Vectors (1A)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Determinant

Determinant of order 2

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \qquad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Determinant of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} a_1 & & \\ & b_2 & b_3 \\ & c_2 & c_3 \end{bmatrix} \qquad \begin{bmatrix} & a_2 & \\ b_1 & & b_3 \\ c_1 & & c_3 \end{bmatrix} \qquad \begin{bmatrix} & & a_3 \\ b_1 & b_2 & \\ c_1 & c_2 & \end{bmatrix}$$

Determinant

Determinant of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} a_1 & & \\ & b_2 & b_3 \\ & c_2 & c_3 \end{bmatrix} \qquad \begin{bmatrix} & a_2 & \\ b_1 & & b_3 \\ c_1 & & c_3 \end{bmatrix} \qquad \begin{bmatrix} & & a_3 \\ b_1 & b_2 & \\ c_1 & c_2 & \end{bmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = + a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Cross Product (1)

Determinant of order 3

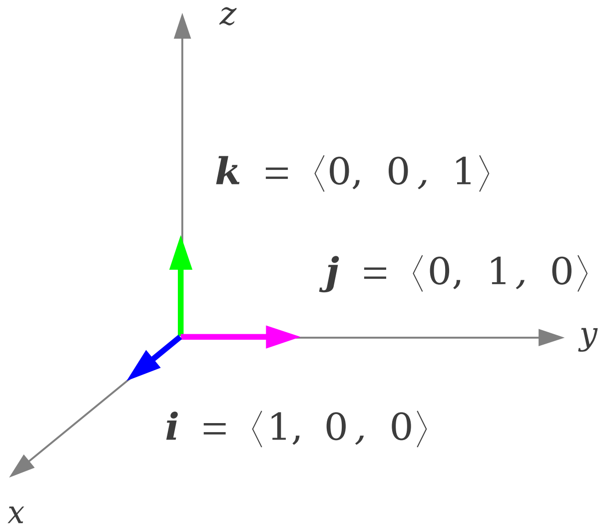
$$\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} = \langle a_1, a_2, a_3 \rangle$$

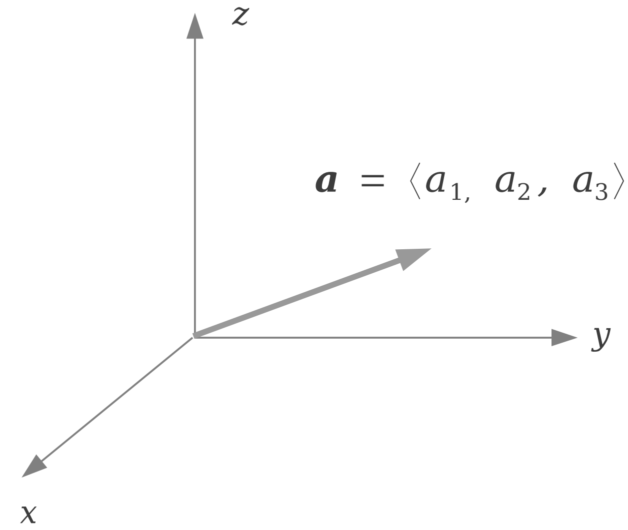
$$\mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k} = \langle b_1, b_2, b_3 \rangle$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{i} + (a_1 b_2 - a_2 b_1)\mathbf{i}$$

Cross Product (2)





$$\mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{k}$$

⇐ normal to $\mathbf{i}$ & $\mathbf{j}$ ⇒

$$\mathbf{j} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i}$$

⇐ normal to $\mathbf{j}$ & $\mathbf{k}$ ⇒

$$\mathbf{k} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j}$$

⇐ normal to $\mathbf{k}$ & $\mathbf{i}$ ⇒

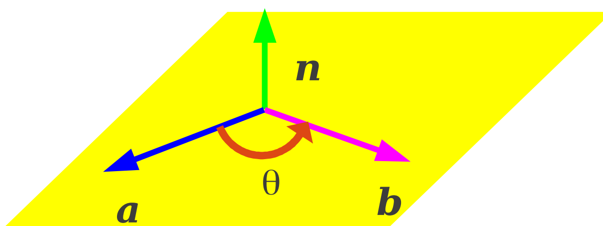
$$\mathbf{i} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\mathbf{j}$$

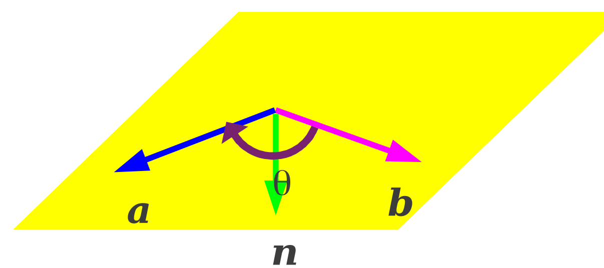
Right Hand Rule

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

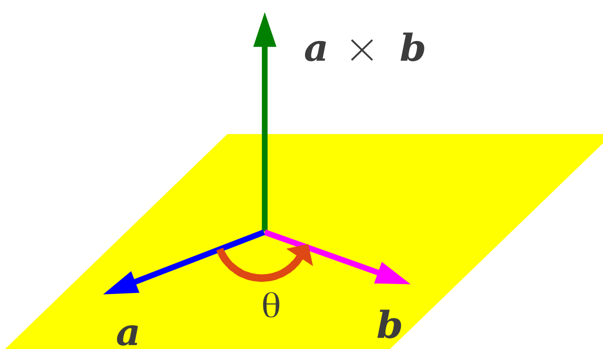
$$\boldsymbol{b} \times \boldsymbol{a} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

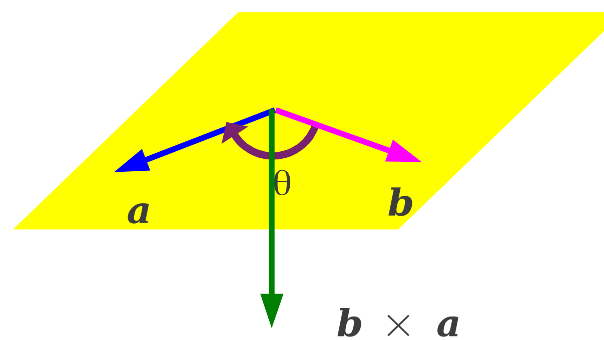
Normal direction $\boldsymbol{n}$





Magnitude = $\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\| \; \|\boldsymbol{b}\| \; \sin\theta$





