Complex Trig & TrigH (H.1)

20160901

Analyticity

e^{iz}, e^{-iz} entire function

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$ entire function

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$ entire function

$\sin z = 0$ only for real numbers $z = n\pi$

$\cos z = 0$ only for real numbers $z = (2n+1)\pi/2$

$\tan z = \dfrac{\sin z}{\cos z}$ $\quad \sec z = \dfrac{1}{\cos z}$ analytic except $z = (2n+1)\pi/2$

$\cot z = \dfrac{\cos z}{\sin z}$ $\quad \csc z = \dfrac{1}{\sin z}$ analytic except $z = n\pi$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\frac{d}{dz}\sin z = \frac{ie^{iz} + ie^{-iz}}{2i} = \cos z$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\frac{d}{dz}\cos z = \frac{ie^{iz} - ie^{-iz}}{2} = -\sin z$$

$$\tan z = \frac{\sin z}{\cos z}$$

$$\frac{d}{dz}\tan z = \frac{\cos^2 z + \sin^2 z}{\cos^2 z} = \sec^2 z$$

$$\cot z = \frac{\cos z}{\sin z}$$

$$\frac{d}{dz}\cot z = \frac{-\sin^2 z - \cos^2 z}{\sin^2 z} = -\csc^2 z$$

$$\sec z = \frac{1}{\cos z}$$

$$\frac{d}{dz}\sec z = \frac{\sin z}{\cos^2 z} = \sec z \tan z$$

$$\csc z = \frac{1}{\sin z}$$

$$\frac{d}{dz}\csc z = \frac{-\cos z}{\sin^2 z} = -\csc z \cot z$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\sin(-z) = \frac{e^{-iz} - e^{+iz}}{2i} = -\sin z$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\cos(-z) = \frac{e^{-iz} + e^{+iz}}{2} = \cos z$$

$$\tan z = \frac{\sin z}{\cos z}$$

$$\tan(-z) = \frac{-\sin z}{\cos z} = -\tan z$$

$$\sin^2 z = \frac{e^{i2z} + e^{-i2z} - 2}{-4}$$

$$\cos^2 z = \frac{e^{i2z} + e^{-i2z} + 2}{+4}$$

$$\sin^2 z + \cos^2 z = 1$$

$$\cos(z_1+z_2) + i\sin(z_1+z_2) = e^{i(z_1+z_2)}$$

$$e^{iz_1} \cdot e^{iz_2}$$
$$= [\cos(z_1) + i\sin(z_1)][\cos(z_2) + i\sin(z_2)]$$
$$= [\cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2)] + i[\cos(z_1)\sin(z_2) + \sin(z_1)\cos(z_2)]$$

$$\cos(z_1 \pm z_2) = [\cos(z_1)\cos(z_2) \mp \sin(z_1)\sin(z_2)]$$
$$\sin(z_1 \pm z_2) = [\sin(z_1)\cos(z_2) \pm \cos(z_1)\sin(z_2)]$$

$$\sin(z+z) = \sin(z)\cos(z) + \cos(z)\sin(z)$$
$$\sin(2z) = 2\sin(z)\cos(z)$$

$$\cos(z+z) = \cos(z)\cos(z) - \sin(z)\sin(z)$$
$$\cos(2z) = \cos^2(z) - \sin^2(z)$$

$$\sinh y = \frac{e^{y} - e^{-y}}{2}$$

$$\cosh y = \frac{e^{y} + e^{-y}}{2}$$

$$\sin(z) = \sin(x+iy) = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$2 * \left[\boxed{e^{ix}\, e^{-y}} - \boxed{e^{-ix}\, e^{y}} \right]$$

$$= \left[\begin{array}{c} \boxed{e^{ix}\, e^{y}} + \boxed{e^{ix}\, e^{-y}} \\ - \boxed{e^{-ix}\, e^{y}} - \boxed{e^{-ix}\, e^{-y}} \end{array} \right] - \left[\begin{array}{c} \boxed{e^{ix}\, e^{y}} - \boxed{e^{ix}\, e^{-y}} \\ \boxed{e^{-ix}\, e^{y}} - \boxed{e^{-ix}\, e^{-y}} \end{array} \right]$$

$$= (e^{ix} - e^{-ix})(e^{y} + e^{-y}) - (e^{ix} + e^{-ix})(e^{y} - e^{-y})$$

$$\sin(x+iy) = \frac{(e^{ix} - e^{-ix})}{2i} \frac{(e^{y} + e^{-y})}{2} - \frac{(e^{ix} + e^{-ix})}{2i} \frac{(e^{y} - e^{-y})}{2}$$

$$\boxed{\sin(x+iy) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)}$$

$$\sinh y = \frac{e^{y} - e^{-y}}{2}$$

$$\cosh y = \frac{e^{y} + e^{-y}}{2}$$

$$\cos(z) = \cos(x+iy) = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2}$$

$$2 * \left[\boxed{e^{ix} e^{-y}} + \boxed{e^{-ix} e^{y}} \right]$$

$$= \left[\begin{array}{c} \boxed{e^{ix} e^{y}} + \boxed{e^{ix} e^{-y}} \\ \boxed{e^{-ix} e^{y}} + \boxed{e^{-ix} e^{-y}} \end{array} \right] - \left[\begin{array}{c} \boxed{e^{ix} e^{y}} - \boxed{e^{ix} e^{-y}} \\ - \boxed{e^{-ix} e^{y}} + \boxed{e^{-ix} e^{-y}} \end{array} \right]$$

$$= (e^{ix} + e^{-ix})(e^{y} + e^{-y}) - (e^{ix} - e^{-ix})(e^{y} - e^{-y})$$

$$\cos(x+iy) = \frac{(e^{ix} + e^{-ix})}{2} \frac{(e^{y} + e^{-y})}{2} - \frac{(e^{ix} - e^{-ix})}{2} \frac{(e^{y} - e^{-y})}{2}$$

$$\boxed{\cos(x+iy) = \cos(x) \cosh(y) - i \sin(x) \sinh(y)}$$

$$\sinh y = \frac{e^y - e^{-y}}{2} \qquad \sinh^2 y = \frac{1}{4}\left(e^{2y} + e^{-2y} - 2\right)$$

$$\cosh y = \frac{e^y + e^{-y}}{2} \qquad \cosh^2 y = \frac{1}{4}\left(e^{2y} + e^{-2y} + 2\right)$$

$$\cosh^2 y - \sinh^2 y = 1$$

$$\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

$$|\sin(x+iy)|^2 = \underline{\sin^2(x)\cosh^2(y)} + \underline{\cos^2(x)\sinh^2(y)}$$

$$(1-\sin^2(x))(\cosh^2 y - 1)$$

$$\cosh^2 y - \sin^2(x)\cosh^2 y - 1 + \sin^2(x)$$

$$\underline{-\sin^2(x)\cosh^2 y} + \cosh^2 y - 1 + \sin^2(x)$$

$$|\sin z|^2 = \sin^2(x) + \sinh^2(y)$$

$$\cos(x+iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

$$|\cos(x+iy)|^2 = \underline{\cos^2(x)\cosh^2(y)} + \sin^2(x)\sinh^2(y)$$

$$(1-\cos^2(x))(\cosh^2 y - 1)$$

$$\cosh^2 y - \cos^2(x)\cosh^2 y - 1 + \cos^2(x)$$

$$\underline{-\cos^2(x)\cosh^2 y} + \cosh^2 y - 1 + \cos^2(x)$$

$$|\cos z|^2 = \cos^2(x) + \sinh^2(y)$$

$$\sin(x+iy) = \sin(x)\boxed{\cosh(y)} + i\cos(x)\boxed{\sinh(y)}$$

$$\cos(x+iy) = \cos(x)\boxed{\cosh(y)} - i\sin(x)\boxed{\sinh(y)}$$

$$|\sin z|^2 = \sin^2(x) + \boxed{\sinh^2(y)}$$

$$|\cos z|^2 = \cos^2(x) + \boxed{\sinh^2(y)}$$

a complex number $z=0 \rightleftarrows |z|^2=0$

$$|\sin z|^2 = \sin^2(x) + \boxed{\sinh^2(y)}$$

$$|\cos z|^2 = \cos^2(x) + \boxed{\sinh^2(y)}$$

$$\sin z = 0 \quad \rightleftarrows \quad \sin^2(x) + \sinh^2(y) = 0$$

$$\sin(x) = 0 \qquad x = n\pi$$

$$\sinh(y) = 0 \qquad y = 0$$

zero $\quad z = n\pi + i\cdot 0 = n\pi, \quad n = 0, \pm 1, \pm 2, \cdots$

$$\cos z \quad \rightleftarrows \quad \cos^2(x) + \sinh^2(y)$$

$$\cos(x) = 0 \qquad x = (2n+1)\frac{\pi}{2}$$

$$\sinh(y) = 0 \qquad y = 0$$

zero $\quad z = (2n+1)\frac{\pi}{2} + i\cdot 0 = (n+\frac{1}{2})\pi, \quad n = 0, \pm 1, \pm 2, \cdots$

for a complex number $z = x + iy$

$$\sinh z = \frac{e^{z} - e^{-z}}{2} \qquad \sinh y = \frac{e^{y} - e^{-y}}{2}$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2} \qquad \cosh y = \frac{e^{y} + e^{-y}}{2}$$

$$\tanh z = \frac{\sinh z}{\cosh z}$$

$$\coth z = \frac{\cosh z}{\sinh z}$$

$$\operatorname{sech} z = \frac{1}{\cosh z}$$

$$\operatorname{csch} z = \frac{1}{\sinh z}$$

$$\sinh z = \frac{e^{z} - e^{-z}}{2} \qquad \frac{d}{dz}\sinh z = \frac{e^{z} + e^{-z}}{2} = \cosh z$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2} \qquad \frac{d}{dz}\cosh z = \frac{e^{z} - e^{-z}}{2} = \sinh z$$

$$\tanh z = \frac{\sinh z}{\cosh z}$$

$$\frac{d}{dz}\tanh z = \frac{\cosh^2 z - \sinh^2 z}{\cosh^2 z} = \operatorname{sech}^2 z$$

$$\coth z = \frac{\cosh z}{\sinh z}$$

$$\frac{d}{dz}\coth z = \frac{\sinh^2 z - \cosh^2 z}{\sinh^2 z} = -\operatorname{csch}^2 z$$

$$\operatorname{sech} z = \frac{1}{\cosh z}$$

$$\frac{d}{dz}\operatorname{sech} z = \frac{-\sinh z}{\cosh^2 z} = -\tanh z \operatorname{sech} z$$

$$\operatorname{csch} z = \frac{1}{\sinh z}$$

$$\frac{d}{dz}\operatorname{csch} z = \frac{-\cosh z}{\sinh^2 z} = -\coth z \operatorname{csch} z$$

$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2}$$

$$\sinh iz = \frac{e^{iz} - e^{-iz}}{2} = i \sin z$$

$$\cosh iz = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

$$\sinh iz = i \sin z \qquad \sin z = -i \sinh iz$$

$$\cosh iz = \cos z \qquad \cos z = \cosh iz$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

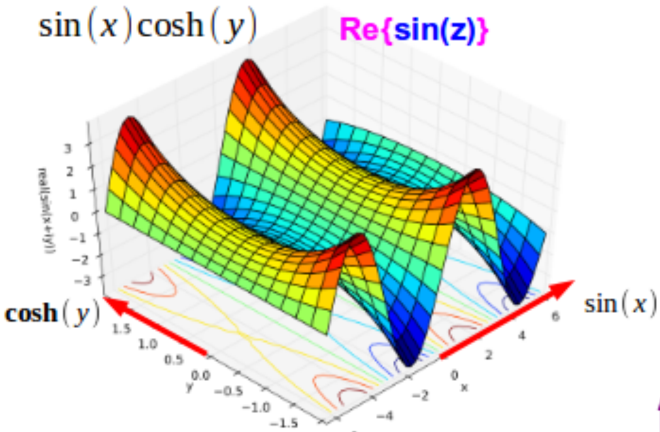
$$\sin iz = \frac{e^{-z} - e^{z}}{2i} = -\frac{1}{i} \sinh z$$

