GENERALIZATION OF THE RÖDSETH–GUPTA THEOREM ON BINARY PARTITIONS

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Abstract. In this article, we give a new proof of the Rödseth–Gupta theorem on binary partitions and give one possible generalization of this theorem.

Keywords: Rödseth–Gupta theorem, binary partitions, Thue–Morse sequence, Rudin–Shapiro sequence.

1. INTRODUCTION

In this work, we consider partitions of the natural number n into the powers of two. The partitions which differ in the order of summands are considered to be identical. Let the number of partitions, i.e., the number of solutions of the equation

$$n = m_s 2^s + m_{s-1} 2^{s-1} + \cdots + m_0 \tag{1}$$

in nonnegative integers be $b(n)$. This function was investigated by L. Euler (1750), A. Tanturi (1918), and K. Mahler (1940) among others. Let, for simplicity, $b(0) = 1$. The generating power series of the sequence $b(n)$ is

$$F(x) = \sum_{i=0}^{\infty} b(n)x^n = \prod_{i=0}^{\infty} (1 - x^{2^i})^{-1}.$$

This function satisfies the functional equation $(1 - x)F(x) = F(x^2)$ and, by comparing coefficients at the same powers of x, we get

$$b(2n + 1) = b(2n), \quad b(2n) = b(2n - 1) + b(n). \tag{2}$$

This has an obvious combinatorial meaning: the number of the partitions of the number $2n$ with $m_0 \neq 0$ is $b(2n - 1)$, the summands of the rest partitions are divisible by 2, hence the number of such partitions is $b(n)$. On the other hand, every partition of the number $2n + 1$ has at least one summand equal to 1, and this gives the first formula of (2). I was not aware of the previous works when in 1997 I proposed for the International Mathematical Olympiad for the school pupils the following problem. Prove that $2^{n^2/4} < b(2^n) < 2^{n^2/2}$ for $n \geqslant 3$. This statement can be proved using the recurrence relations (2). It can be deduced from here that $b(2h) \leqslant hb(h)$ and $b(4h) > 2hb(h)$ for $h \geqslant 2$. K. Mahler investigated the difference equation $(f(z + \omega) - f(z))\omega^{-1} = f(qz)$ and derived the following asymptotic formula for $b(n)$:

$$b(2n) = \mathrm{e}^{\mathrm{O}(1)} \sum_{v=0}^{\infty} \frac{2^{-v(v-1)} n^v}{v!}, \quad n \to \infty.$$

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This gives the exact asymptotic for $\ln b(n)$. N. de Bruin (his results and possible generalizations are given in [6]) sharpened the Mahler results by using the Tauber type theorems. He extracted from the remainder term O(1) some periodical function, thus getting an o(1) term. The first asymptotic term of the logarithm

$$\ln b(n) \sim \frac{(\ln n)^2}{2\ln 2} \tag{3}$$

can be easily calculated using the geometrical method [7]. This result was elementary (combinatorially) proved by G. Eckstein [8]. In 1969, R. F. Churchhouse [1] noted some arithmetical properties of the sequence $b(n)$. He proved some theorems and made a conjecture which now is known as the Rödseth–Gupta theorem.

THEOREM. *For an odd n the following congruence holds*:

$$b(2^{s+2}n) - b(2^s n) \equiv 2^{\mu(s)} (\operatorname{mod} 2^{\mu(s)+1}), \quad \mu(s) = \left[\frac{3s+4}{2}\right]. \tag{4}$$

This fact was independently proved by Ö. Rödseth [2] and H. Gupta [3]. The third proof was given in G. Andrews' book [4]. In our work, we shall give a new proof of the Rödseth–Gupta theorem, which is different from the rest above mentioned ones and is more close to the original proof of Gupta but technically easier. This approach can be generalized, which allows us to write other congruences of type (4). In particular, we prove that for an odd n and $s \geqslant 2$ the expression $b(2^{s+4}n) + 7b(2^{s+2}n) - 8b(2^s n)$ is divisible by exactly $[\frac{3s}{2}+8]$th power of 2 (note that the theorem of Rödseth–Gupta implies the division of this expression only by $[\frac{3s}{2}+6]$th power of 2 and does not say anything about the precision of the division). We end with some other properties of the sequence $b(n)$.