Line Equations (1)

Vector Equation

$$\boldsymbol{r} = \boldsymbol{r}_2 + t\,\boldsymbol{a}$$

Parameter

Direction Vector

$$\boldsymbol{r}_2 = \langle x_2, y_2, z_2 \rangle$$
$$\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$$

Parametric Equation

Component

$$x = x_2 + t\,a_1$$
$$y = y_2 + t\,a_2$$
$$z = z_2 + t\,a_3$$

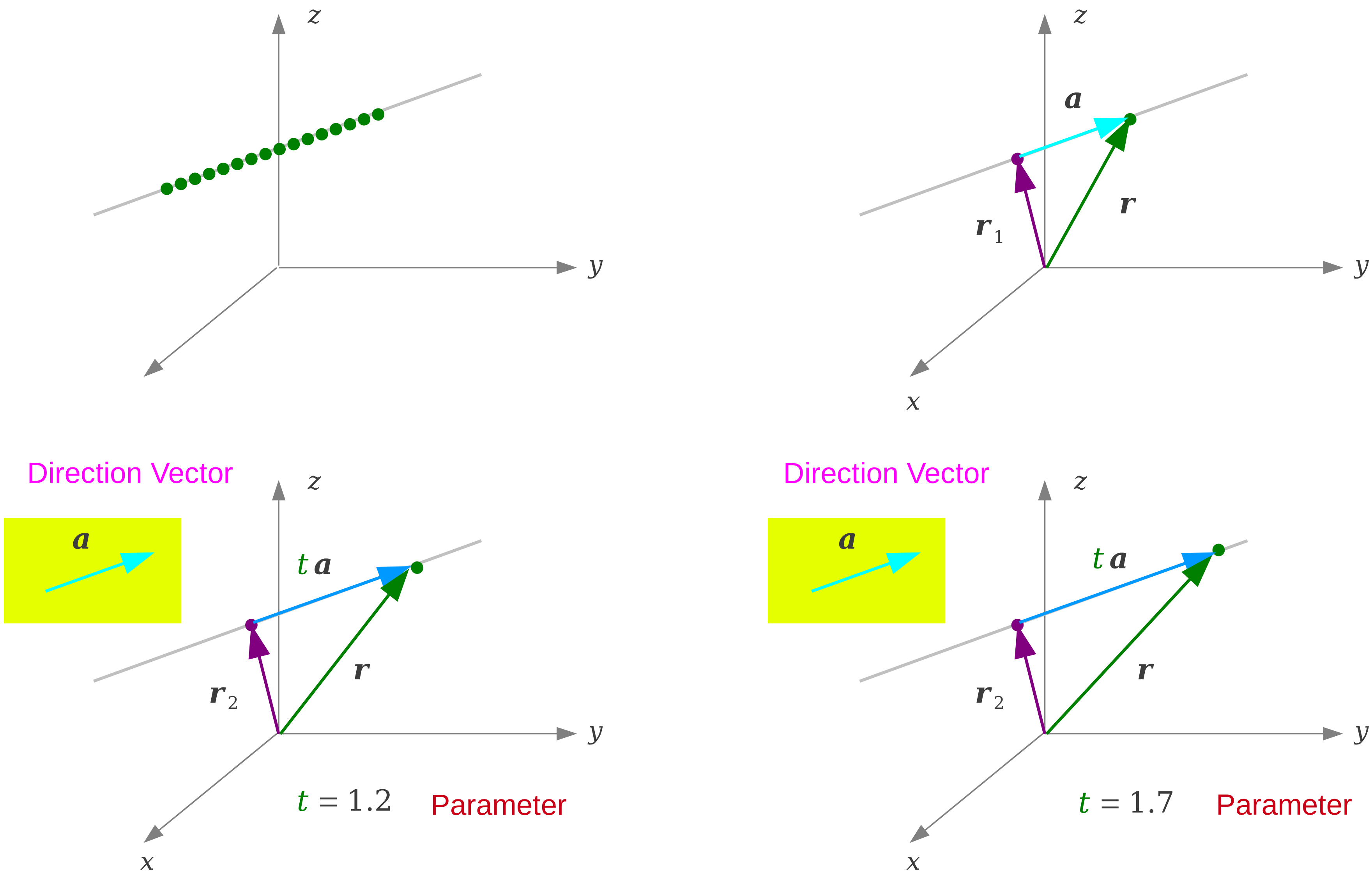
$$t\,a_1 = x - x_2$$
$$t\,a_2 = y - y_2$$
$$t\,a_3 = z - z_2$$