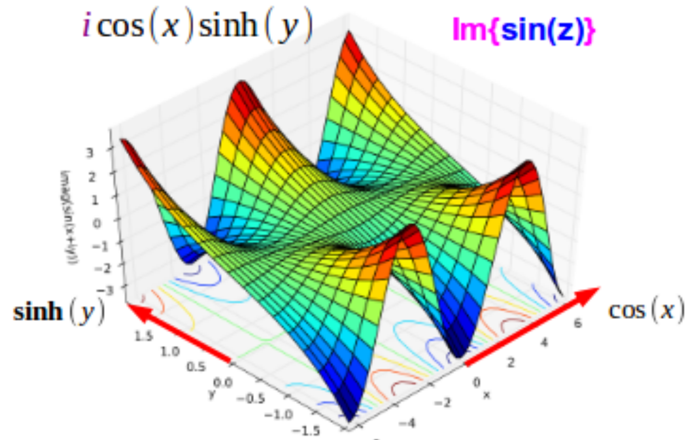
$$\cos iz = \frac{e^{-z} + e^{z}}{2} = \cosh z$$

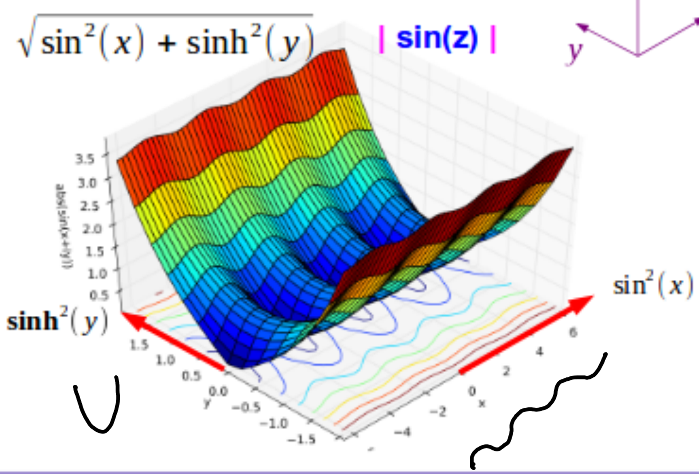
$$\sin iz = i \sinh z \qquad \sinh z = -i \sin iz$$

$$\cos iz = \cosh z \qquad \cosh z = \cos iz$$

Graphs of sin(z)







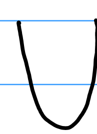
$$\sin(z) = \sin(x + iy)$$

$$= \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

$$|\sin(z)|^2 = \sin^2(x) + \sinh^2(y)$$

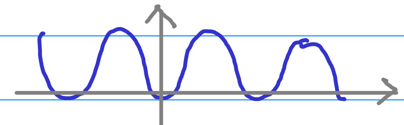
http://en.wikipedia.org/

$$\sinh^2(y) = \frac{1}{4}\left(e^{+2y} + e^{-2y} - 2\right)$$

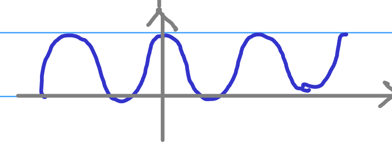


$$\tan\theta = \frac{\cos(x)\ \sinh(y)}{\sin(x)\ \cosh(y)} = \cot x \tanh y$$

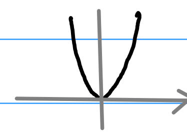
$$\sin^2 x = \frac{1}{2}\left(1 - \cos(2x)\right)$$



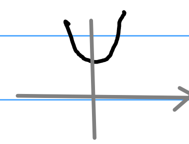
$$\cos^2 x = \frac{1}{2}\left(1 + \cos(2x)\right)$$



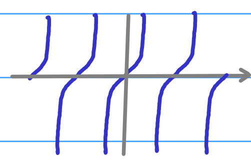
$$\sinh^2(x) = \frac{1}{4}\left(e^{+2x} + e^{-2x} - 2\right)$$



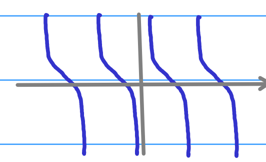
$$\cosh^2(x) = \frac{1}{4}\left(e^{+2x} + e^{-2x} + 2\right)$$



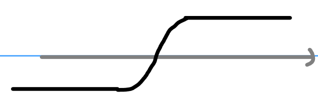
$\tan(x)$



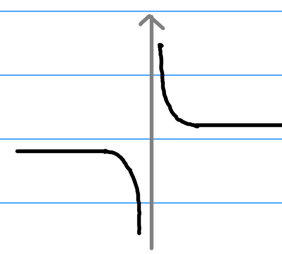
$\cot(x)$



$\tanh(x)$



$\coth(x)$



$$\sin^2 x = \frac{1}{2}\left(1 - \cos(2x)\right)$$

Zero $2x = 0, \pm 2\pi, \pm 4\pi, \cdots$

$x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \cdots$

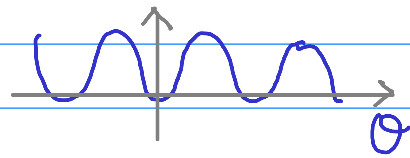
$$\cos^2 x = \frac{1}{2}\left(1 + \cos(2x)\right)$$

Zero $2x = \pm\pi, \pm 3\pi, \pm 5\pi, \cdots$

$x = \pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \cdots$

* $\sin^2(\arg z)$ plot

$$\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$$



arg $\theta = 0, 2\pi \rightarrow \sin^2\theta = 0$ dominantly real

$\theta = \frac{\pi}{2}, \frac{3}{2}\pi \rightarrow \sin^2\theta = 1$ dominantly imag

http://functions.wolfram.com/ElementaryFunctions/Sin/visualizations/5/

red $\theta = \pm 2n\pi$

cyan $\theta = \pm(2n+1)\pi$

the square of the sine of the argument of $\sin(z)$

plot $\boxed{\sin^2\theta}$, $\tan\theta = \cot(x)\tanh(y)$

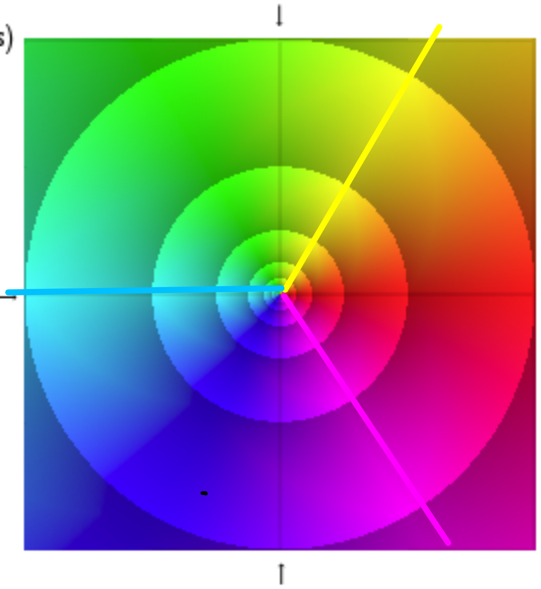
$$\theta = \arg\{\sin(z)\} = \tan^{-1}\{\cot(x)\tanh(y)\}$$

Domain Coloring

The Unit Circle

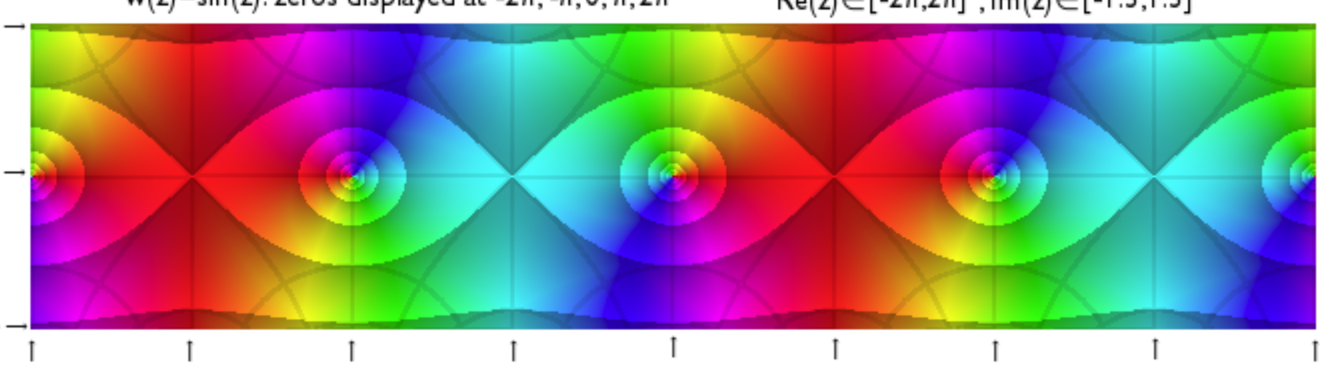
hue to phase/angle/argument legend:

hue	phase (radians)
red	0 mod 2π
yellow	π/3 mod 2π
green	2π/3 mod 2π
cyan	π mod 2π
blue	4π/3 mod 2π
magenta	5π/3 mod 2π

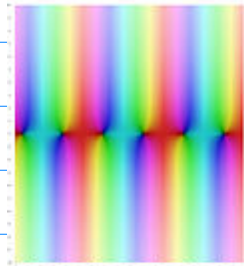


Each discontinuity in intensity occurs when |z|=2^n, for integer n (0,-1,-2,..)

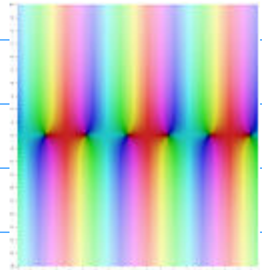
w(z)=sin(z). zeros displayed at -2π, -π, 0, π, 2π   Re(z)∈[-2π,2π] , Im(z)∈[-1.5,1.5]

