COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
63. INTUITIVE GEOMETRY, SZEGED (HUNGARY), 1991

A Characterization of 3-dimensional Convex Sets with an Infinite X-ray Number

K. BEZDEK[1] and T. ZAMFIRESCU

1. Introduction and result

The X-raying and illumination of convex sets were intensively studied during the last decade. Our references at the end of the paper constitute just a very incomplete selection. The problem considered here belongs to this area.

Let $\mathcal{K}^d$ be the space of all closed convex sets with non-empty interior in the d-dimensional Euclidean space $\mathbf{E}^d$, where $d \geq 2$. Let $L \subset \mathbf{E}^d$ be a line through the origin. We say that the point p of $K \in \mathcal{K}^d$ is *X-rayed along* L if the line parallel to L passing through p intersects the interior of K. The *X-ray number of* K is the smallest number of lines such that every point of K is X-rayed along at least one of these lines (see also P. S. Soltan and V. P. Soltan's paper [7]).

Let $\operatorname{int} M$, $\operatorname{bd} M$, $\operatorname{rbd} M$, $\operatorname{vert} M$, $\operatorname{conv} M$, $\overline{M}$ denote the interior, boundary, relative boundary, vertex set, convex hull, closure of M, respectively.

It is easy to see that in $\mathbf{E}^2$ every closed convex set with a non-empty interior possesses an X-ray number not larger than 2. However, in $\mathbf{E}^d$ with $d \geq 3$, there are sets with infinite X-ray numbers. An example can be

[1] The work was supported by Hung. Nat. Found. for Sci. Research No. 326-0413.

obtained in the following way. Take a d-dimensional closed convex cone $C \subset \mathbf{E}^d$ with balls as $(d-1)$-dimensional sections orthogonal to the axis. Then let H be a hyperplane that meets $\operatorname{int} C$ and is parallel to a ray in $\operatorname{bd} C$. Denoting by H^+ the closed halfspace bounded by H which contains the apex of C, it is not hard to see that the X-ray number of $C \cap H^+$ is infinite: In order to X-ray the points of $H \cap \operatorname{bd} C$ we need infinitely many directions. Thus, it is natural to raise the following problem.

Problem 1. *Find a characterization of those members of $\mathcal{K}^d$ which have a finite X-ray number ($d \geq 3$).*

We will solve here this problem in the case $d = 3$.

If H_1, H_2 are hyperplanes, A_1, A_2 affine subspaces of $\mathbf{E}^d$, and K an element of $\mathcal{K}^d$ verifying $(H_1 \cup H_2) \cap \operatorname{int} K = \emptyset$, then $\angle(H_1, K, H_2)$ means the measure of the dihedral angle between H_1 and H_2 containing $\operatorname{int} K$, and $\angle(A_1, A_2)$ means the smallest measure of an angle between a line in A_1 and a line in A_2.

We say that $K \in \mathcal{K}^d$ is *asymptotically singular* if there are a line L and two sequences of not necessarily distinct supporting hyperplanes $\{H_n\}_{n=1}^{\infty}$ and $\{H'_n\}_{n=1}^{\infty}$ such that

(i) H_n is parallel to L for each n,

(ii) $H_n \cap H'_n \cap K \neq \emptyset$ for each n,

(iii) $\angle(H_n, K, H'_n) \to 0$,

(iv) $\angle(L, H_n \cap H'_n) \to 0$.

There are essentially two different cases: If the hyperplanes H_n are all identical we say that K has the *singular face* $H_n \cap K$. If the hyperplanes H_n are all distinct we say that K has the *singular sequence of faces* $\{H_n \cap H'_n \cap K\}_{n=1}^{\infty}$.

Our main result is the following.

Theorem. *The X-ray number of $K \in \mathcal{K}^3$ is infinite if and only if K is asymptotically singular.*

Obviously, if $K \in \mathcal{K}^d$ is compact, i.e. a convex body, then the X-ray number of K is at least d. This bound is sharp because it is attained by any smooth convex body. Moreover, as the X-ray number of K is at most as large as its illumination number, $(d+1)^d$ is an upper bound, not sharp though (see [5]). The following set has a quite large X-ray number: Let

$Q \subset \mathbf{E}^d$ be a d-dimensional cube and F a $(d-2)$-dimensional face of Q. Then the X-ray number of $\mathrm{conv}(\mathrm{vert}\, Q - \mathrm{vert}\, F)$ is $3 \cdot 2^{d-2}$.