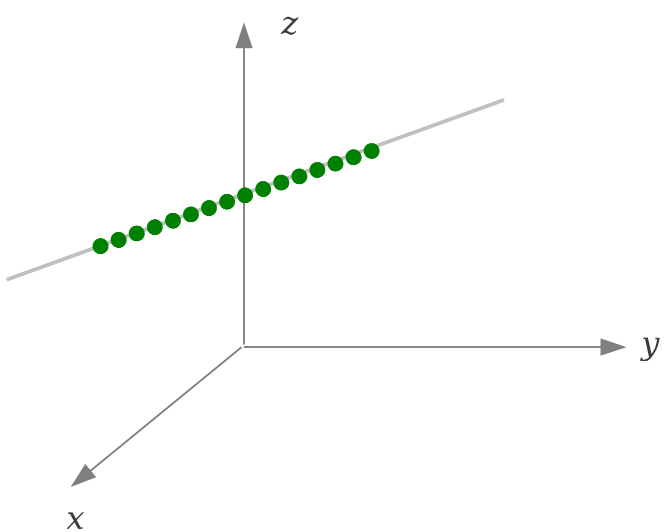
Symmetric Equation

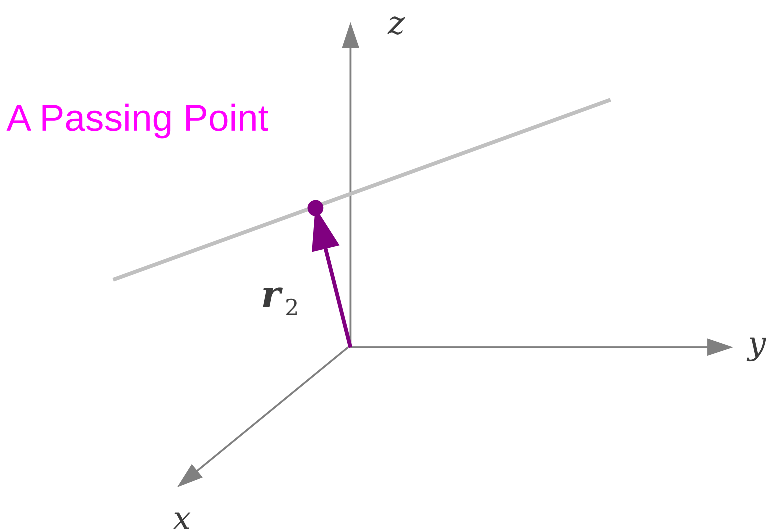
$$t = \frac{x - x_2}{a_1} = \frac{y - y_2}{a_2} = \frac{z - z_2}{a_3}$$

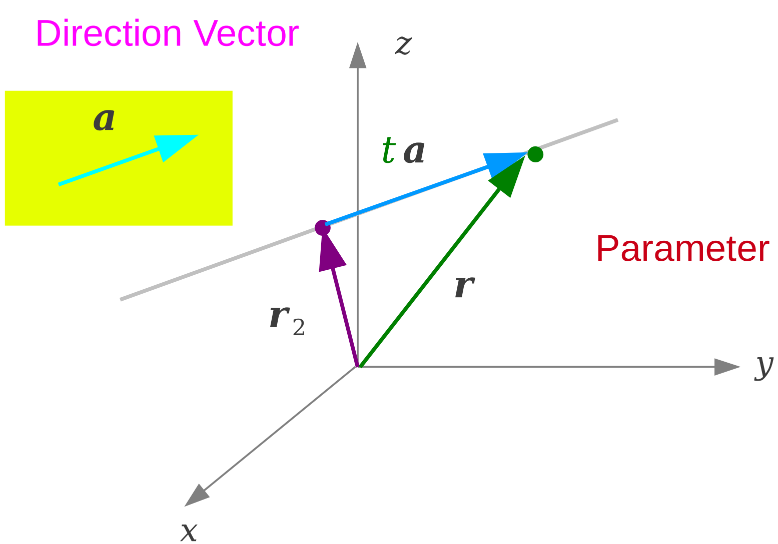
Elimination of parameter

Line Equations (2)









$$\boldsymbol{r} = \boldsymbol{r}_2 + t\,\boldsymbol{a}$$

$$\boldsymbol{r}_2 = \langle x_2, y_2, z_2 \rangle$$
$$\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$$

Plane Equations (1)

Vector equation

$$\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}_1) = 0$$

Normal Vector

$$\boldsymbol{r} = \langle x, y, z \rangle$$

$$\boldsymbol{r}_1 = \langle x_1, y_1, z_1 \rangle$$

$$\boldsymbol{n} = \langle a, b, c \rangle$$

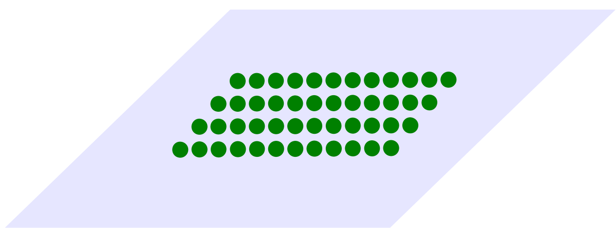
$$\boldsymbol{r} - \boldsymbol{r}_1 = \langle x - x_1, \; y - y_1, \; z - z_1 \rangle$$

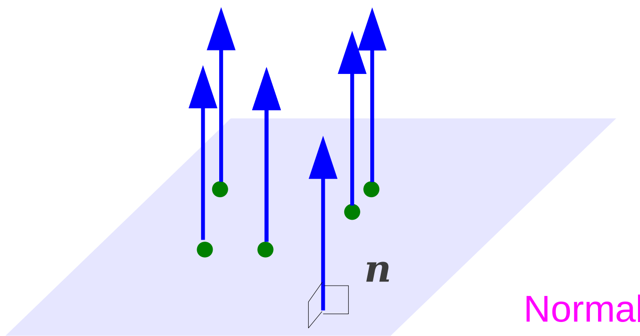
$$\boldsymbol{n} = \langle \quad a, \quad b, \quad c \rangle$$

Cartesian equation

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

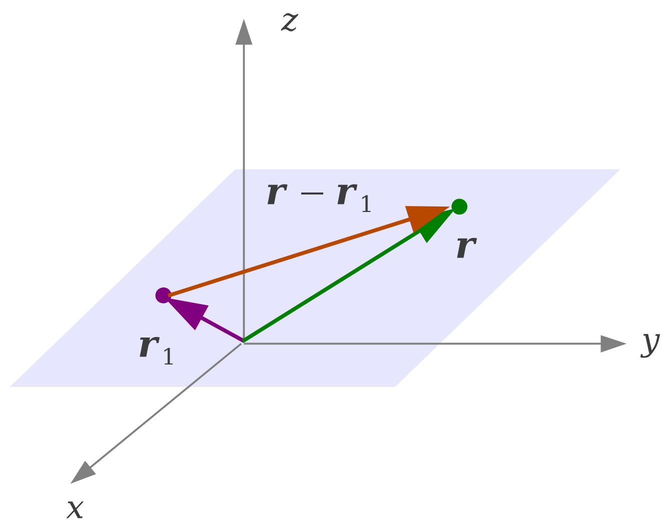
Plane Equations (2)