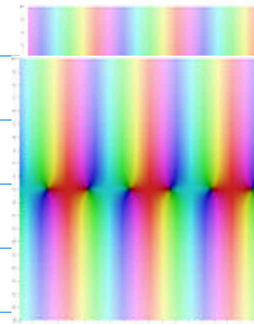


sin z



cos z

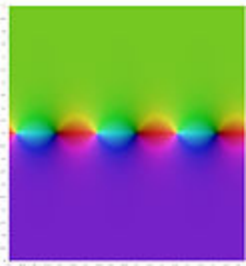




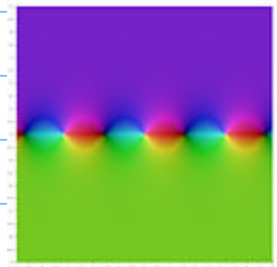
sin z

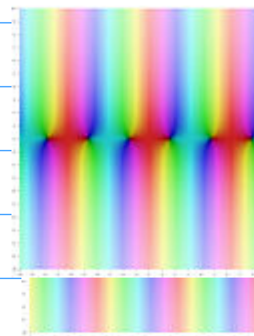
cos z

tan z



cot z

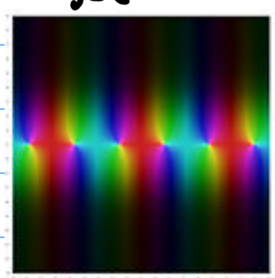




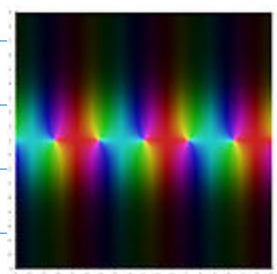
sin z

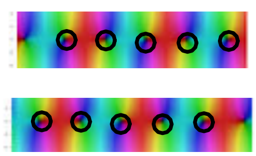
cos z

sec z



csc z





sin z

cos z

https://en.wikipedia.org/wiki/Trigonometric_functions

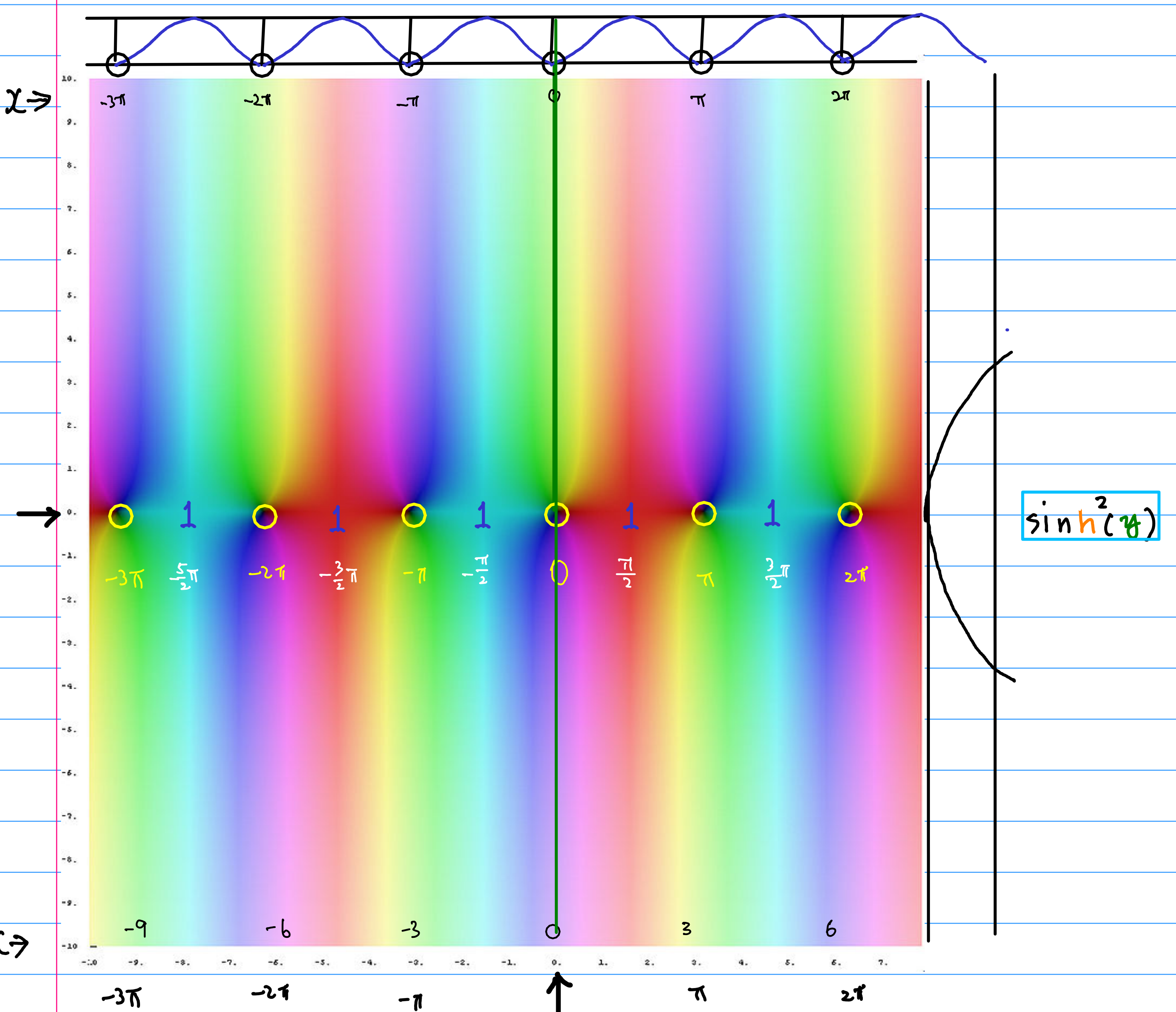
① $|\sin z|$ brightness

$$|\sin z|^2 = \sin^2(x) + \boxed{\sinh^2(y)}$$

zero $\longleftarrow$ $y=0$ & $x = 0, \pm\pi, \pm 2\pi, \cdots$ $\quad \sinh(0)=0$

1 $\longleftarrow$ $y=0$ & $x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \cdots$ $\quad \sinh(0)=0$

$\sin^2(x)$



① $\arg(\sin z)$

$$\sin(x+iy) = \sin(x)\boxed{\cosh(y)} + i\cos(x)\boxed{\sinh(y)}$$

$$\theta = \arg\{\sin(z)\}$$

$$\tan\theta = \cot(x)\tanh(y) \Leftarrow \frac{\cos(x)\boxed{\sinh(y)}}{\sin(x)\boxed{\cosh(y)}}$$

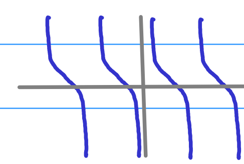
$$\begin{cases} \cot(x) = \pm\infty \\ \tan\theta = \pm\infty \end{cases} \qquad \begin{aligned} x &= 0, \pm\pi, \pm 2\pi, \cdots \\ \theta &= \pm\tfrac{\pi}{2}, \pm\tfrac{3}{2}\pi, \cdots \end{aligned}$$

$$\begin{cases} \cot(x) = 0 \\ \tan\theta = 0 \end{cases} \qquad \begin{aligned} x &= \pm\tfrac{\pi}{2}, \pm\tfrac{3}{2}\pi, \cdots \\ \theta &= 0, \pm\pi, \pm 2\pi, \cdots \end{aligned}$$

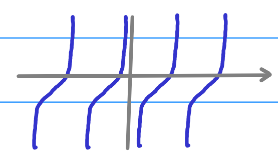
$$\tanh(y) = \begin{cases} +1 & (y>1) \\ -1 & (y<-1) \end{cases}$$

$$\tan\theta = \begin{cases} +\cot(x) & (y>1) \\ -\cot(x) & (y<-1) \end{cases}$$

$$\tan\theta = +\cot(x)$$



$$\tan\theta = -\cot(x)$$



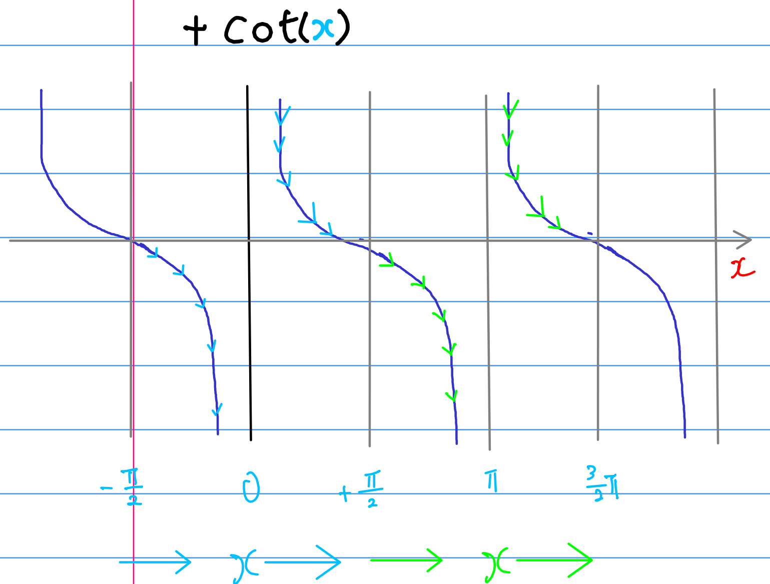
$$\boxed{\theta = \arg\{\sin(z)\}}$$

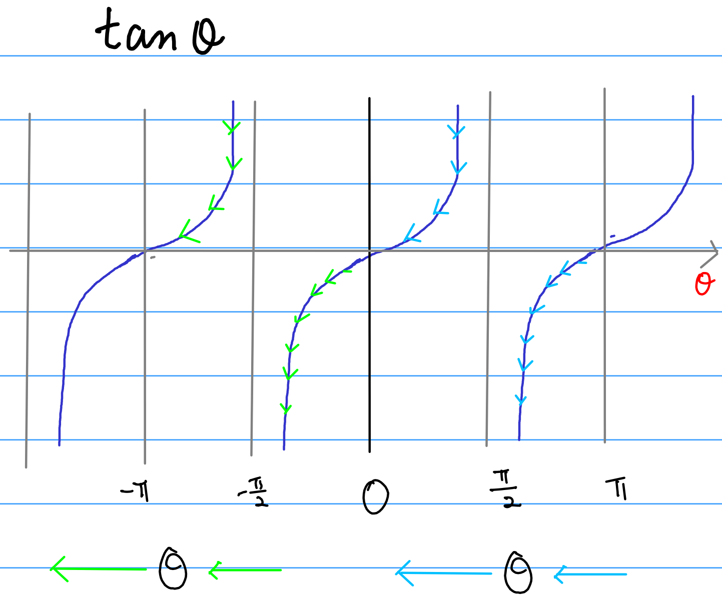
$\tan\theta = 0 \qquad x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \cdots$

$\tan\theta = \pm\infty \qquad x = 0, \pm\pi, \pm 2\pi, \cdots$

① $\arg(\sin z)$ $\tan\theta = +\cot(x)$ $(y>1)$

2

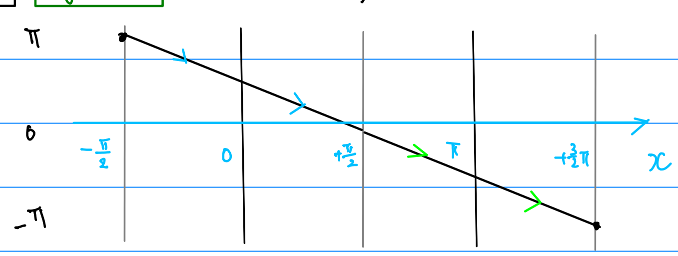




$x \in [-\frac{\pi}{2}, \frac{3}{2}\pi]$ $\longrightarrow$

$\theta \in [-\pi, +\pi]$ $\longleftarrow$

$\boxed{\theta \text{ trend}}$ $\boxed{y>1}$ $\tan\theta = \cot(x)$



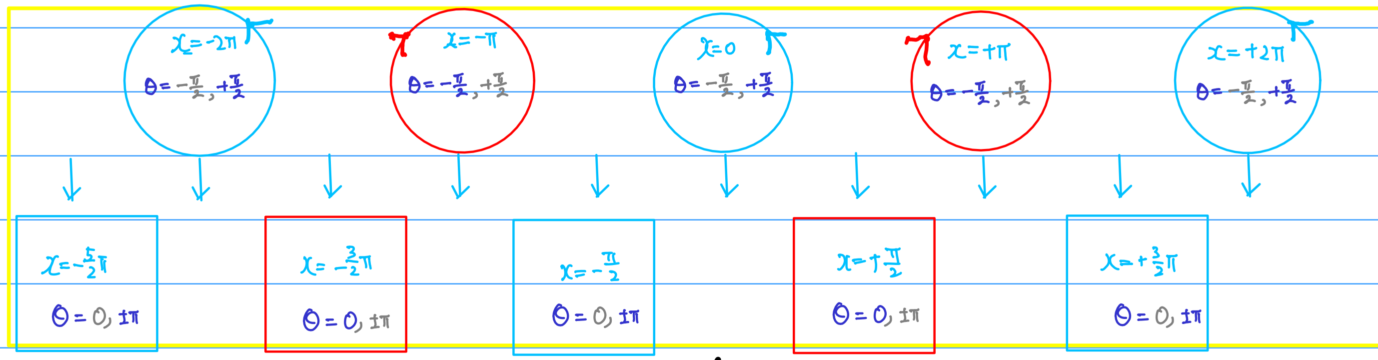
① $\arg(\sin z)$ $\tan\theta = \begin{cases} +\cot(x) & y>1 \\ -\cot(x) & y<1 \end{cases}$

