b}_2, \ldots, \mathbf{b}_6$ and arbitrary vectors $\mathbf{b}_1, \mathbf{b}_7, \mathbf{b}_8, \mathbf{b}_9$. Even though it turns out that $\mathbf{b}_1$ and $\mathbf{b}_7$ must be decomposable, the remaining vectors $\mathbf{b}_8$ and $\mathbf{b}_9$ have to be indecomposable. (b) An explicit example of a non-faithful orthogonal representation of the triangle hypergraph with only decomposable vectors resulting in the multiple occurrence of $\mathbf{s} \otimes \mathbf{t}$.

VII. STEERING (IN)DECOMPOSABILITY

If the physical means are restricted to real spaces the existence of "plain" planes which contain only non-zero vectors of either one of the two categories – factorizable (aka decomposable) and indecomposable – and the associated orthogonal planes which are of the same types allows a sort of "steering" into such "plain" planes. In this way one party controlling the source as well as the (two elementary) observables spanning the "original plane" can, in a directed manner, signal factorizable or entangled states towards a second party at the receiving end.

For the sake of an example take two factorizable vectors spanning a type $(0,0)$ plane, and the associated orthogonal plane which is also of type $(0,0)$, containing only factorizable vectors. To be more explicit consider a four-port generalized beam splitter [17] associated with the output states corresponding to the vectors $\left(1,0,0,0\right)^\intercal$, $\left(0,1,0,0\right)^\intercal$, $\left(0,0,1,0\right)^\intercal$, and $\left(0,0,0,1\right)^\intercal$, respectively. Suppose the first party called Alice controls the source and the first two ports associated with $\left(1,0,0,0\right)^\intercal$ and $\left(0,1,0,0\right)^\intercal$, and the second party called Bob controls the last two ports associated with $\left(0,0,1,0\right)^\intercal$ and $\left(0,0,0,1\right)^\intercal$. If Alice makes sure that she is sending and receiving no other states then she can be sure that Bob, no matter what he does on "his side" of the output ports, will end up with a factorizable state.

If, on the other hand, only plane types $(1,1)$ are involved – say one plane spanned by $(1/\sqrt{2})\left(1,0,0,1\right)^\intercal$ and $(1/\sqrt{2})\left(0,1,1,0\right)^\intercal$ on Alice's state emission and her beam splitter ports, and $(1/\sqrt{2})\left(1,0,0,-1\right)^\intercal$ and $(1/\sqrt{2})\left(0,1,-1,0\right)^\intercal$ on Bob's ports – Alice can be sure that Bob, no matter what he does on "his side" of the output ports, will end up with an entangled state.

Note that is suffices for Alice to generate the respective states and observe her shares of the ports. In that way one can imagine a type of BB84 [18] protocol which, instead of random shared sequences of bits, render shared factorizable and entangled states.

We close this investigation into the (in)decomposability of vectors in planes (aka two-dimensional subspaces) of four-dimensional Hilbert spaces by noting that their structure exhibits a richness which might not be obvious at first glance. There exist planes consisting of purely decomposable vectors. Nevertheless, in general indecomposability and thus physical entanglement and the encoding of relational properties by quantum states "prevails" and occurs more often than separability associated with well defined individual, separable states.

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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