2. THE RÖDSETH–GUPTA THEOREM

Define by $b_s(n)$ the number of the partitions of the number n into the powers of 2 not exceeding s. The generating power series is

$$H_s(n) = \sum_{n=0}^{\infty} b_s(n)x^n = \prod_{i=0}^{s} (1 - x^{2^i})^{-1}.$$

It is clear that $(1-x)H_{s+1}(x) = H_s(x^2)$ and by comparing the coefficients we get $b_{s+1}(2n) - b_{s+1}(2n-1) = b_s(n)$, $b_{s+1}(2n+1) = b_{s+1}(2n)$, i.e., $b_{s+1}(2n) = \sum_{i=0}^{n} b_s(i)$. This allows us to find the explicit expression of $b_s(n)$:

$$b_0(n) = 1, \qquad b_1(n) = \left[\frac{n}{2}\right] + 1, \qquad b_2(n) = \frac{1}{4}\left[\frac{n}{2}\right]^2 + \left[\frac{n}{2}\right] + \frac{4 - \varepsilon_1(n)}{4},$$

where $\varepsilon_1(n)$ is the digit at the power 2^1 of the binary expansion of the natural number n. Define a function $f_s(n) = b_s(2^s n)$. The number of the partitions of $2^{s+1}n$ into the powers of 2 with exponent not exceeding $s+1$, where 2^{s+1} appears as a summand exactly t times, is equal to $b_s(2^{s+1}(n-t))$; hence

$$f_{s+1}(n) = b_{s+1}(2^{s+1}n) = \sum_{t=0}^{n} b_s(2^{s+1}(n-t)) = f_s(0) + f_s(2) + \cdots + f_s(2n). \tag{5}$$

Equality (5) is also a conclusion of the formula $H_{s+1}(x) = (\sum_{i=0}^{\infty} x^i) H_s(x^2)$. Thus, we can find the expression of $f_s(n)$ recurrently. First, we have $f_0(n) = b_0(n) = 1$. Hence,

$$
\begin{aligned}
f_0(n) &= 1\\
f_1(n) &= n+1\\
f_2(n) &= (n+1)^2\\
f_3(n) &= \frac{4}{3}(n+1)^3 - \frac{1}{3}(n+1)\\
f_4(n) &= \frac{8}{3}(n+1)^4 - \frac{5}{3}(n+1)^2\\
f_5(n) &= \frac{2^7}{15}(n+1)^5 - \frac{28}{3}(n+1)^3 + \frac{9}{5}(n+1).
\end{aligned}
\tag{6}
$$

We see that $f_s(n)$ is a linear combination of $(n+1)^s$, $(n+1)^{s-2}, \ldots$. Let $D_s = \sum_{k=0}^{n}(2k+1)^s$. We will proceed by induction. Our assumption (i.e., that $f_k(n)$ is a distinct linear combination of $(n+1)^k, (n+1)^{k-2}, \ldots$) is true for $k \leqslant 5$. Suppose that it is true for T. We will prove that it is true for $T+1$. By assumption, $f_T(n) = \sum_{l, 0 \leqslant 2l \leqslant T} a_{T,l}(n+1)^{T-2l}$. Then (5) yields that $f_{T+1}(n) = \sum_{l, 0 \leqslant 2l \leqslant T} a_{T,l} D_{T-2l}$. Thus, it is enough to prove that D_s is a linear combination of $(n+1)^{s+1}, (n+1)^{s-1}, \ldots$, for all natural s. This is a sequel of the two following lemmas.

LEMMA 1. *The natural numbers D_s satisfy the identity*

$$
\sum_{0 \leqslant j \leqslant [\frac{s}{2}]} D_{s-2j} \cdot \binom{s+1}{2j+1} = 2^s (n+1)^{s+1}.
$$

Proof. We have

$$
\begin{aligned}
(k+1)^{s+1} - k^{s+1} &= \left(\left(k+\frac{1}{2}\right)+\frac{1}{2}\right)^{s+1} - \left(\left(k+\frac{1}{2}\right)-\frac{1}{2}\right)^{s+1}\\
&= \sum_{0 \leqslant j \leqslant [\frac{s}{2}]} 2\left(k+\frac{1}{2}\right)^{s-2j}\left(\frac{1}{2}\right)^{2j+1}\binom{s+1}{2j+1}.
\end{aligned}
$$

Hence, $2^s((k+1)^{s+1} - k^{s+1}) = \sum_{0 \leqslant j \leqslant [\frac{s}{2}]}(2k+1)^{s-2j}\binom{s+1}{2j+1}$ and summing over k from 0 to n we obtain the equality of the lemma.