3

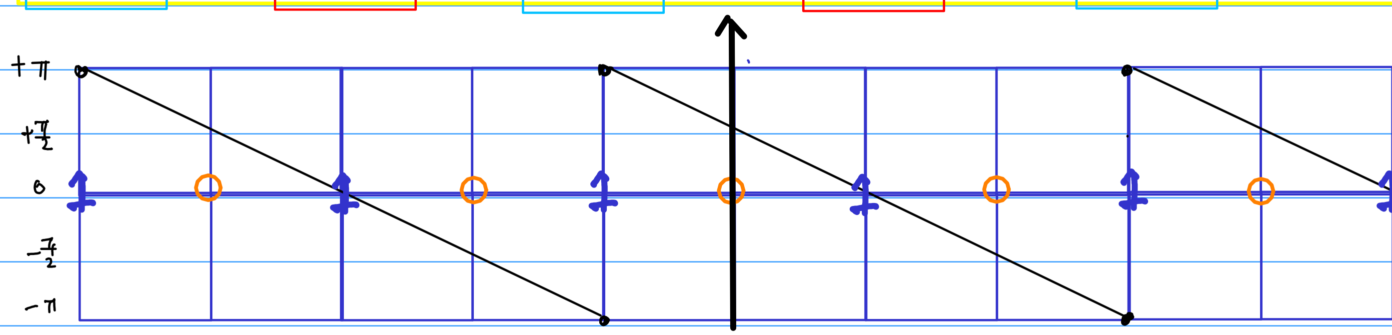
$$\begin{cases} \cot(x) = 0 \\ \tan\theta = 0 \end{cases} \qquad x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \cdots \qquad \theta = 0, \pm\pi, \pm 2\pi, \cdots$$

$$\begin{cases} \cot(x) = \pm\infty \\ \tan\theta = \pm\infty \end{cases} \qquad x = 0, \pm\pi, \pm 2\pi, \cdots \qquad \theta = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \cdots$$

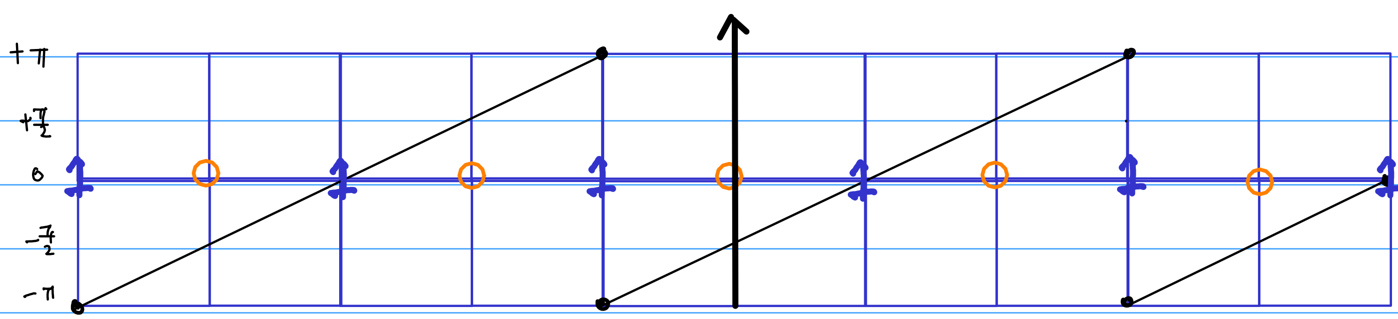
$y>1$



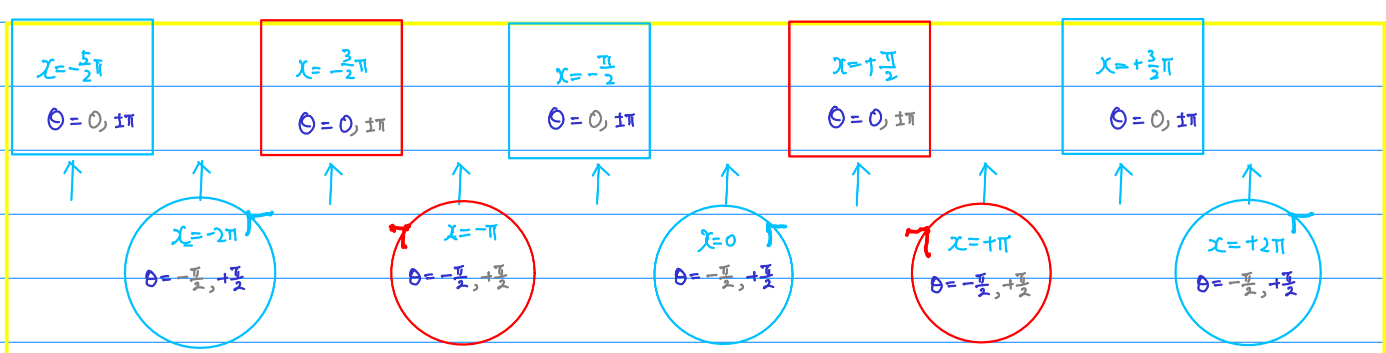
θ trend with x varying



θ trend with x varying



$y<-1$

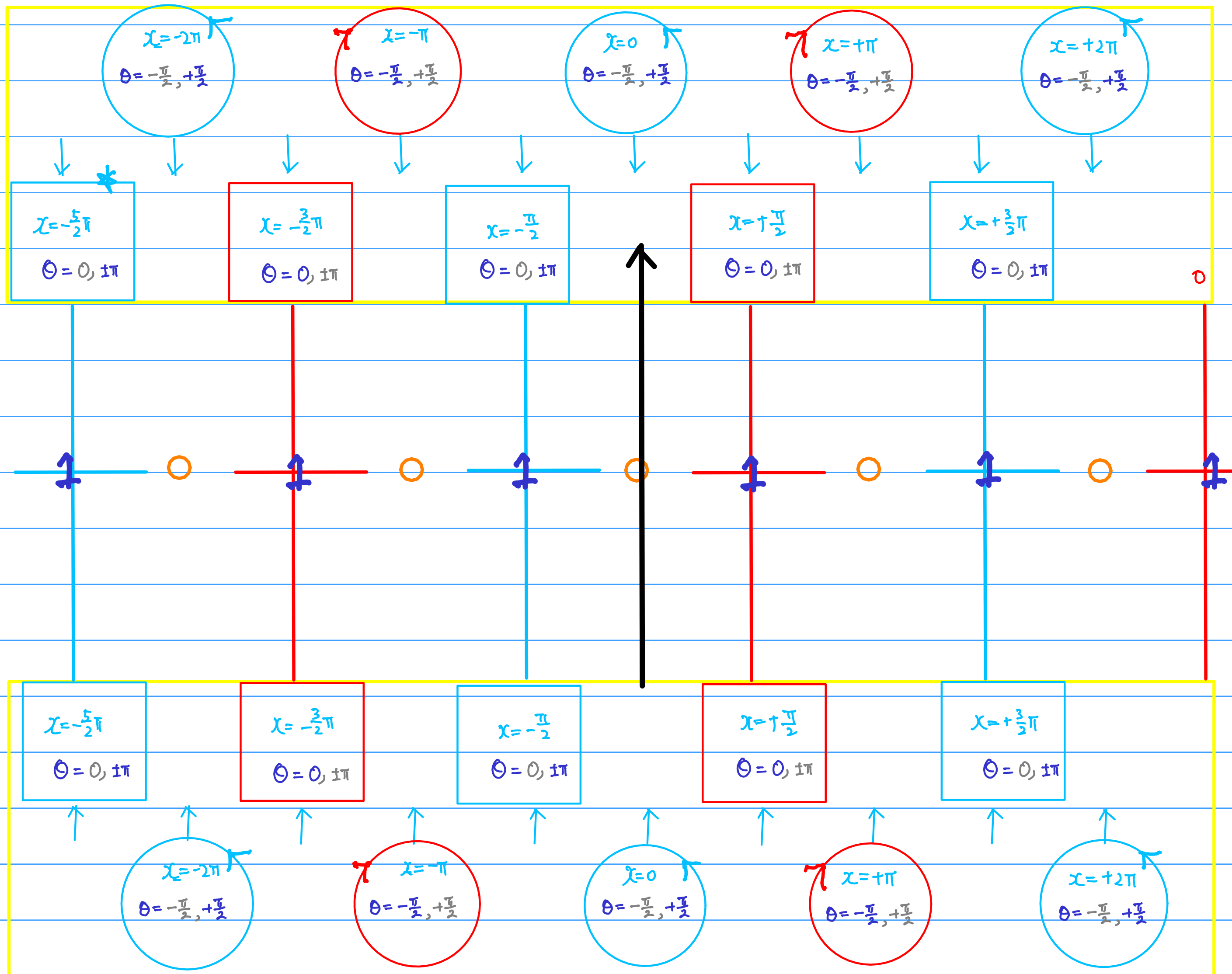


① $\arg(\sin z)$ $\theta = 0$ & $\theta = \pm\pi$ $|y|>1$

4

$y>1$

$y<-1$

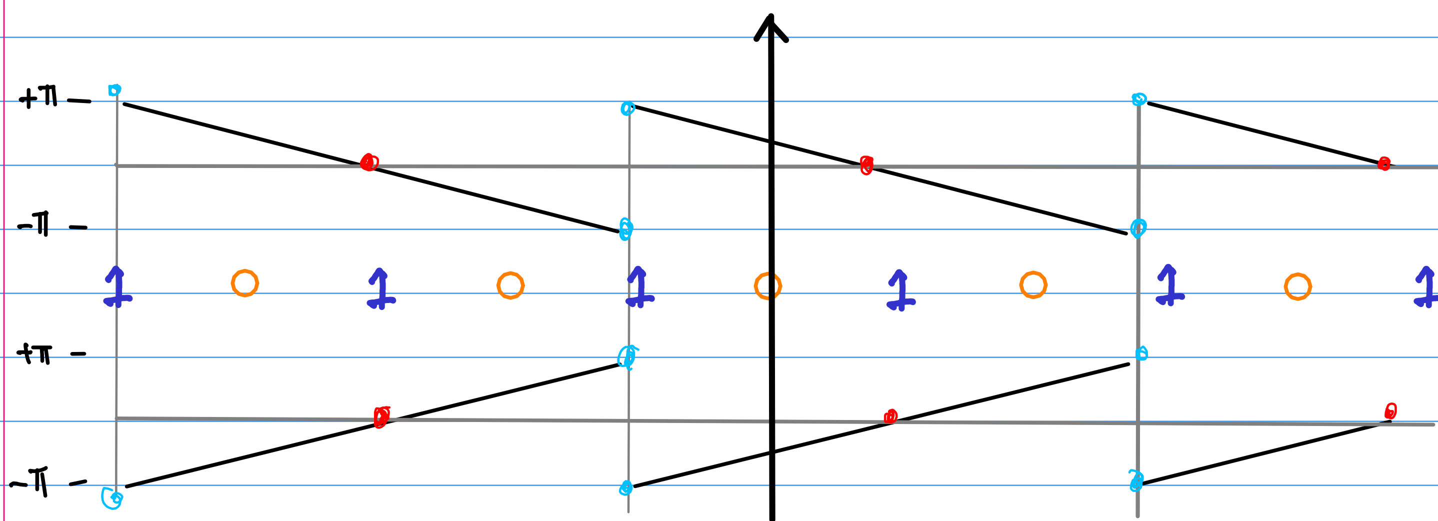


discontinuous θ ← $x = \dots -\frac{5}{2}\pi, -\frac{\pi}{2}, +\frac{3}{2}\pi, +\frac{7}{2}\pi \dots$

