Undetermined Coefficients (3A)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Finding a Particular Solution
- Undetermined Coefficients

Particular Solutions

DEQ

$$a\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + cy = g(x) \longleftarrow y_p$$

particular solution
by a conjecture

(I) FORM Rule

(II) Multiplication Rule

When coefficients are constant

And

$g(x) =$

- ***A constant*** or ------------------------- k
- ***A polynomial*** or ------------------------- $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$
- ***An exponential function*** or --------------- $e^{\alpha x}$
- ***A sine and cosine functions*** or --------- $\sin(\beta x) \quad \cos(\beta x)$
- ***Finite sum and products of the above functions*** ------ $e^{\alpha x}\sin(\beta x) + x^2$

And

$$g(x) \neq \quad \ln x \quad \frac{1}{x} \quad \tan x \quad \sin^{-1} x$$

Form Rule

DEQ

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g(x) \longleftarrow y_p$$

particular solution
by a conjecture

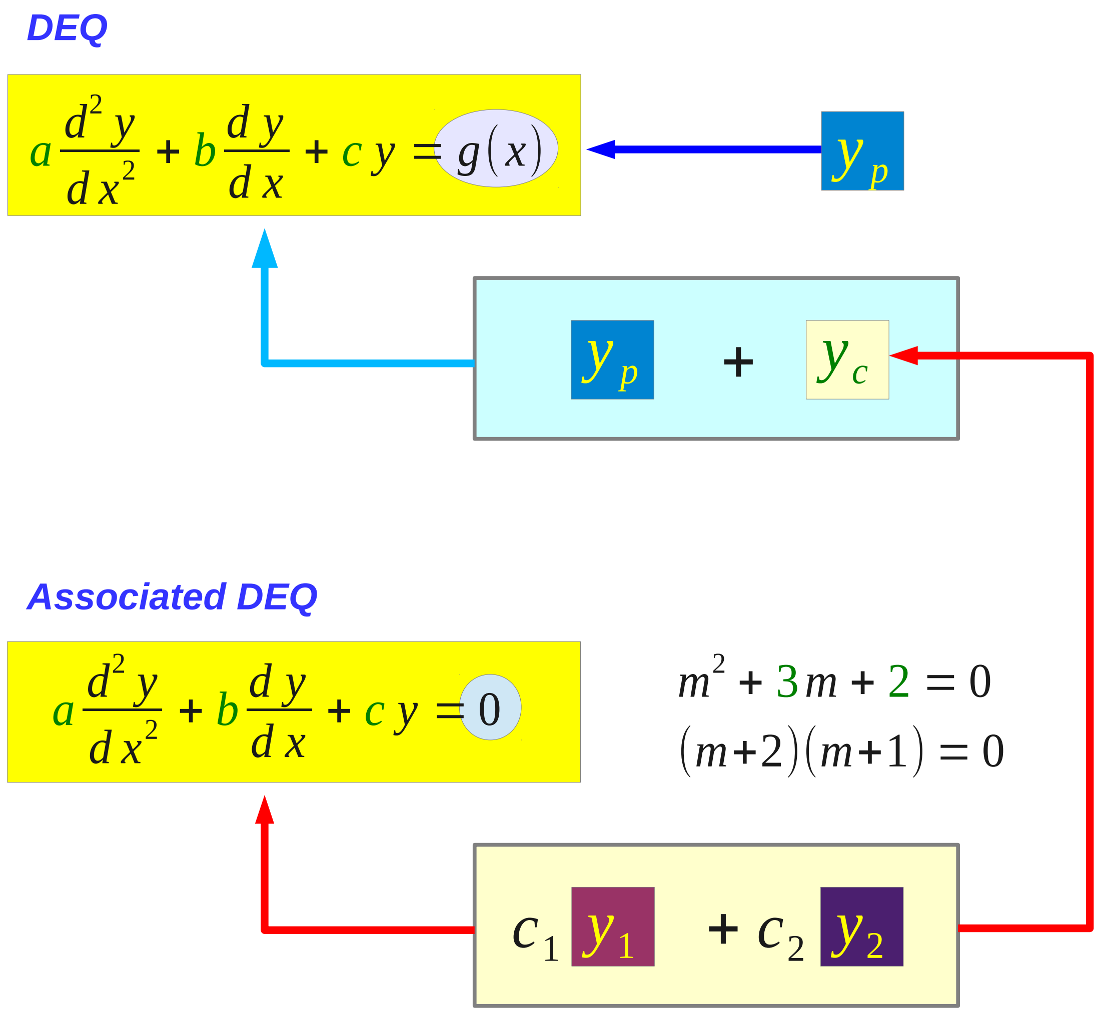
(I) FORM Rule

(II) Multiplication Rule

When coefficients are constant

$g(x) = 2$	$y_p = A$
$g(x) = 3x+4$	$y_p = Ax+B$
$g(x) = 6x^2-7$	$y_p = Ax^2+Bx+C$
$g(x) = \sin 8x$	$y_p = A\cos 8x + B\sin 8x$
$g(x) = \cos 9x$	$y_p = A\cos 9x + B\sin 9x$
$g(x) = e^{10x}$	$y_p = Ae^{10x}$
$g(x) = xe^{11x}$	$y_p = (Ax+B)e^{11x}$
$g(x) = e^{11x}\sin 12x$	$y_p = Ae^{11x}\sin 12x + Be^{11x}\cos 12x$
$g(x) = 5x\sin(3x)$	$y_p = (Ax+B)\cos(3x) + (Cx+D)\sin(3x)$

Form Rule Example



$$m^2 + 3m + 2 = 0$$
$$(m+2)(m+1) = 0$$

$$y_p = Ax + B$$
$$y_p' = A$$
$$y_p'' = 0$$

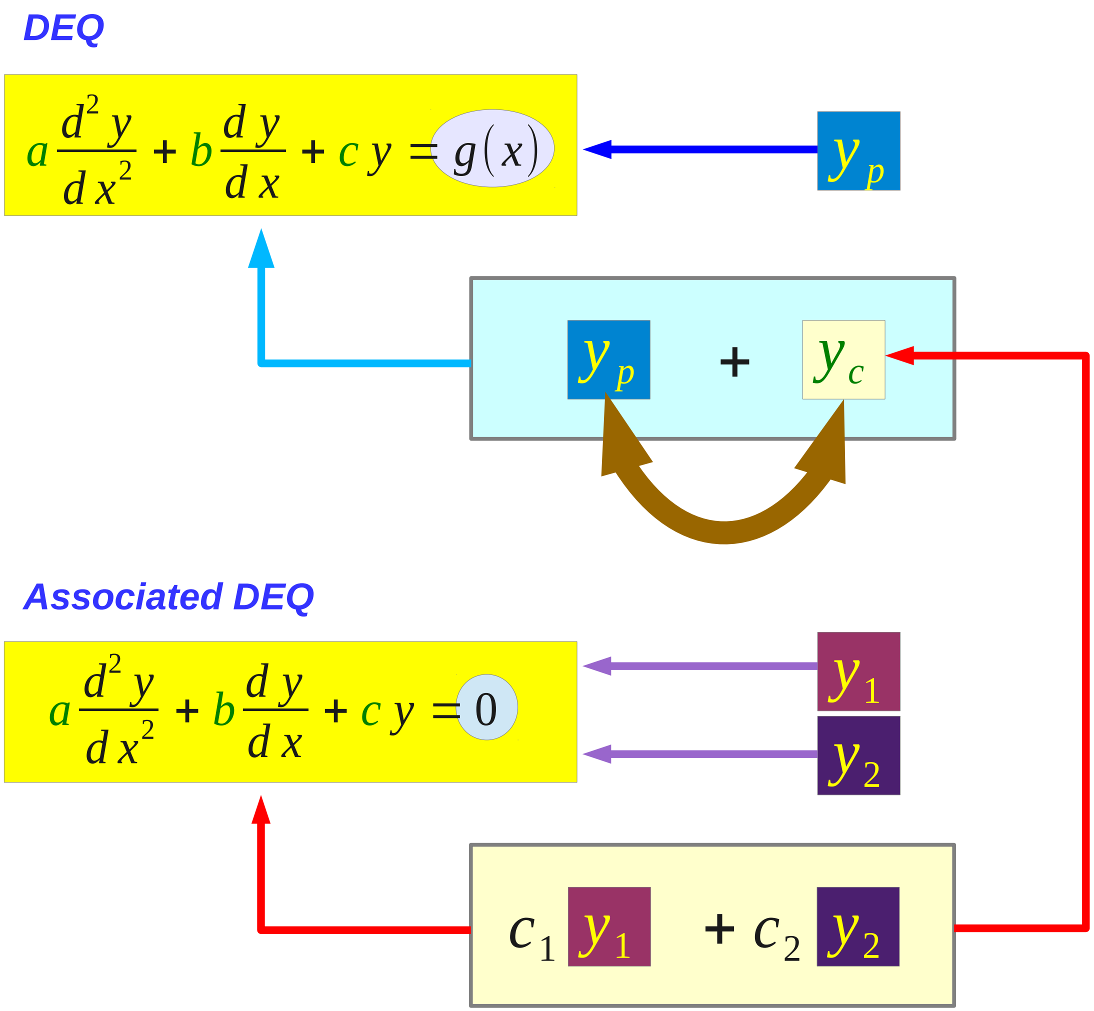
$$y_p'' + 3y_p' + 2y_p$$
$$= 3A + 2(Ax + B)$$
$$= 2Ax + 3A + 2B$$
$$= x$$