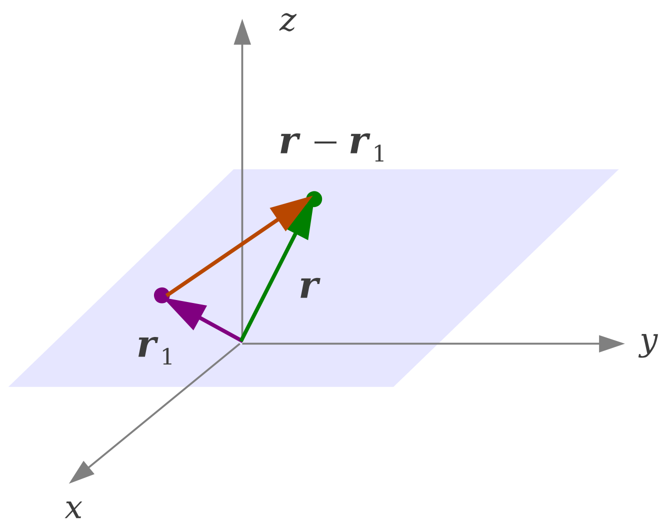




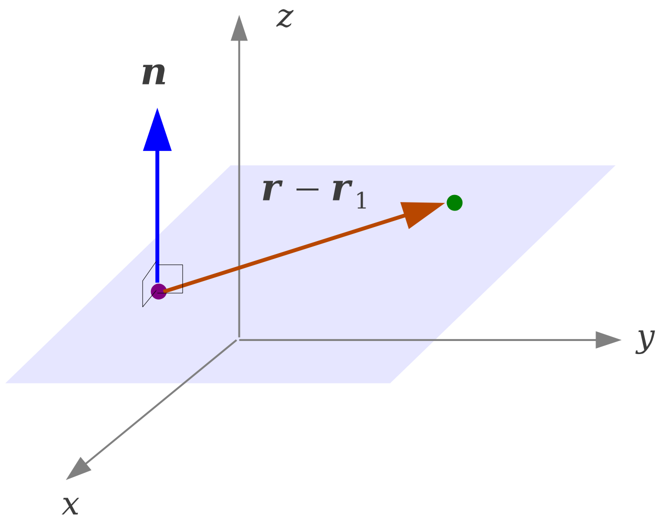
Normal Vector

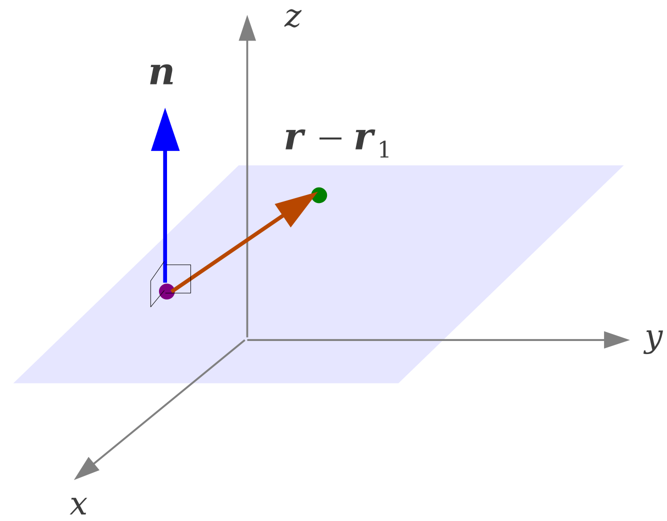
No Parameter

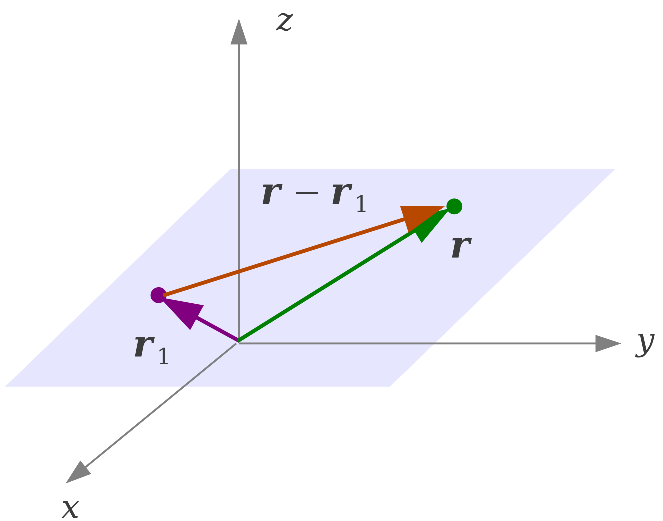




Plane Equations (3)







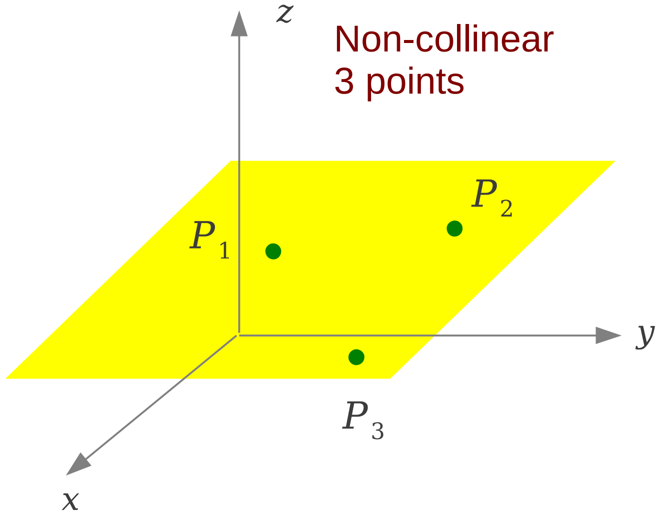
$$\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}_1) = 0$$