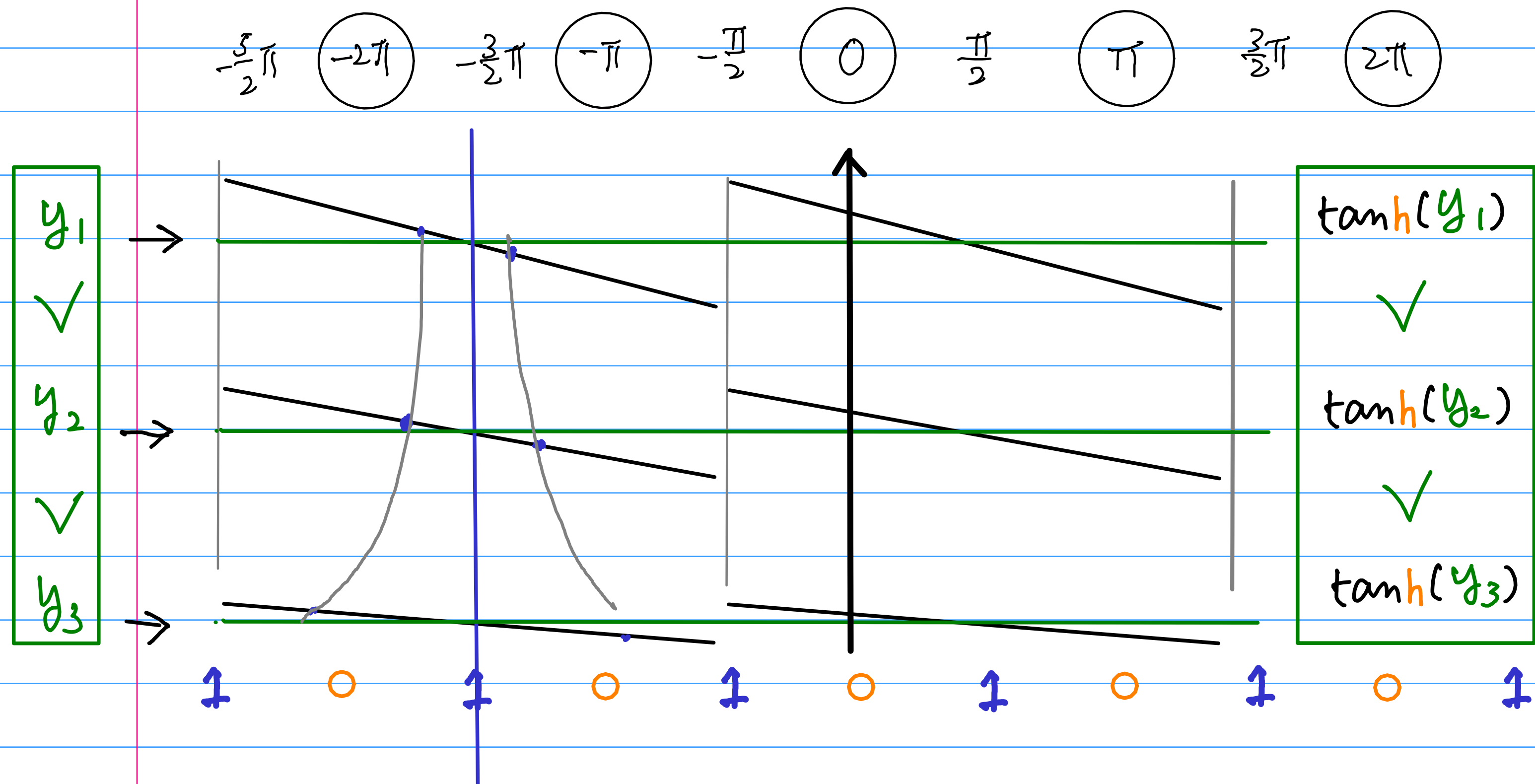
$y>1$

θ trend with x varying

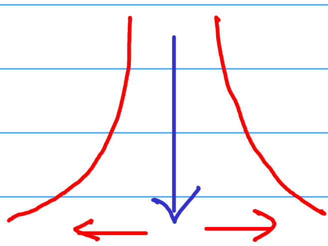
$y<-1$



① arg(sin z) θ shifts as $y \leftarrow 0$ $0 < y < 1$



$$\boxed{\tan\theta = \cot(x)\tanh(y)}$$



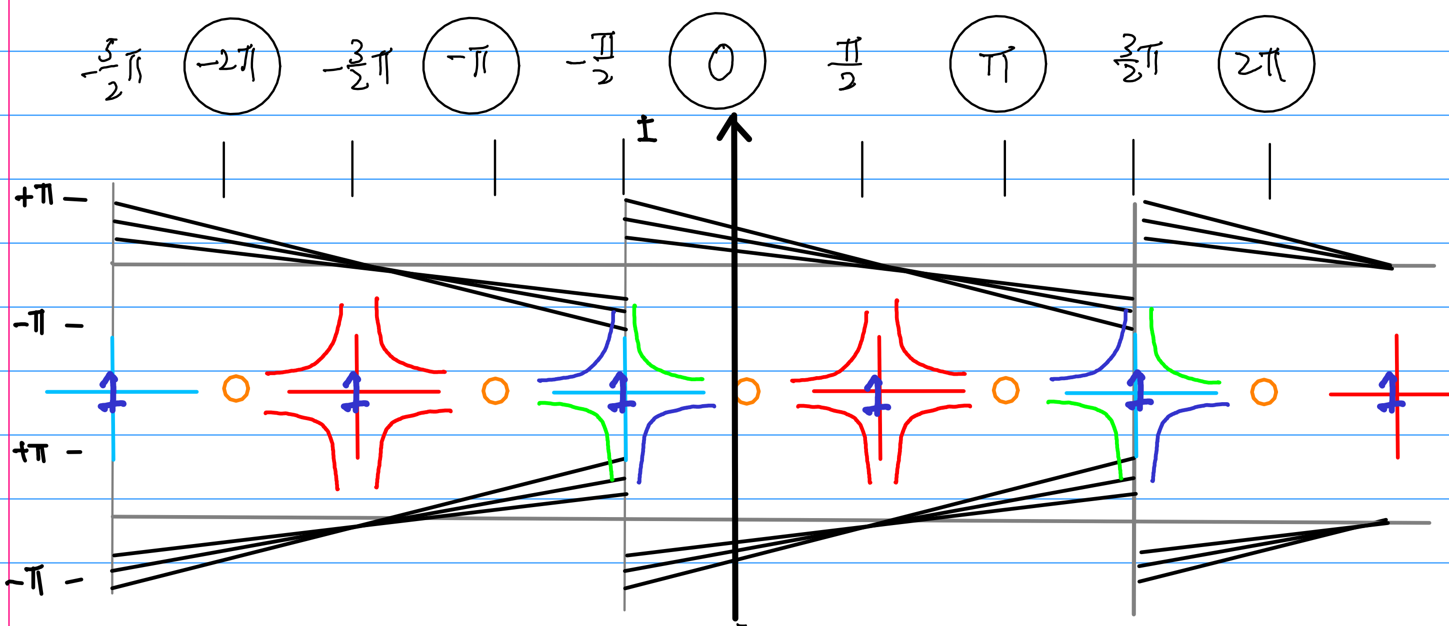
move away
as $y \leftarrow 0$

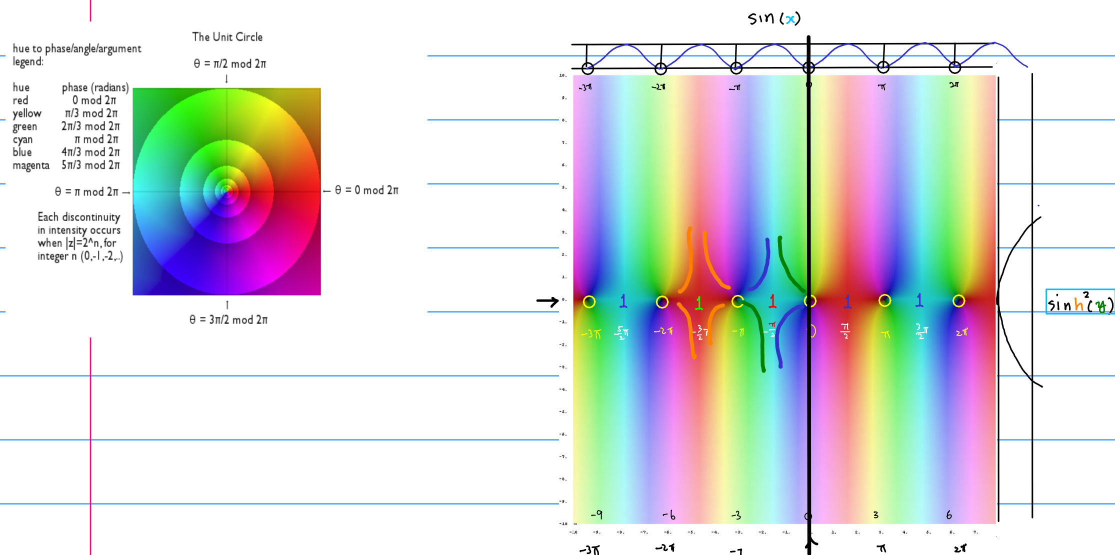
make the slope smaller

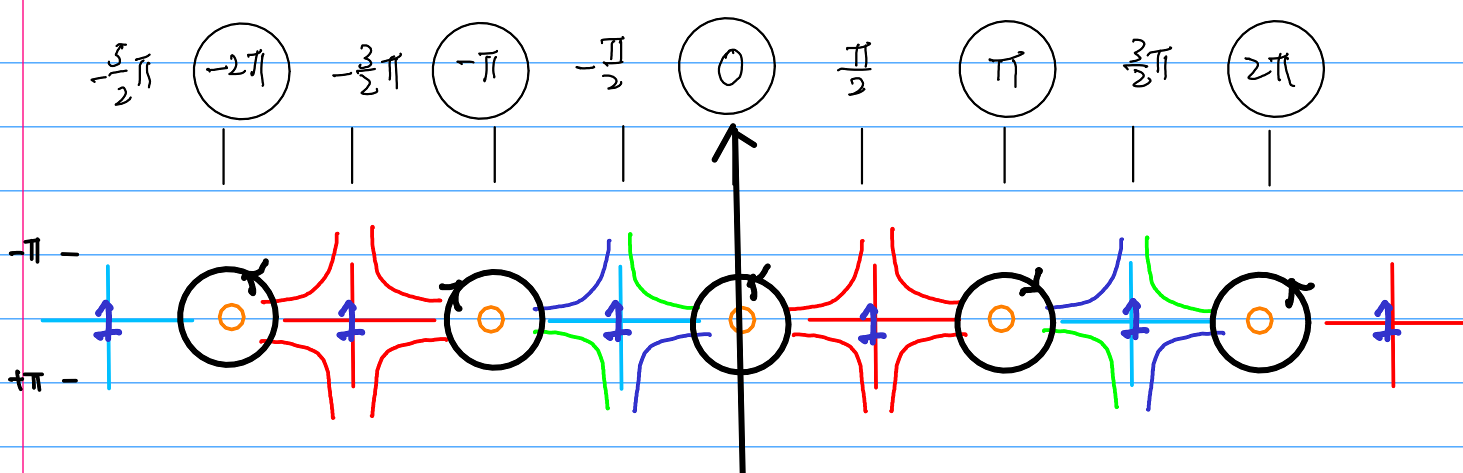
$1 > y_1 > y_2 > y_3$
$1 > \tanh(y_1) > \tanh(y_2) > \tanh(y_3)$