Now using this lemma we can express D_s as a linear combination of the powers of $(n+1)$.

LEMMA 2. *There exist constants c_i for which*

$$
D_s = \sum_{0 \leqslant i \leqslant [\frac{s}{2}]} c_i \frac{s!}{(s-2i+1)!} 2^{s-2i} (n+1)^{s-2i+1}. \tag{7}
$$

Furthermore, $c_i = \frac{r_i}{(2i+1)!}$ where r_i is a rational number with odd numerator and odd denominator.

Proof. Let us take in Lemma 1 $s-2i$ instead of s, multiply both sides by $c_i\frac{(s+1)!}{(s-2i+1)!}$ and sum over i. We obtain

$$\begin{aligned}
&\sum_{0\leqslant i+j\leqslant[\frac{s}{2}]} D_{s-2i-2j}\cdot c_i\cdot\frac{(s+1)!}{(2j+1)!(s-2i-2j)!}\\
&\quad=\sum_{0\leqslant n\leqslant[\frac{s}{2}]} D_{s-2n}\frac{(s+1)!}{(s-2n)!}\cdot\sum_{i+j=n}\frac{c_i}{(2j+1)!}\\
&\quad=\sum_{0\leqslant i\leqslant[\frac{s}{2}]} c_i\frac{(s+1)!}{(s-2i+1)!}2^{s-2i}(n+1)^{s-2i+1}.
\end{aligned}\tag{8}$$

Hence, we can define $c_0=1$ and then recurrently

$$\sum_{i=0}^{n}\frac{c_i}{(2n-2i+1)!}=0,\quad n\geqslant 1.\tag{9}$$

Dividing Eq. (8) by $s+1$, we get the first statement of the lemma. Thus, $c_0=1$, $c_1=-\frac{1}{3!}$, $c_2=\frac{7}{3\cdot 5!}$, $c_3=-\frac{31}{3\cdot 7!}$. The second statement of the lemma is correct for $i=0$. Suppose that it is correct for $i\leqslant n-1$, $n\geqslant 1$, i.e., $c_i=\frac{r_i}{(2i+1)!}$. Hence, Eq. (9) multiplied by $(2n+1)!$ gives

$$\sum_{i=0}^{n-1}\frac{r_i}{2n-2i+1}\binom{2n+1}{2i+1}=-(2n+1)!c_n.$$

The sum on the left-hand side is a rational number and it has an odd denominator, the parity of numerator coincides with the parity of the number $\sum_{i=0}^{n-1}\binom{2n+1}{2i+1}=2^{2n}-1$ (since r_i by assumption are rational numbers with odd numerators and odd denominators), and this number is odd. The lemma is proved.

Sum (7) can be written as

$$D_s=\sum_{0\leqslant i\leqslant[\frac{s}{2}]}\frac{r_i}{2i+1}\binom{s}{2i}\frac{1}{s-2i+1}2^{s-2i}(n+1)^{s-2i+1}.$$

Define by $\pi(t)$ the power of 2 in the canonical expression of the rational number t. It is easy to check that for the integer $t\geqslant 4$ we have the inequality $t-\pi(t+1)\geqslant 4$ and, therefore, all the coefficients of sum (7) at the powers of $n+1$ not less than 5

$$\text{are divisible by } 2^4.\tag{10}$$

Let for convenience write down the two last terms of (7):

$$\begin{aligned}
D_{2s}&=\cdots+\frac{4}{3}r_{s-1}\cdot s\cdot(n+1)^3+\frac{r_s}{2s+1}(n+1),\\
D_{2s+1}&=\cdots+\frac{2}{3}r_{s-1}(2s+1)\cdot s\cdot(n+1)^4+r_s(n+1)^2.
\end{aligned}\tag{11}$$

Note that

$$f_3(n) - f_1(n) = \frac{4}{3}(n+1)^3 - \frac{4}{3}(n+1)$$

$$f_4(n) - f_2(n) = \frac{8}{3}(n+1)^4 - \frac{8}{3}(n+1)^2$$

$$f_5(n) - f_3(n) = \frac{2^7}{15}(n+1)^5 - \frac{32}{3}(n+1)^3 + \frac{32}{15}(n+1).$$

This gives a hint to formulate the following lemma.