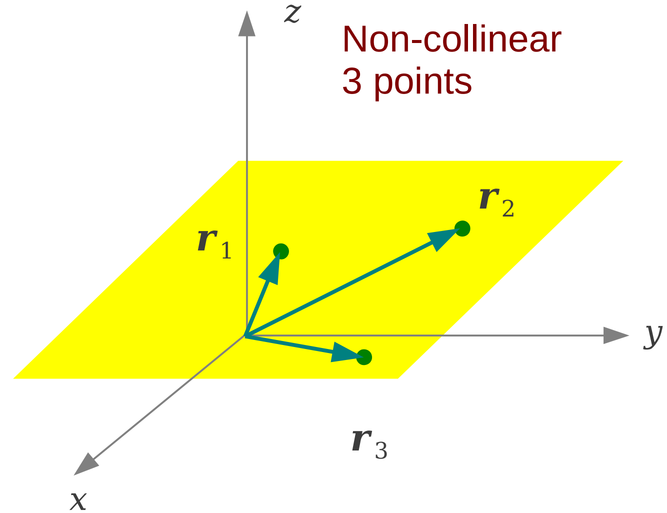
$$\boldsymbol{r} = \langle x, y, z \rangle$$

$$\boldsymbol{r}_1 = \langle x_1, y_1, z_1 \rangle$$

$$\boldsymbol{n} = \langle a, b, c \rangle$$

Normal Vector & 3 Points



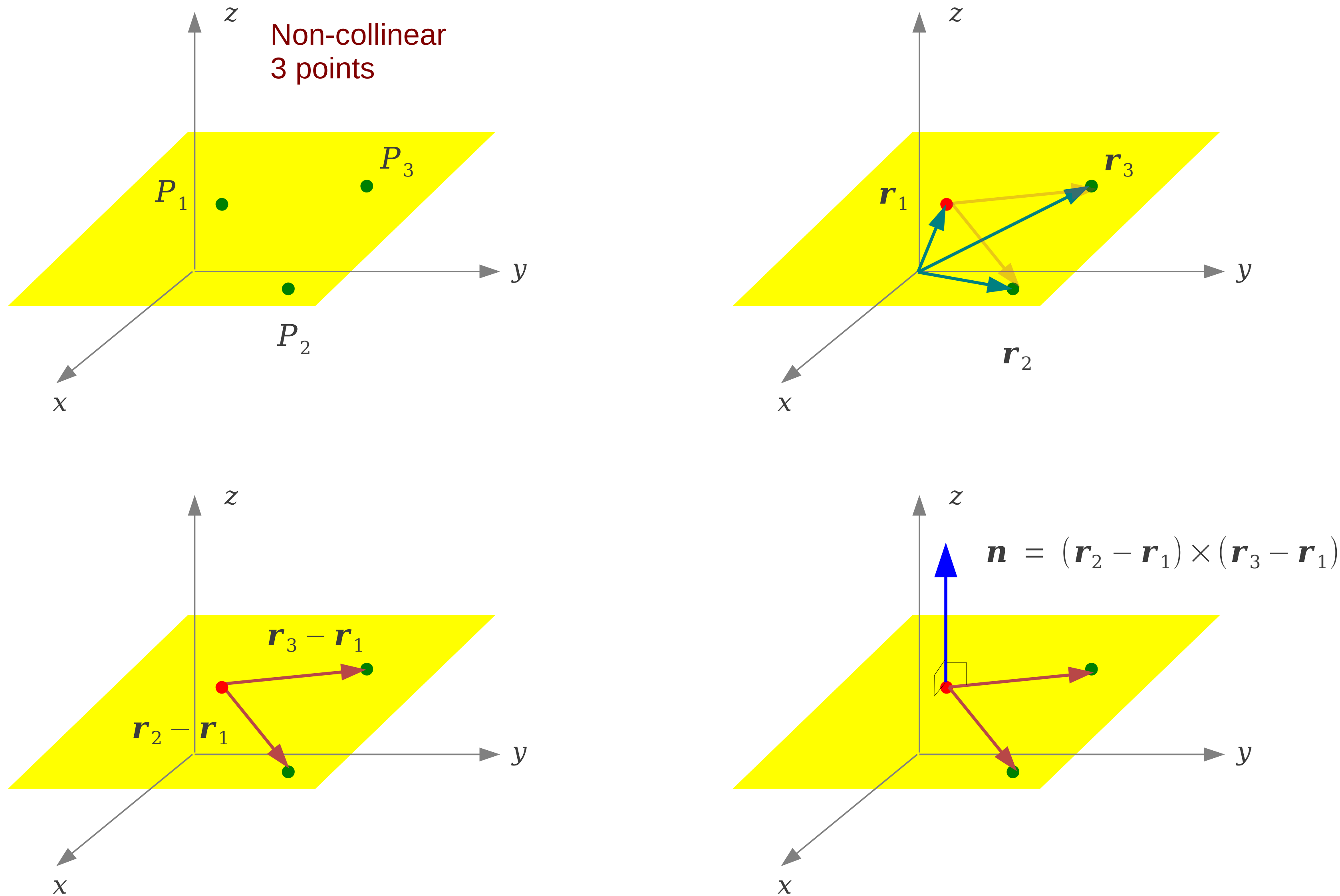


Graph of a plane

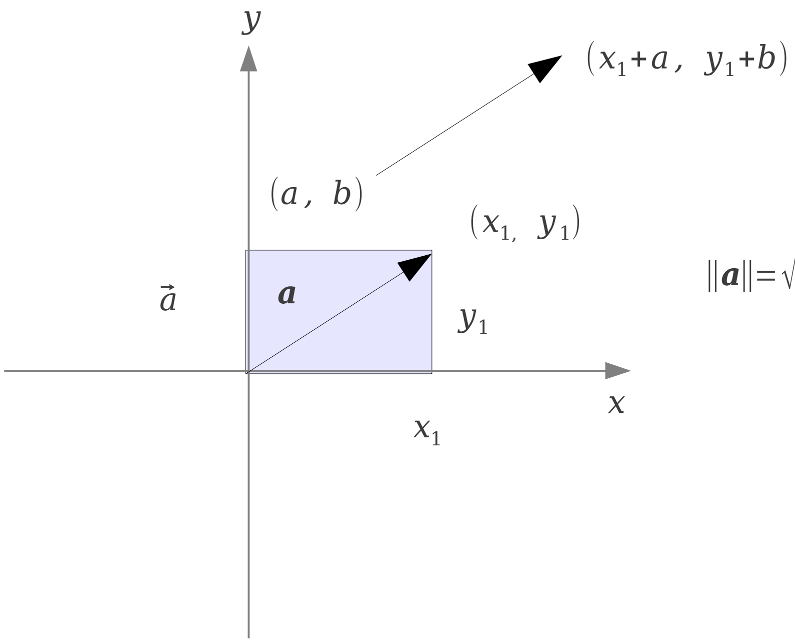
Line intersection of two planes

Point of intersection of a line and plane