① ang(sin z) CCW / CW at zeros

6



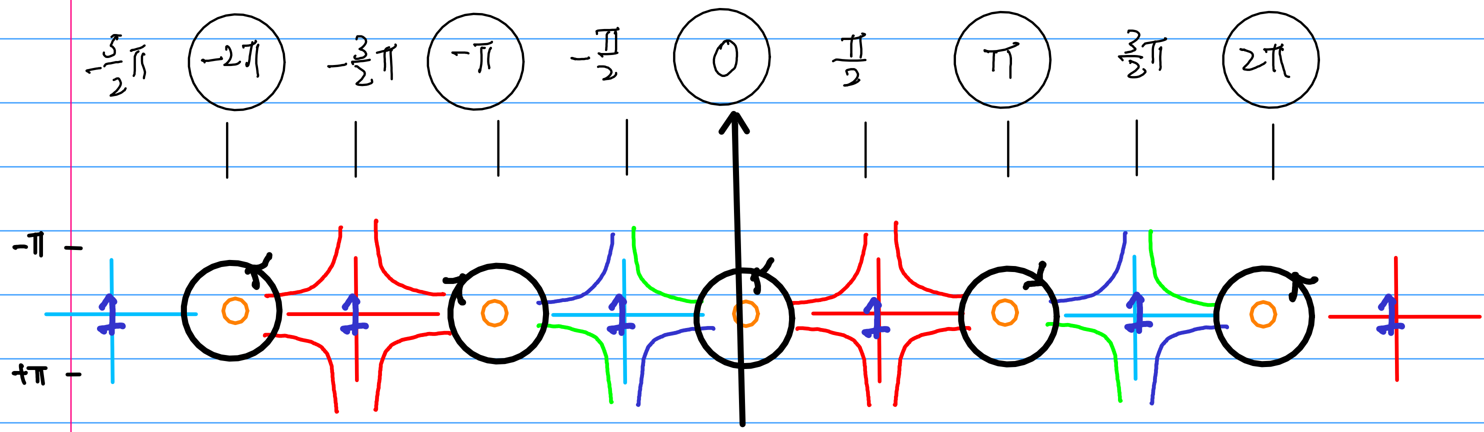




① $\arg(\sin z)$

$$\begin{cases} \text{CCW} & (x = 0 + 2n\pi) \\ \text{CW} & (x = \pi + 2n\pi) \end{cases}$$

7



zeros	$-\frac{5}{2}\pi$	-2π	$-\frac{3}{2}\pi$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π
CCW	$-\frac{5}{2}\pi$	-2π	$-\frac{3}{2}\pi$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π
CW	$-\frac{5}{2}\pi$	-2π	$-\frac{3}{2}\pi$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π

* arrow toward near one 1

https://en.wikipedia.org/wiki/Trigonometric_functions

①₈ $\arg(\sin z)$

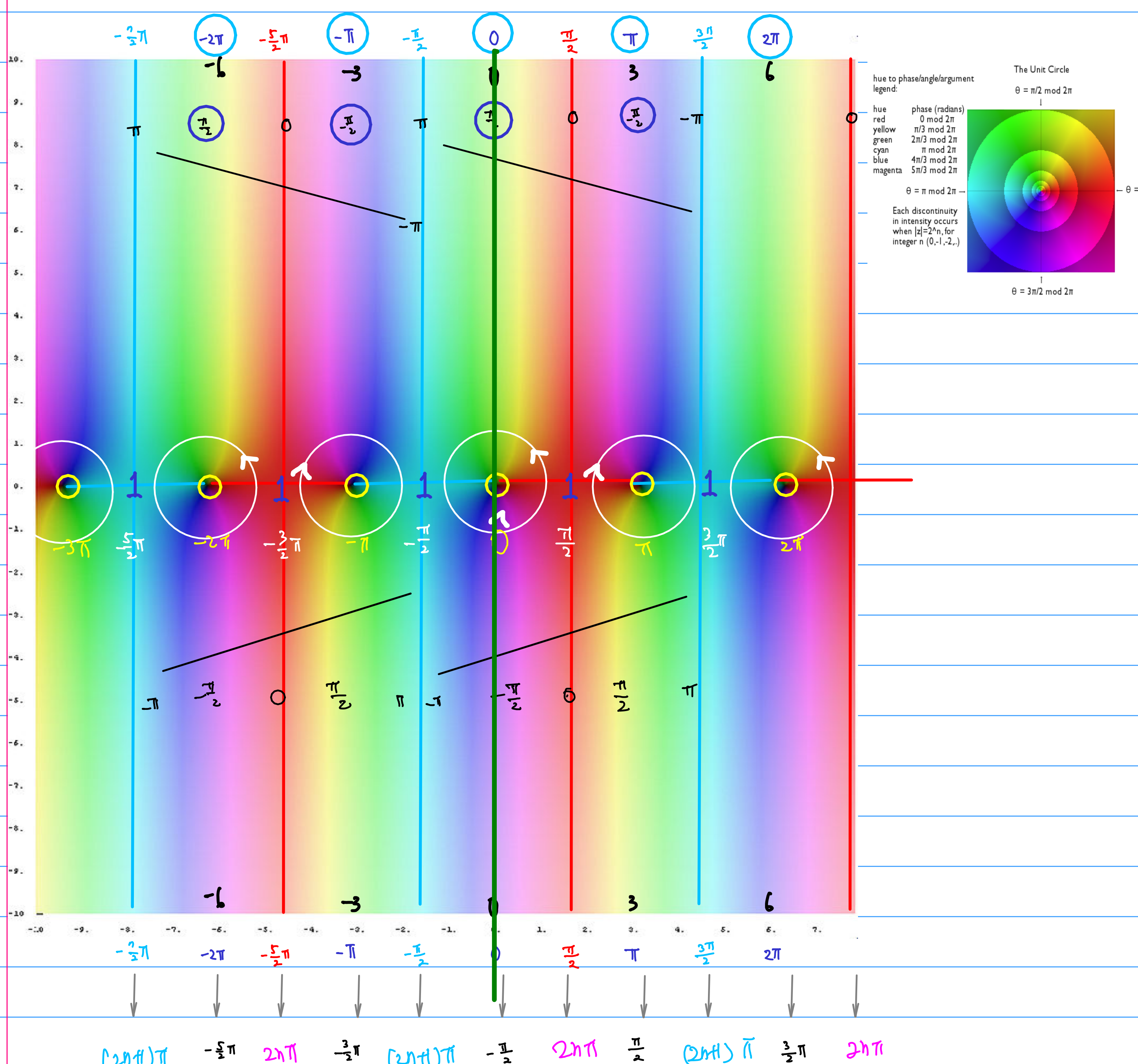
$$\tan\theta = \cot(x)\tanh(y)$$

$\cot(x) = \pm\infty$ $\quad x = 0, \pm\pi, \pm 2\pi, \cdots$

$\tan\theta = \pm\infty$ $\quad \boxed{\theta = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi}$

$\cot(x) = 0$ $\quad x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \cdots$

$\tan\theta = 0$ $\quad \boxed{\theta = 0, \pm\pi}$



② $|\cos z|$ brightness

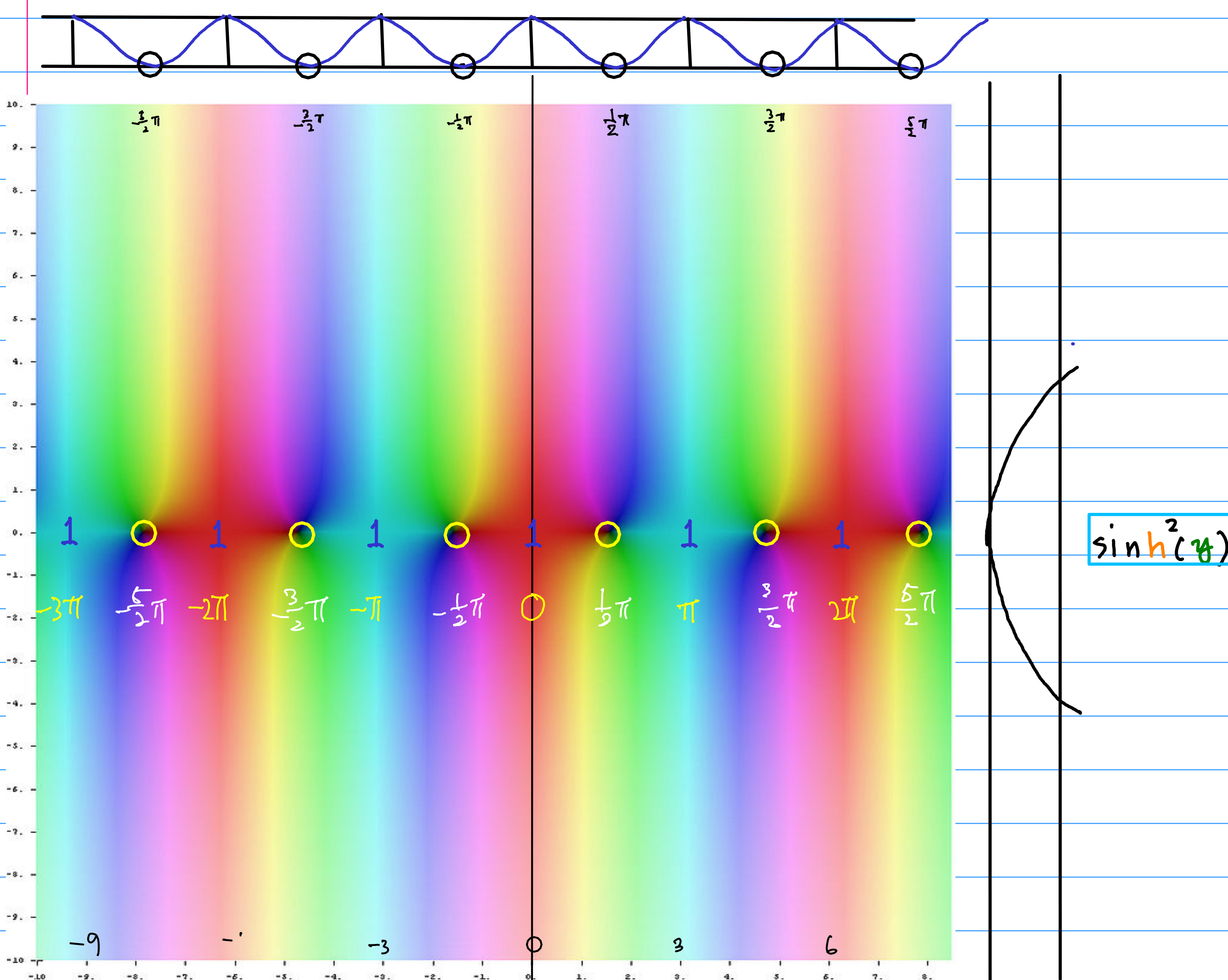
https://en.wikipedia.org/wiki/Trigonometric_functions

$$|\cos z|^2 = \cos^2(x) + \boxed{\sinh^2(y)}$$

zero $\longleftarrow$ $y=0$ & $x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \cdots$ $\sinh(0)=0$