LEMMA 3. *Let, for* $k \geqslant 1$,

$$f_{2k+2} - f_{2k} = a_{2k+2}^{(k)}(n+1)^{2k+2} + \cdots + a_4^{(k)}(n+1)^4 + a_2^{(k)}(n+1)^2,$$

$$f_{2k+1} - f_{2k-1} = b_{2k+1}^{(k)}(n+1)^{2k+1} + \cdots + b_3^{(k)}(n+1)^3 + b_1^{(k)}(n+1).$$

Then

$$\begin{aligned} &\pi(a_2^{(k)}) = \pi(a_4^{(k)}) = 3k, \qquad \pi(a_{2i}^{(k)}) \geqslant 3k+5, \\ &\pi(b_1^{(k)}) = \pi(b_3^{(k)}) = 3k-1, \qquad \pi(b_{2i-1}^{(k)}) \geqslant 3k+1, \quad \text{if } i > 2. \end{aligned} \tag{12}$$

Proof. Suppose that the second formula of (12) is correct. Then $f_{2k+2}(n) - f_{2k}(n) = b_{2k+1}^{(k)}D_{2k+1} + \cdots + b_3^{(k)}D_3 + b_1^{(k)}D_1$. Let us express D via (7). Then every power of $n+1$ not less than 5 will appear with the coefficients for which $\pi \geqslant 3k+1+4 = 3k+5$ (we use ((10) and (12)). Further, (11) gives $a_2^{(k)} = b_1^{(k)}r_0 + b_3^{(k)}r_1 + \cdots + b_{2k+1}^{(k)}r_k$. For all the summands, except for the first two, we have $\pi \geqslant 3k+1$. Further, $f_{2k+1}(0) - f_{2k-1}(0) = b_{2k+1}^{(k)} + \cdots + b_3^{(k)} + b_1^{(k)} = 0$, hence, $\pi(b_3^{(k)} + b_1^{(k)}) \geqslant 3k+1$ and, therefore, $\pi(b_1^{(k)} - b_3^{(k)}) = \pi((b_1^{(k)} + b_3^{(k)}) - 2b_3^{(k)}) = 3k$. Since $r_0 = 1$, $r_1 = -1$, we have $\pi(a_2^{(k)}) = 3k$. Similarly, (11) gives

$$a_4^{(k)} = b_3^{(k)}\frac{2}{3}\cdot 3\cdot r_0 + \cdots + b_{2k+1}^{(k)}\frac{2}{3}(2k+1)kr_{k-1}.$$

For all the summands except the first, $\pi \geqslant 3k+2$, which gives $\pi(a_4^{(k)}) = 3k$. Suppose that the first formula of (12) is correct. Then

$$f_{2k+3}(n) - f_{2k+1}(n) = a_{2k+2}^{(k)}D_{2k+2} + \cdots + a_4^{(k)}D_4 + a_2^{(k)}D_2.$$

Every power of $n+1$ not less than 7 will appear with the coefficients for which $\pi \geqslant 3k+5+4 > 3k+4$ (we use (12) and (10) again). The fifth power will appear with similar coefficients except the one which is obtained from $a_4^{(k)}D_4$ (and for this coefficient we have $\pi = 3k+4$). Further, (11) gives

$$b_1^{(k+1)} = a_2^{(k)}\frac{r_1}{3} + a_4^{(k)}\frac{r_2}{5} + \cdots + a_{2k+2}^{(k)}\frac{r_{k+1}}{2k+3}.$$

For all the summands, except the first two, $\pi \geqslant 3k+5$. The first two are $-\frac{1}{3}a_2^{(k)} + \frac{7}{15}a_4^{(k)} = -\frac{1}{3}(a_2^{(k)} + a_4^{(k)}) + \frac{4}{5}a_4^{(k)}$, and since similarly $\pi(a_2^{(k)} + a_4^{(k)}) \geqslant 3k+5$, we have $\pi(b_1^{(k+1)}) = 3k+2$. Equality (11) gives

$$b_3^{(k)} = \frac{4}{3}r_0a_2^{(k)} + \frac{4}{3}r_1\cdot 2a_4^{(k)} + \cdots + \frac{4}{3}r_k(k+1)a_{2k+2}^{(k)},$$

and, therefore, $\pi(b_3^{(k+1)}) = 3k+2$. The lemma is proved.