Normal Vector & 3 Points

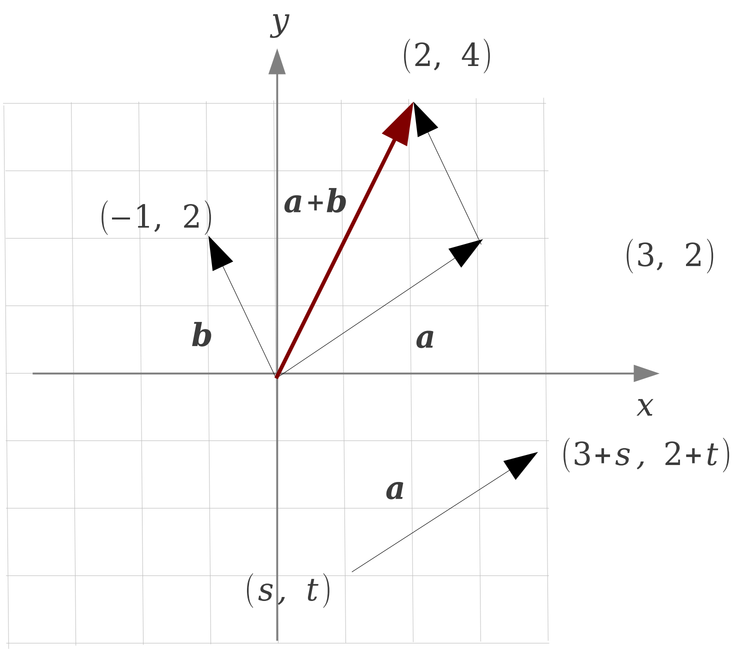


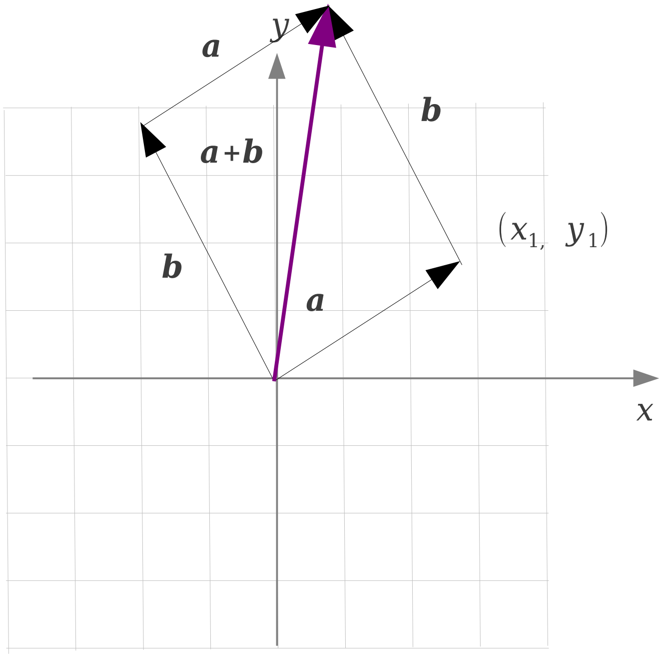
Vectors



$\|\boldsymbol{a}\| = \sqrt{x_1^2 + y_1^2}$

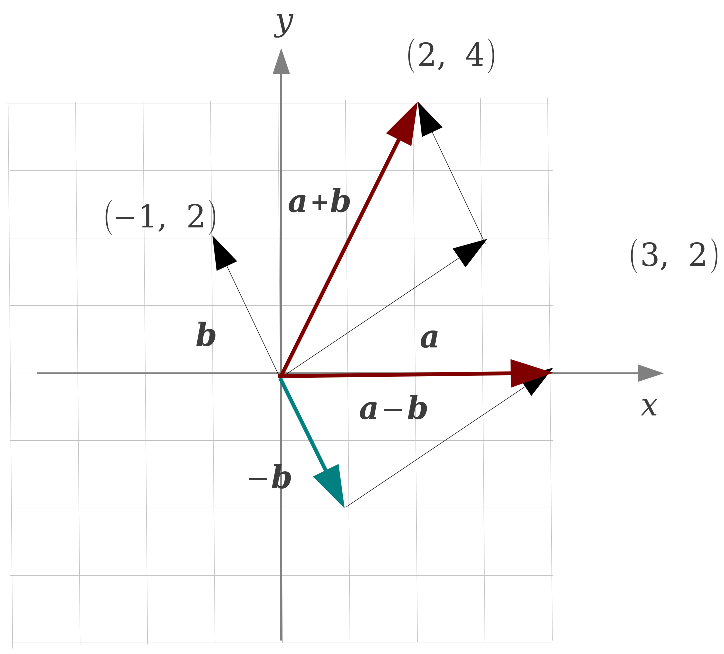
Vector Addition

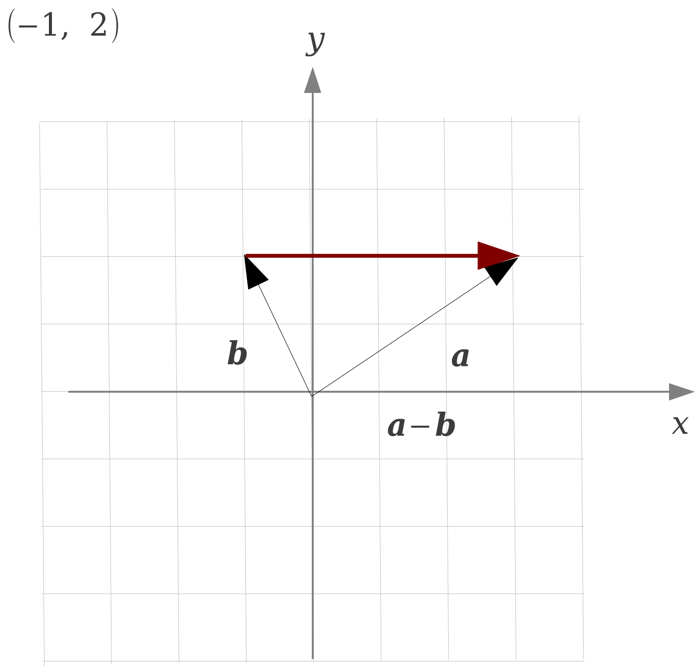




$$\|\boldsymbol{a}\| = \sqrt{x_1^2 + y_1^2}$$

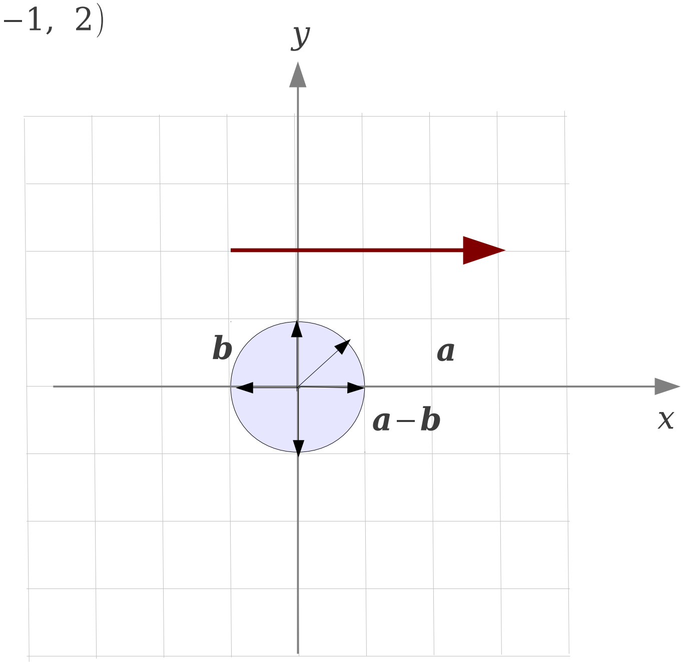
Vector Subtraction



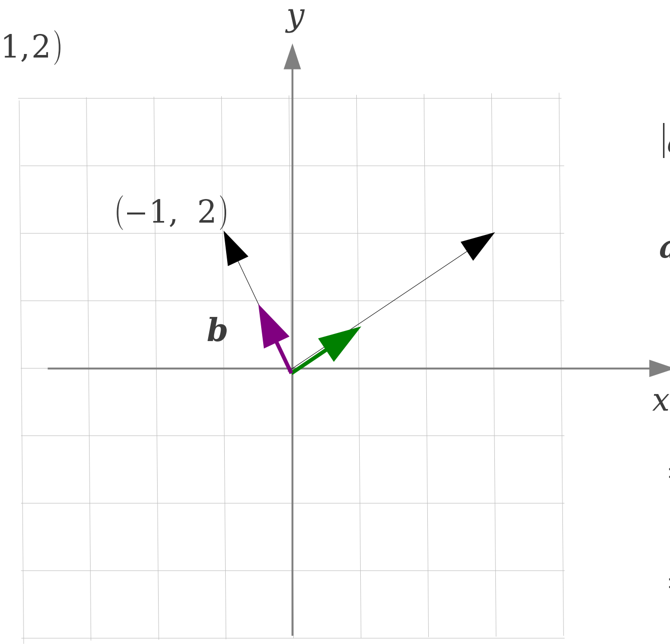