$$2A = 1 \qquad A = \tfrac{1}{2}$$
$$3A + 2B = 0 \qquad B = -\tfrac{3}{4}$$

$$y_p = \tfrac{1}{2}x - \tfrac{3}{4}$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + \tfrac{1}{2}x - \tfrac{3}{4}$$

Multiplication Rule



use $y_p = x^n y_1$ $\quad$ $y_p = x^n y_2$

if $y_p = y_1$ $\quad$ $y_p = y_2$

When y_p contains a term which is the same term in y_c

Use y_p multiplied by x^n

n is the smallest positive integer that eliminates the duplication

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Multiplication Rule Example (1)

$$y'' - 2y' + y = 2e^x$$

$$y_p = Ae^x \Rightarrow Axe^x \Rightarrow Ax^2e^x$$

$$y_1 = e^x \qquad y_2 = xe^x$$

$$y'' - 2y' + y = 0$$

$$y'' - 2y' + y = 6xe^x$$

$$y_p = Axe^x \Rightarrow Ax^2e^x \Rightarrow Ax^3e^x$$

$$2Ae^x \neq 6xe^x$$

$$y_1 = e^x \qquad y_2 = xe^x$$

$$y'' - 2y' + y = 0$$

$$y_p = x(Ax + B)e^x \rightarrow Bxe^x$$

$$y_p = x^2(Ax + B)e^x$$

Multiplication Rule Example (2)

$$y' + 4y = e^{x}\sin(2t) + 2t\cos(2t)$$

$$y_p(t) = e^{x}(A\cos(2t) + B\sin(2t)) + (Ct + D)\cos(2t) + (Et + F)\sin(2t)$$

$$y_p(t) = e^{x}(A\cos(2t) + B\sin(2t)) + \mathbf{t}(Ct + D)\cos(2t) + \mathbf{t}(Et + F)\sin(2t)$$

$$y_h(t) = c_1 e^{+i2t} + c_2 e^{-i2t}$$

$$= (c_3\cos(2t) + c_4\sin(2t))$$

$$y'' + 5y' + 6y = t^2 e^{-3t}$$

$$y_p(t) = (At^2 + Bt + C)e^{-3t}$$

$$y_p(t) = \mathbf{t}(At^2 + Bt + C)e^{-3t}$$

$$y_h = c_1 e^{-2t} + c_2 e^{-3t}$$

Superposition (1)