Problem 2. *Is the X-ray number of any convex body in $\mathbf{E}^d$ at most $3 \cdot 2^{d-2}$?*

2. Proofs

Let F be a face of $K \in \mathcal{K}^d$, where $d \geq 3$. The *spherical image* $\nu(F)$ of the face F is the set of all points x in the $(d-1)$-dimensional unit sphere $S^{d-1} \subset \mathbf{E}^d$ centered at the origin $\mathbf{0}$ of $\mathbf{E}^d$ such that the supporting hyperplane of K with outer normal x contains F. It is easy to see that $\nu(F)$ is compact and spherically convex. Moreover, the spherical images of distinct faces of K have disjoint relative interiors.

The following simple lemma, the proof of which we leave to the reader, will be tacitly used throughout the rest of the paper.

Lemma 1. *Let $d \geq 3$, $K \in \mathcal{K}^d$, $p \in \mathrm{bd}\, K$ and F be a face of K of smallest dimension which contains p. Then p is X-rayed along the line $L \ni \mathbf{0}$ if and only if $L^{\perp} \cap \nu(F) = \emptyset$. Therefore, the X-ray number of K is the smallest number of $(d-2)$-dimensional great spheres in S^{d-1} such that the spherical image of every face of K is disjoint from at least one of those great spheres.*

From now on let $K \in \mathcal{K}^3$ be unbounded. Then $\nu(K)$ lies in a closed hemisphere $S^2_+ \subset S^2$, with a great circle S^1 as boundary. Put

$$T = S^1 \cap \nu(K).$$

Lemma 2. *If for the point $p \in T$ there is no sequence of faces $\{F_n\}_{n=1}^{\infty}$ such that*

$$\nu(F_n) \cap S^1 \neq \emptyset$$

for each n and the distance from both p and $-p$ to $\nu(F_n)$ tends to zero, then there is a neighbourhood D_p of p and four great circles in S^2 such that for every face F of K, for which $\nu(F)$ meets $D_p \cap S^1$, one of the circles is disjoint from $\nu(F)$.

Proof. For any open disk D_p around p in S^2, let $A_p = D_p \cap S^1$. Under the hypotheses of the lemma, there is an open disk D_p around p such that no $\nu(F)$ meets both A_p and $-D_p$. Let $x_1, x_2 \in S^1$ be the endpoints of the arc $\overline{A_p}$. Choose the great circle S_i passing through x_i, $-x_i$ $(i = 1, 2)$ such that $S_1 \cap S_2 \cap S^2_+ \subset -D_p$. At most one 2-dimensional spherical image $\nu(F_1)$ simultaneously meets A_p, S_1 and S_2. No 1-dimensional spherical image $\nu(F)$ simultaneously meets A_p, S_1 and S_2 except for the case that $\nu(F) \subset S^1$, and we may have at most one such spherical image $\nu(F_2)$. Thus, the circles S_1, S_2, a great circle disjoint from $\nu(F_1)$ and another one disjoint from $\nu(F_2)$ satisfy the requirements of the lemma.

Proof of the Theorem. We prove only the non-straightforward implication. Assume that the X-ray number of $K \in \mathcal{K}^3$ is infinite and prove that K is asymptotically singular. There are two possible cases.

Case I: For each $p \in T$, there is no sequence $\{\nu(F_n)\}_{n=1}^{\infty}$ such that $\nu(F_n) \cap S^1 \neq \emptyset$ for each n and the distance from both p and $-p$ to $\nu(F_n)$ tends to zero.

In this case, by Lemma 2, for each $p \in T$ there are 4 great circles and an open disk D_p around p such that every $\nu(F)$ meeting $D_p \cap S^1$ is disjoint from at least one of the circles. Clearly $\{D_p\}_{p \in T}$ covers T and we select a finite subcovering $\{D_p\}_{n \leq n_0}$. Thus $4n_0$ great circles and S^1 take care of the spherical images of all faces of K, but this contradicts the assumption.

Case II: For some $p \in T$, there is a sequence $\{\nu(F_n)\}_{n=1}^{\infty}$ such that $\nu(F_n) \cap S^1 \neq \emptyset$ for each n and the distance from both p and $-p$ to $\nu(F_n)$ tends to zero.

This can happen for at most one pair of points $(p, -p)$. Indeed, if this happened for another pair of points $(q, -q)$ too then, necessarily, for some large m and n, the relative interior of $\nu(F_m)$ (to which p and $-p$ are close) would meet the relative interior of $\nu(F'_n)$ (to which q and $-q$ are close), a contradiction.