Unit Vector

$(-1,\ 2)$



$\boldsymbol{b} = (-1, 2)$



$|\boldsymbol{a}| = \sqrt{3^2 + 2^2} = \sqrt{13}$

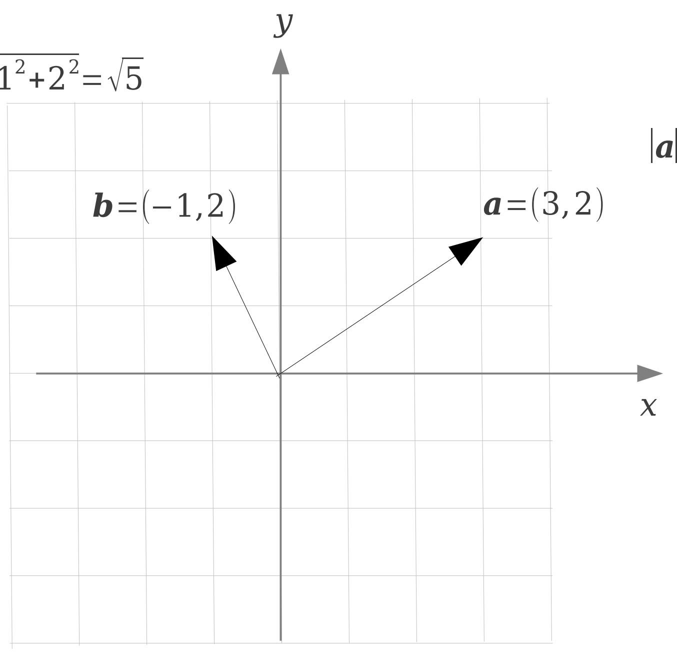
$\boldsymbol{a} = (3, 2)$

$= \left(\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\right)$

$= \sqrt{\frac{3^2}{13} + \frac{2^2}{13}}$

Vector Magnitude

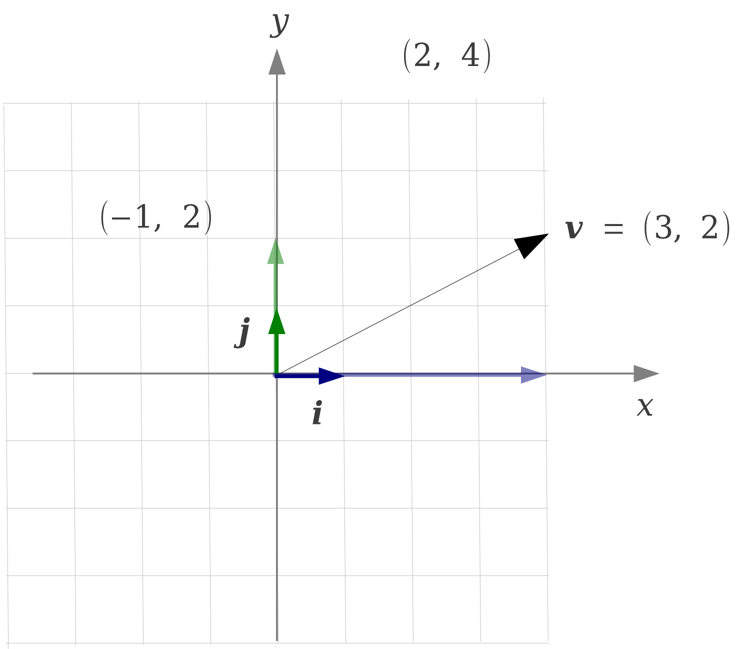
$|\boldsymbol{b}| = \sqrt{-1^2 + 2^2} = \sqrt{5}$