DEQ

$$\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + c\,y = 2x^2 + 3 + \cos 8x$$

$$(2x^2 + 3) + (\cos 8x)$$

additive

$$\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + c\,y = 0 \quad \leftarrow \quad y_c$$

$$\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + c\,y = 2x^2 + 3 \quad \leftarrow \quad y_{p1}$$

$$\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + c\,y = \cos 8x \quad \leftarrow \quad y_{p2}$$

$$\frac{d^2 y_c}{dx^2} + b\frac{dy_c}{dx} + c\,y_c = 0$$

$$\frac{d^2 y_{p1}}{dx^2} + b\frac{dy_{p1}}{dx} + c\,y_{p1} = (2x^2 + 3)$$

$$\frac{d^2 y_{P2}}{dx^2} + b\frac{dy_{p2}}{dx} + c\,y_{P2} = \cos 8x$$

$$\frac{d^2}{dx^2}[y_c + y_{p1} + y_{p2}] + b\frac{d}{dx}[y_c + y_{p1} + y_{p2}] + c[y_c + y_{p1} + y_{p2}] = 2x^2 + 3 + \cos 8x$$

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Superposition (2)

DEQ

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c\,y = (2x^2+3)\cdot\cos 8x$$

$$y_p = (Ax^2+Bx+C)\cdot(\cos 8x+\sin 8x)$$

multiplicative

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c\,y = 0 \quad \leftarrow \quad y_c$$

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c\,y = (2x^2+3) \quad \leftarrow \quad y_{p1}$$

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c\,y = \cos 8x \quad \leftarrow \quad y_{p2}$$

$$\frac{d^2}{dx^2}[y_c+y_{p1}\cdot y_{p2}] + b\frac{d}{dx}[y_c+y_{p1}\cdot y_{p2}] + c\,[y_c+y_{p1}\cdot y_{p2}] = (2x^2+3)\cdot\cos 8x$$

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Finite Number of Derivative Functions

$$y = x e^{mx}$$

$$\dot{y} = e^{mx} + m x e^{mx}$$

$$\ddot{y} = m e^{mx} + m(e^{mx} + m x e^{mx}) = 2 m e^{mx} + m^2 x e^{mx}$$

$$\dddot{y} = 2 m e^{mx} + m^2(e^{mx} + m x e^{mx}) = (m^2 + 2m) e^{mx} + m^3 x e^{mx}$$

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$$\{e^{mx}, \; x e^{mx}\}$$

two kinds

$$\{e^{mx}, \; x e^{mx}\}$$

linearly independent functions

$$y = 2x^2 + 3x + 4$$

$$\dot{y} = 4x + 3$$

$$\ddot{y} = 4$$

$$\dddot{y} = 0$$

$$\{2x^2 + 3x + 4, \; 4x + 3, \; 4\}$$

three kinds

$$\{A x^2 + B x + C\}$$

linearly independent functions

Infinite Number of Derivative Functions

$$y = +x^{-1}$$

$$\dot{y} = -x^{-2}$$

$$\ddot{y} = +2x^{-3}$$

$$\dddot{y} = -6x^{-4}$$

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•

$$y = \ln x$$

$$y = +x^{-1}$$

$$\dot{y} = -x^{-2}$$

$$\ddot{y} = +2x^{-3}$$

$$\dddot{y} = -6x^{-4}$$

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These kinds of functions are not suitable for the undetermined coefficient method

the form of a particular solution is a linear combination of all linearly independent functions that are generated by repeated differentiation of g(x) input function

References

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