1 $\longleftarrow$ $y=0$ & $x = 0, \pm\pi, \pm 2\pi, \cdots$ $\sinh(0)=0$

$\cos^2(x)$



② $\arg(\cos z)$

$$\cos(x+iy) = \cos(x)\boxed{\cosh(y)} - i\sin(x)\boxed{\sinh(y)}$$

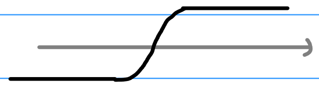
$$\theta = \arg\{\cos(z)\}$$

$$\tan\theta = -\tan(x)\tanh(y) \Leftarrow -\frac{\sin(x)\boxed{\sinh(y)}}{\cos(x)\boxed{\cosh(y)}}$$

$$\begin{cases} \tan(x) = \pm\infty & x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \cdots \\ \tan\theta = \pm\infty & \theta = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \cdots \end{cases}$$

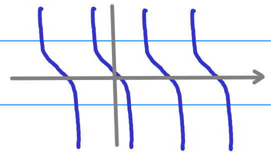
$$\begin{cases} \tan(x) = 0 & x = 0, \pm\pi, \pm 2\pi, \cdots \\ \tan\theta = 0 & \theta = 0, \pm\pi, \pm 2\pi, \cdots \end{cases}$$

$$\tanh(y) = \begin{cases} +1 & (y>1) \\ -1 & (y<-1) \end{cases}$$

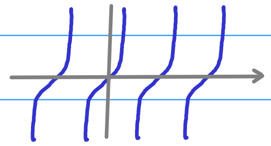


$$\tan\theta = \begin{cases} -\tan(x) & (y>1) \\ +\tan(x) & (y<-1) \end{cases}$$

$$\tan\theta = -\tan(x)$$



$$\tan\theta = +\tan(x)$$



$$\boxed{\theta = \arg\{\cos(z)\}}$$

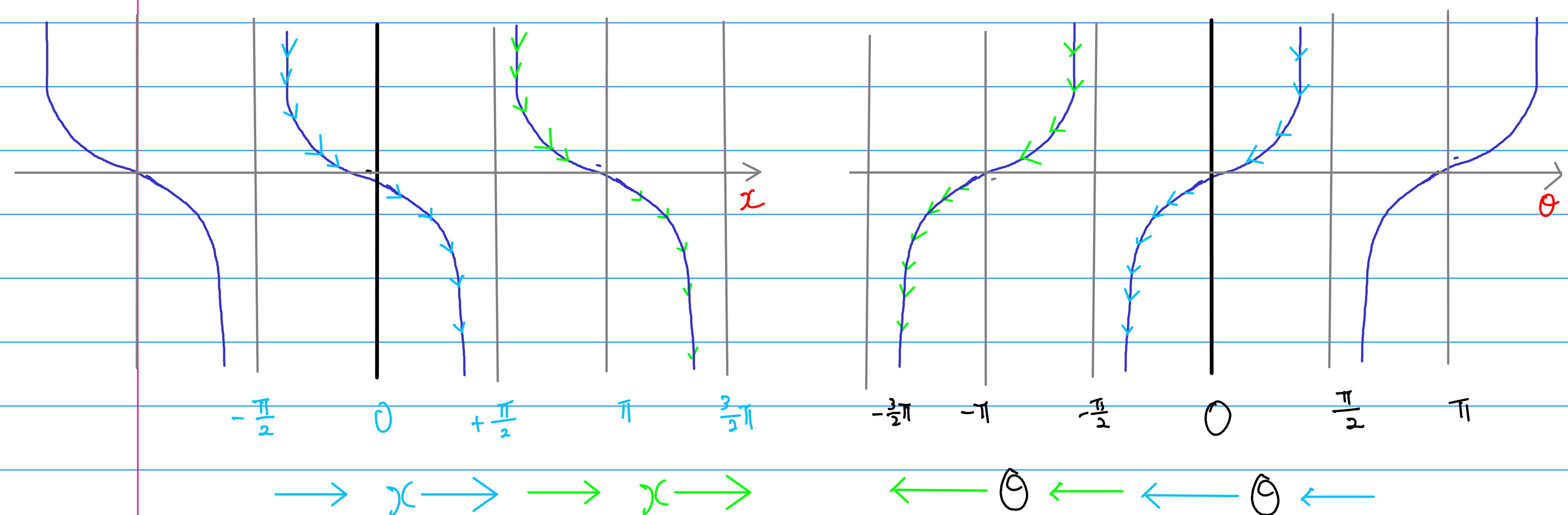
$\tan\theta = 0 \quad x = 0, \pm\pi, \pm 2\pi, \cdots$

$\tan\theta = \pm\infty \quad x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \cdots$

② $\arg(\cos z)$ $\tan\theta = -\tan(x)$ $(y>1)$

2

$-\tan(x)$ $\tan\theta$



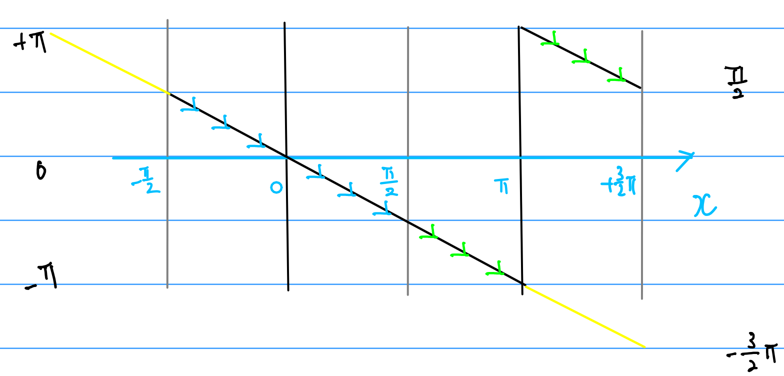
$x \in \left[-\frac{\pi}{2}, \frac{3}{2}\pi\right]$ $\longrightarrow$

$\theta \in \left[-\frac{3}{2}\pi, +\frac{\pi}{2}\right]$ $\longleftarrow$

$\left[-\frac{3\pi}{2}, -\pi\right] \cup \left[-\pi, \frac{\pi}{2}\right]$

$\left[\frac{\pi}{2}, \pi\right] \cup \left[-\pi, \frac{\pi}{2}\right]$

$\boxed{\theta \text{ trend}}$ $\boxed{y>1}$ $\tan\theta = -\tan(x)$



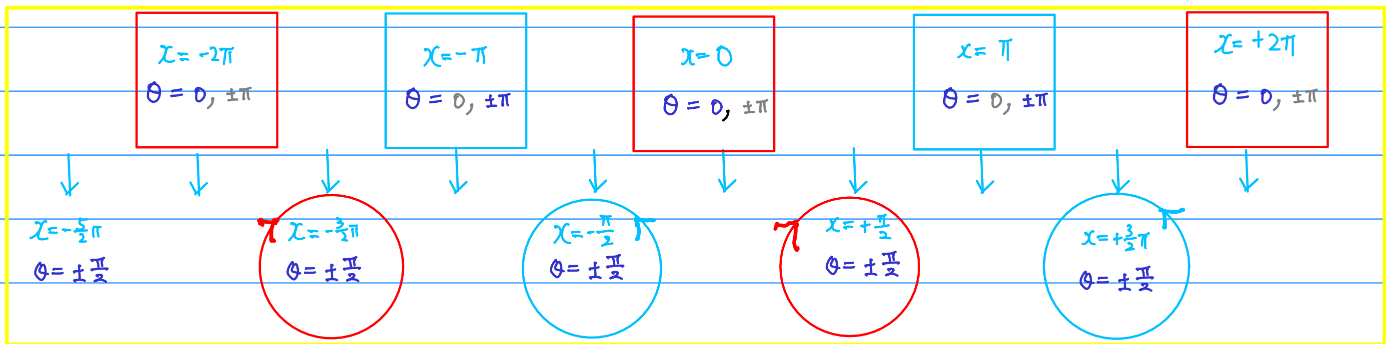
② $\arg(\cos z)$ $\tan\theta = \begin{cases} -\tan(x) & y>1 \\ +\tan(x) & y<1 \end{cases}$

3

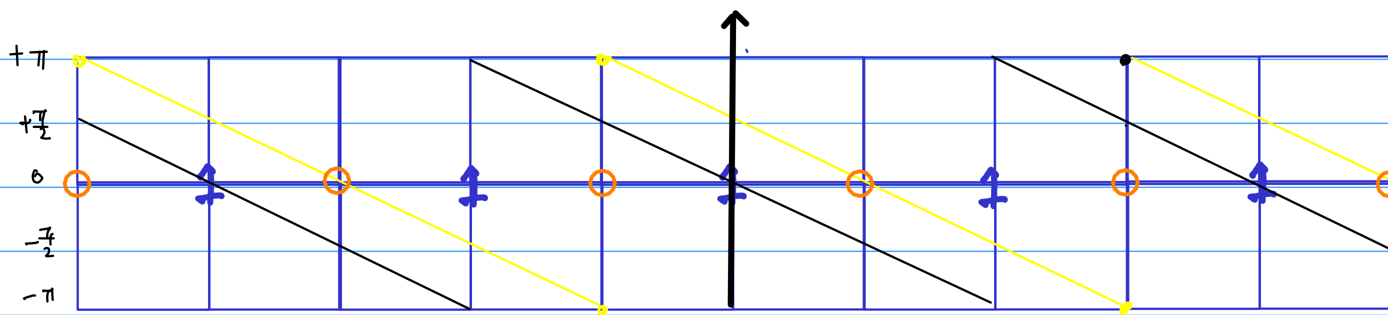
$\begin{cases} \tan(x) = \pm\infty \\ \tan\theta = \pm\infty \end{cases}$ $x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \cdots$
$\theta = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \cdots$

$\begin{cases} \tan(x) = 0 \\ \tan\theta = 0 \end{cases}$ $x = 0, \pm\pi, \pm 2\pi, \cdots$
$\theta = 0, \pm\pi, \pm 2\pi, \cdots$