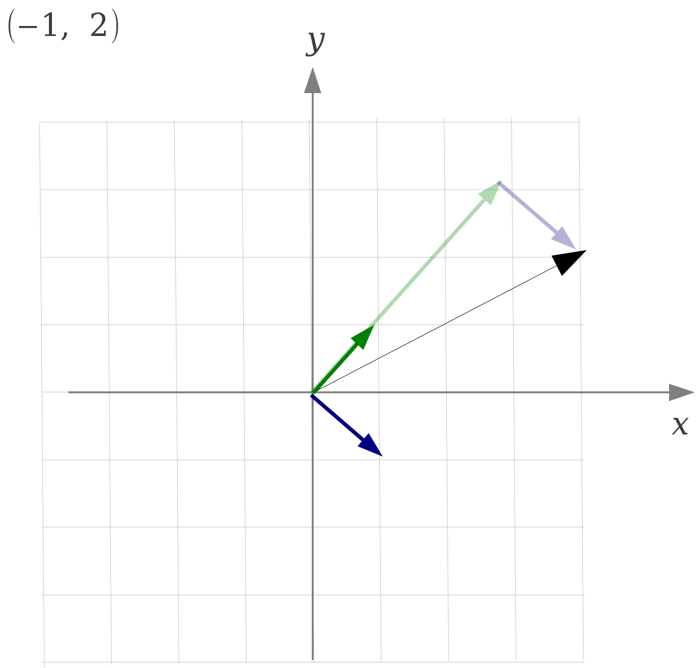


$|\boldsymbol{a}| = \sqrt{3^2 + 2^2} = \sqrt{13}$

$\boldsymbol{a} \cdot \boldsymbol{b} = 3 \cdot -1 + 2 \cdot 2 = -3 + 4 = 1$

Basis





$\boldsymbol{v} = 3\boldsymbol{i} + 2\boldsymbol{j}$

$a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix} = 0 \Rightarrow a = b = 0$

$a\begin{bmatrix}1\\1\end{bmatrix} + b\begin{bmatrix}1\\-1\end{bmatrix} = 0 \Rightarrow a = b = 0$

basis

basis

Vector Triple Product

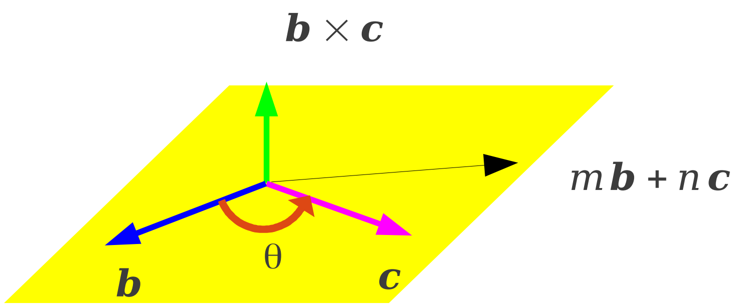
$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$$

⇨ Perpendicular to $\boldsymbol{b} \times \boldsymbol{c}$

⇨ Any vector perpendicular to $\boldsymbol{b} \times \boldsymbol{c}$
lies in the plane perpendicular to $\boldsymbol{b} \times \boldsymbol{c}$

⇨ lies in the plane of $\boldsymbol{b}$ and $\boldsymbol{c}$

Perpendicular to the plane of



$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \Rightarrow m\boldsymbol{b} + n\boldsymbol{c}$$

$$(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$$

Scalar Triple Product

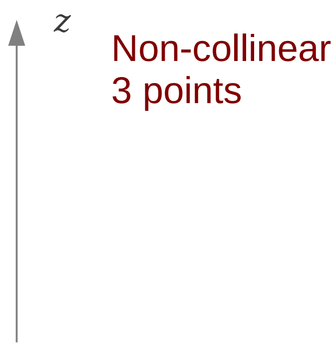
$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$$

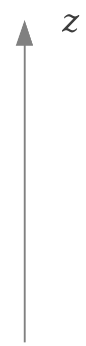
$$\boldsymbol{a} = a_1 \boldsymbol{i} + a_2 \boldsymbol{j} + a_3 \boldsymbol{k}$$
$$\boldsymbol{b} = b_1 \boldsymbol{i} + b_2 \boldsymbol{j} + b_3 \boldsymbol{k}$$
$$\boldsymbol{c} = c_1 \boldsymbol{i} + c_2 \boldsymbol{j} + c_3 \boldsymbol{k}$$

$$\boldsymbol{b} \times \boldsymbol{c} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = (a_1 \boldsymbol{i} + a_2 \boldsymbol{j} + a_3 \boldsymbol{k}) \cdot \left(\boldsymbol{i}\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right)$$

$$= \left(a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$





$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Gram-Schmidt Orthogonalization Process

$$\mathbf{v}_1 = \mathbf{u}_1$$

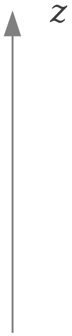
$$\mathbf{v}_2 = \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2$$

$$\mathbf{v}_4 = \mathbf{u}_4 - \left(\frac{\mathbf{u}_4 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_4 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 - \left(\frac{\mathbf{u}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\right)\mathbf{v}_3$$

Normal Vector & 3 Points

Non-collinear
3 points



References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[4] D.G. Zill, “Advanced Engineering Mathematics”