The proof of the Rödseth–Gupta theorem. We derive from Eq. (12) that for any odd number n $2^{3k+2} | (f_{2k+2}(n) - f_{2k}(n))$ and $2^{3k} | (f_{2k+1}(n) - f_{2k-1}(n))$, therefore $2^{\mu(k)} | f_{k+2}(n) - f_k(n)$ (see (4)) and the division is precise if and only if

$$n \equiv 1 (\operatorname{mod} 4). \tag{13}$$

Note that every partition of the number $2^s n$ can be expressed as $2^s n = \sum_{a \leqslant s} 2^a + \sum_{b>s} 2^b = \sum_{a \leqslant s} 2^a + 2^{s+1} t$. It is clear that the second sum can be written in $b(t)$ ways. Consequently,

$$b(2^s n) = \sum_{0 \leqslant t \leqslant [\frac{n}{2}]} b_s\big(2^s(n-2t)\big) \cdot b(t) = \sum_{0 \leqslant t \leqslant [\frac{n}{2}]} f_s(n-2t) \cdot b(t)$$

and, therefore,

$$b(2^{s+2} n) - b(2^s n) = \sum_{0 \leqslant t \leqslant [\frac{n}{2}]} \big(f_{s+2}(n-2t) - f_s(n-2t)\big) \cdot b(t).$$

If n is odd, then the first multiplier is divisible by $2^{\mu(s)}$. Further, $b(0) = b(1) = 1$, and $b(t)$ is even for $t \geqslant 2$. Hence,

$$b(2^{s+2} n) - b(2^s n) \equiv \big(f_{s+2}(n) - f_s(n)\big) + \big(f_{s+2}(n-2) + f_s(n-2)\big) \big(\operatorname{mod} 2^{\mu(s)+1}\big).$$

(This equality is true for $n = 1$, since $f_s(-1) = 0$). It remains to note that one of the numbers n or $n - 2 \equiv 1 (\operatorname{mod} 4)$ and the other $\equiv 3 (\operatorname{mod} 4)$. Now we can use (13) to obtain (4). The theorem is proved.

3. GENERALIZATION OF THE THEOREM

We succeeded in proving the Rödseth–Gupta theorem because, for small k, the value of π of the coefficients at small powers of $n+1$ of the polynomials $f_{k+2} - f_k$ is more than zero. Hence, for the polynomials $f_{k+3} - f_{k+1}$ the value of π increases. We can investigate other linear combinations of f_k and investigate the value of π of the coefficients at small powers of $n+1$. The generalization is possible in all the proofs of the Rödseth–Gupta theorems but this has not been done so far. One of the possible realizations of this idea is as follows.

Define by U_k some linear combination of the polynomials $f_k, f_{k+2}, \ldots, f_{k+2t}$:

$$U_k = \sum_{i=0}^{t} \theta_i f_{k+2i}, \quad \text{where } \theta_i \in \mathbb{Z}, \quad \sum_{i=0}^{t} \theta_i = 0. \tag{14}$$

In other words, U_k is some linear combination of $(n+1)^{k+2t}$, $(n+1)^{k+2t-2}, \ldots$.

THEOREM 1. *Let*

$$U_k = \cdots + c(n+1)^{4+\delta_k} + b(n+1)^{2+\delta_k} + a(n+1)^{\delta_k},$$

where $\delta_k = 2$ if k is even and $\delta_k = 1$ if k is odd. Suppose that for some k the following conditions hold:

$$\pi(a) = \pi(b) = \tau, \qquad \pi(d) \geqslant \tau + 2 \textit{ for the rest coefficients } c, \ldots.$$

Then for any odd n the expression

$$A_s = \sum_{i=0}^{t} \theta_i \cdot b(2^{s+2i} n) \tag{15}$$

is divisible by the $[\frac{3(s-k)+2\tau+2+\delta_k}{2}]$th power of two.