Suppose that $p \in \nu(F_n)$ for infinitely many n. Then a subsequence of $\{\nu(F_n)\}_{n=1}^{\infty}$ converges to a halfcircle S^* with endpoints p, $-p$. This implies that K has the singular face with outernormal unit vector p (for L orthogonal to the plane of S^*). The same happens if $-p \in \nu(F_n)$ for infinitely many n. So we may suppose $p, -p \notin \nu(F_n)$ for all n (otherwise take a subsequence). Let

$$p_n \in \nu(F_n) \cap S^1$$

and suppose without loss of generality that $p_n \to p$. Then there are points $q_n \in \nu(F_n)$ such that $q_n \to -p$.

We need the following statement. If a sequence of spherical images converges to a halfcircle, then the spherical image of the whole set is included in a closed hemisphere having that halfcircle on its boundary. In order to see this take an arbitrary interior point, say O, of the given set $K \in \mathcal{K}^3$ and let $C \subset \mathbf{E}^3$ be the union of (closed) halflines emanating, from O and lying in K. Obviously, C is a closed convex cone with apex O. Let $L \subset \mathbf{E}^3$ be the line passing through O and being orthogonal to the plane of the halfcircle that is the limit of the sequence $\{\nu(F'_n)\}_{n=1}^{\infty}$ of the spherical images of the faces F'_n of K. Obviously, K has to be unbounded i.e. K as well as C contain at least one halfline and so the spherical image of K is included in a closed hemisphere. Moreover, the dimension of C is at most 2, otherwise the spherical image of K would have a spherical diameter $< \pi$, a contradiction. Let $p'_n \in F'_n$ for $n = 1, 2, \ldots$ As the sequence $\{p'_n\}_{n=1}^{\infty}$ cannot be bounded in $\mathbf{E}^3$ there exists a subsequence, say $\{p'_n\}_{n=1}^{\infty}$ itself, such that the (closed) halflines emanating from O and passing through the points p'_n tend to a halfline say, ℓ emanating from O and lying in C. Finally, let the supporting hyperplanes H_n, H'_n of K be chosen such that they pass through the point p'_n with $\angle(H_n, K, H'_n) \to 0$. As C has to be contained in the dihedral angle between H_n and H'_n containing $\operatorname{int} K$, we get that ℓ is an extreme halfline of C. Finally, observe that $\ell \subset L$. This completes the proof of the statement.

Now, some subsequence of $\{\nu(F_n)\}_{n=1}^{\infty}$ converges to a halfcircle S'. If $S' \subset S^1$ then the subsequence is a singular sequence of faces (for properly choosen H_n and H'_n with $F_n = H_n \cap H'_n \cap K$ and for L orthogonal to the plane of S^1). If $S' \not\subset S^1$ then there is a closed hemisphere S^+ with $\operatorname{rbd} S^+ = S'$ containing, in fact all spherical images of faces of K. Also, in the present case $p, -p \notin \nu(K)$. Therefore a great circle C with

$$C \cap S^2_+ \cap S^+ = \{p, -p\}$$

misses all spherical images of faces of K. But this again contradicts the infinity of the X-ray number of K.

The proof is finished.

References

[1] K. Bezdek, The problem of illumination of the boundary of a convex body by affine subspaces, *Mathematika* **38** (1991), 362–375.

[2] K. Bezdek, Hadwiger-Levi's covering problem revisited, in: New Trends in Discrete and Computational Geometry (Editor: J. Pach), Springer Verlag (1993), 199–233.

[3] V. G. Boltyanskii and I. Z. Gohberg, Results and problems in combinatorial geometry, Cambridge University Press, Cambridge (1985).

[4] R. J. Gardner and P. McMullen, On Hammer's X-ray problem, *J. London Math. Soc.* **21** (1980), 171–175.

[5] M. Lassak, Covering the boundary of a convex set by tiles, *Proc. Amer. Math. Soc.* **104** (1988), 269–272.

[6] J. O'Rourke, Art Gallery Theorems and Algorithms, Oxford University Press, New York – Oxford (1987).

[7] P. S. Soltan and V. P. Soltan, On the X-raying of convex bodies, *Soviet Math. Dokl.* **33** (1986), 42–44.

[8] A. Volčič, A three point solution to Hammer's X-ray problem, *J. London Math. Soc.* **34** (1986), 340–359.

[9] A. Volčič and T. Zamfirescu, Ghosts are scarce, *J. London Math. Soc.* **40** (1989), 171–178.

Károly Bezdek
Department of Geometry,
Eötvös Loránd University,
1088 Budapest, Rákóczi út 5,
Hungary.

Tudor Zamfirescu
Department of Mathematics,
University of Dortmund,
Postfach 500500
D-4600 Dortmund
Germany.