Proof. The proof follows from the argument of Lemma 3 and subsequent considerations, therefore we shall only sketch it here. Let

$$\begin{aligned} U_{2l-1} &= \cdots + b_5^{(l)}(n+1)^5 + b_3^{(l)}(n+1)^3 + b_1^{(l)}(n+1), \\ U_{2l} &= \cdots + a_6^{(l)}(n+1)^6 + a_4^{(l)}(n+1)^4 + a_2^{(l)}(n+1)^2. \end{aligned} \tag{16}$$

Suppose that for l we have $\pi(a_2^{(l)}) = \pi(a_4^{(l)}) = \gamma$ and $\pi(a_{2i}^{(l)}) \geqslant \gamma + 2$, for $i > 2$. Then $U_{2l+1} = \cdots + a_6^{(l)} D_6 + a_4^{(l)} D_4 + a_2^{(l)} D_2$. Every power of $n+1 \geqslant 7$ will appear in U_{2l+1} with the coefficients for which $\pi \geqslant \gamma + 2 + 4$ (from (10)). The fifth power will appear with similar coefficients with the exception of a coefficient which is obtained from $a_4^{(l)} D_4$ (for it $\pi = \gamma + 4$). In all these cases $\pi(b_{2i+1}^{(l+1)}) \geqslant \gamma + 4$, $i \geqslant 2$. Further,

$$b_1^{(l+1)} = a_2^{(l)} \frac{r_1}{3} + a_4^{(l)} \frac{r_2}{5} + \cdots + a_{2l+2t}^{(l)} \frac{r_{l+t}}{2l+2t+1}$$

(from (11)). In Lemma 3 for π of the coefficients $a_{2i}^{(l)}$ we had a bigger reserve. Nevertheless,

$$a_2^{(l)} \frac{r_1}{3} + a_4^{(l)} \frac{r_2}{5} = -\frac{1}{3}\big(a_2^{(l)} + a_4^{(l)}\big) + \frac{4}{5} a_4^{(l)} = \frac{1}{3} \sum_{i=3}^{k+t} a_{2i}^{(l)} + \frac{4}{5} a_4^{(l)}$$

((14) and (16) for $n = 0$ gives $U_{2l}(0) = \sum_{i=0}^{k+t} a_{2i}^{(l)} = 0$). Substituting this expression into the formula for $b_1^{(l+1)}$ we obtain that the coefficients at $a_{2i}^{(l)}$ for $i \geqslant 3$ are equal to $(\frac{r_i}{2i+1} + \frac{1}{3})$, hence, for all the summands of the sum (except the first $\frac{4}{5} a_4^{(l)}$), the value of π will be $\geqslant \gamma + 2 + 1 = \gamma + 3$. Thus, $\pi(b_1^{(l+1)}) = \gamma + 2$. Further,

$$b_3^{(l+1)} = \frac{4}{3} r_0 a_2^{(l)} + \frac{4}{3} r_1 \cdot 2 \cdot a_4^{(l)} + \cdots + \frac{4}{3} r_{l+t-1} (l+t) a_{2l+2t}^{(l)}$$

(we use (10) again) and this gives $\pi(b_3^{(l+1)}) = \gamma + 2$.

Now if for some l we have $\pi(b_1^{(l)}) = \pi(b_3^{(l)}) = \gamma$, $\pi(b_{2i+1}^{(l)}) \geqslant \gamma + 2$ for $i \geqslant 2$, then as in Lemma 3 $\pi(a_2^{(l)}) = \pi(a_4^{(l)}) = \gamma + 1$, $\pi(a_{2i}^{(l)}) \geqslant \gamma + 3$ for $i > 2$. We see that when passing from U_k to U_{k+1} the value of π of the two last coefficients at $n+1$ increases by 2 if k is even and by 1 if k is odd. What remains is an easy exercise. For the last two coefficients of U_{k+2l}, the value of π equals $\tau + 3l$ whereas for the remining ones $\pi \geqslant \tau + 3l + 2$. For the last two coefficients U_{k+2l+1} the value of π equals $\tau + 3l + \delta_k$ whereas for the remaining ones $\pi \geqslant \tau + 3l + \delta_k + 2$.

Similarly, for odd n the number $U_{k+2l}(n)$ is divisible by $2^{\tau+3l+\delta_k}$, $U_{k+2l+1}(n)$ is divisible by $2^{\tau+3l+\delta_k+\delta_{k+1}} = 2^{\tau+3l+3}$, and the division is exact if and only if $n \equiv 1 (\mathrm{mod}\, 4)$. The argument following Lemma 3 can be repeated without any change. It remains to note that the correspondence $k + 2l \to \tau + 3l + \delta_k$, $k + 2l + 1 \to \tau + 3l + 3$ coincides with $s \to [\frac{3(s-k)+2\tau+2+\delta_k}{2}]$ for $s \geqslant k$. The theorem is proved.

4. CONCLUSIONS

1. The Rödseth–Gupta theorem is a trivial corollary of Theorem 1.
2. Calculations show that

$$f_6 + 7f_4 - 8f_2 = \frac{2^{11}}{45}(n+1)^6 - \frac{2^9}{9}(n+1)^4 + \frac{2^9}{45}(n+1)^2.$$

Hence, we can apply the theorem with $k = 2$, $\tau = 9$. Therefore, for odd n we have the following claim: $b(2^{s+4}n) + 7b(2^{s+2}n) - 8b(2^s n)$ is exactly divisible by the $[\frac{3s}{2}] + 8$ power of two for $s \geqslant 2$.

To end, we shall list some more properties of the sequence $b(n)$ (this generalizes the results obtained by Churchhouse.) Some elementary properties of this sequence were proved in [1]. Namely:

1. $b(n) = 0 (\bmod 2)$ for all $n \geqslant 2$.
2. $b(n) = 0 (\bmod 4)$ if and only if n or $n - 1 = 4^m (2k + 1)$, $m \geqslant 1$.
3. $b(n) \neq 0 (\bmod 8)$ for all n.

Let $w(n)$ be the Thue–Morse sequence with $w(0) = 0$ and $w(1) = 1$ and let $\tau(n)$ be the Rudin–Shapiro sequence with $\tau(0) = 0$ and $\tau(3) = 1$. It is easy to see that $w(n) = 0$ if the sum of the digits of n in the binary expansion is even and $w(n) = 1$ if this sum is odd. In the same way, $\tau(n) = 0$ if in the binary expansion of the number n there is an even number of the blocks of 11 and $\tau(n) = 1$, otherwise. Then, using the mathematical induction on the argument (in the first three cases), we can easily prove the following theorem.

THEOREM 2.

1. $b(2^{2s+1}(2n + 1)) \equiv 10 + 20w(n) + 8w([n/2]) + 16\tau(n) (\bmod 32)$ *for* $s \geqslant 1$.
2. $b(4n + 2) \equiv 2 + 4w(n) + 8w([n/2]) + 16\tau(n) (\bmod 32)$.
3. $b(2^{2s}(2n + 1)) \equiv 4 + 8w([n/2]) + 16w(n) (\bmod 32)$ *for* $s \geqslant 1$.
4. *For every* s, $3 \leqslant s \leqslant 14$, $s \neq 8$, *infinitely many* $b(n)$ *are divisible by* s.

REFERENCES

1. R. F. Churchhouse, Congruence properties of the binary partition function, *Proc. Cambridge Philos. Soc.*, **66**, 371-376 (1969).
2. Ö. Rödseth, Some arithmetical properties of m-ary partitions, *Proc. Cambridge Philos. Soc.*, **68**, 447–453 (1970).
3. H. Gupta, Proof of the Churchhouse conjecture concerning binary partitions, *Proc. Cambridge Philos. Soc.*, **70**, 53–56 (1971).
4. G. Andrews, *Theory of Partitions* (in Russian), Nauka (1982).
5. G. Andrews, Congruence properties of the m-ary partition function, *J. Number Theory*, **3**, 104–110 (1970).
6. W. B. Pennington, On Mahler partition problem, *Ann. of Math.*, **57**, 531–546 (1952).
7. G. Alkauskas, The beginnings of significant mathematical results in the school children competitions (in Lithuanian), *Alpha Plus Omega*, **4**, 94–98 (1997).
8. G. Eckstein, On an olympiad problem, *Romanian Math. Mag.*, **7**, 15–16 (1998).

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