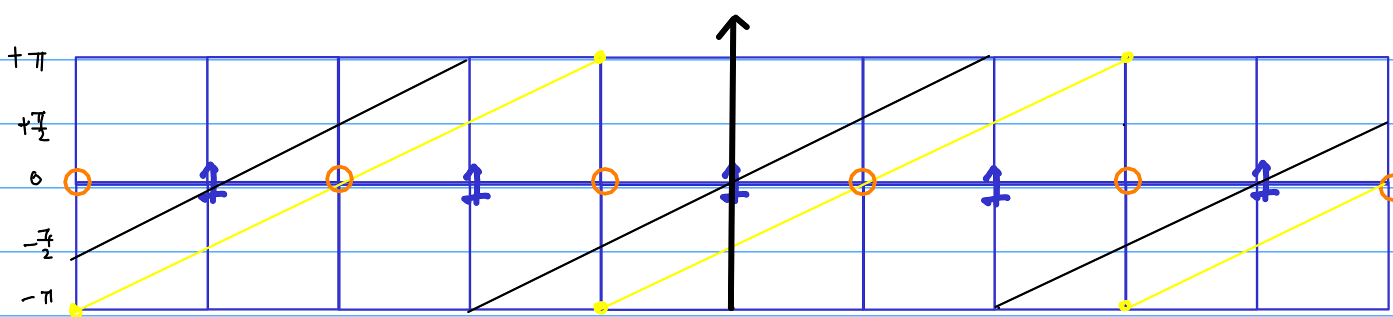
$y>1$



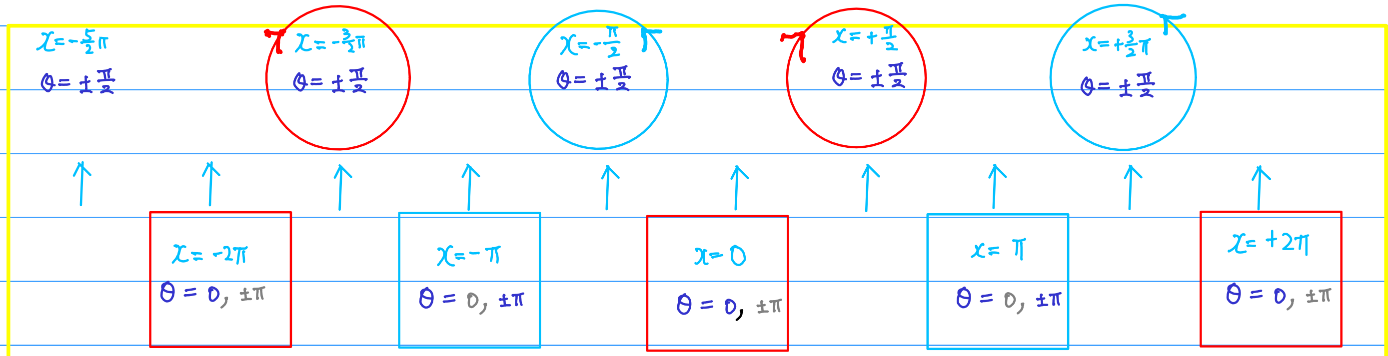
θ trend with x varying



θ trend with x varying

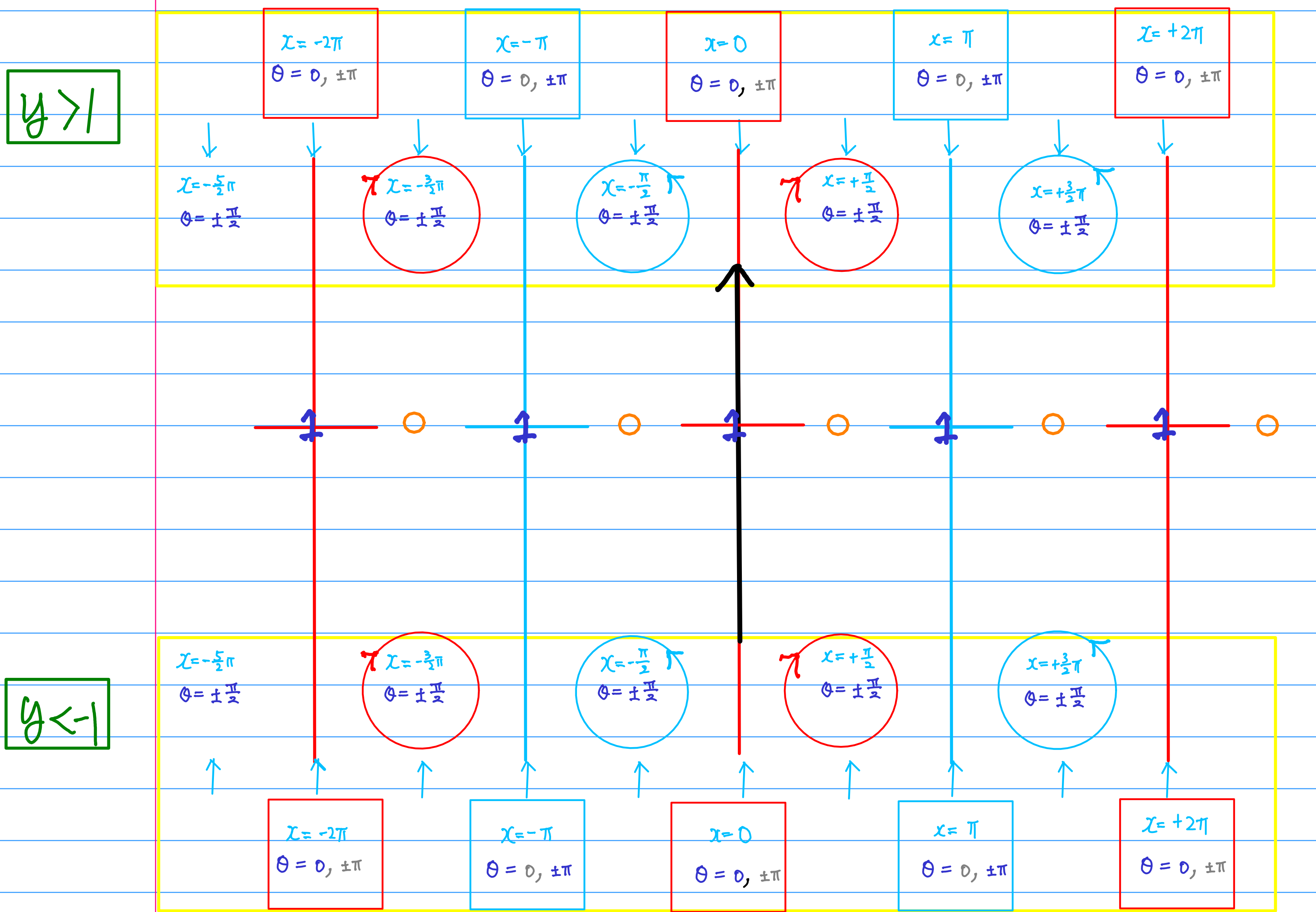


$y<-1$

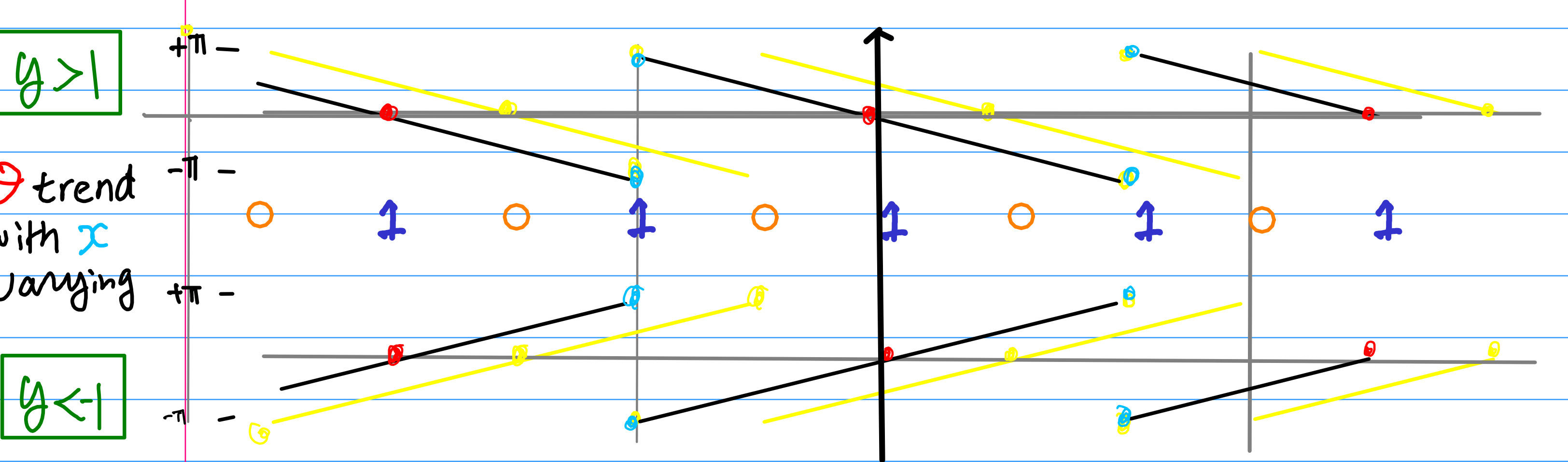


② $\arg(\cos z)$ $\theta = 0$ & $\theta = \pm\pi$ $|y| > 1$

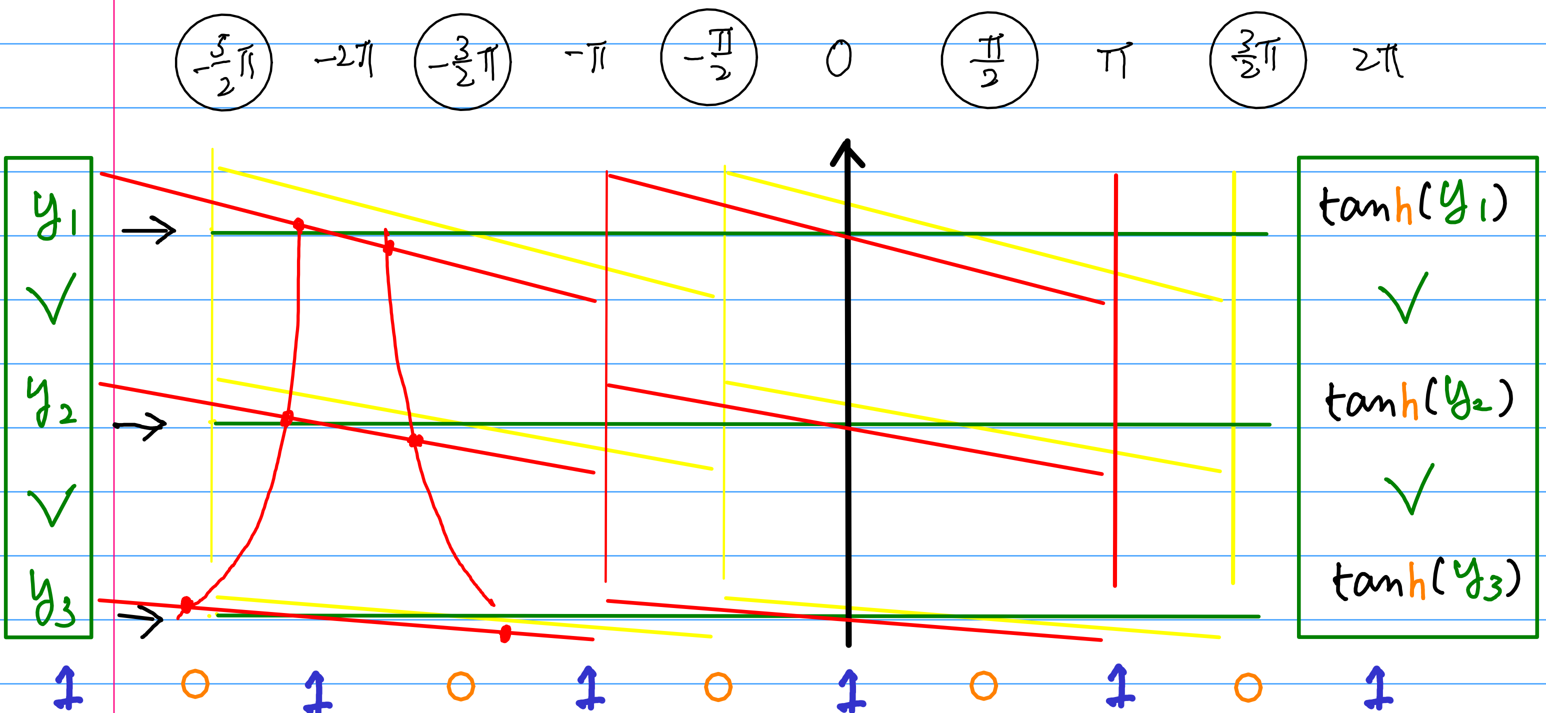
4



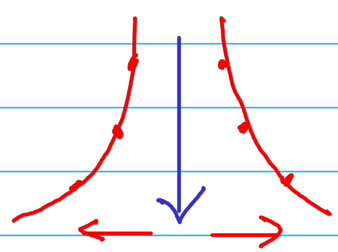
discontinuous θ ← $x = \cdots -3\pi, -\pi, +\pi, +3\pi, \cdots$



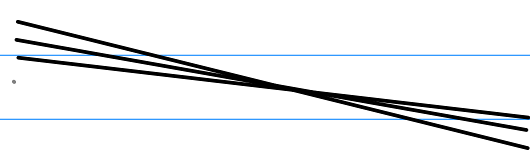
② $\arg(\cos z)$ θ shifts as $y \leftarrow 0$ $0 < y < 1$



$$\boxed{\tan\theta = -\tan(x)\tanh(y)}$$



move away as $y \leftarrow 0$

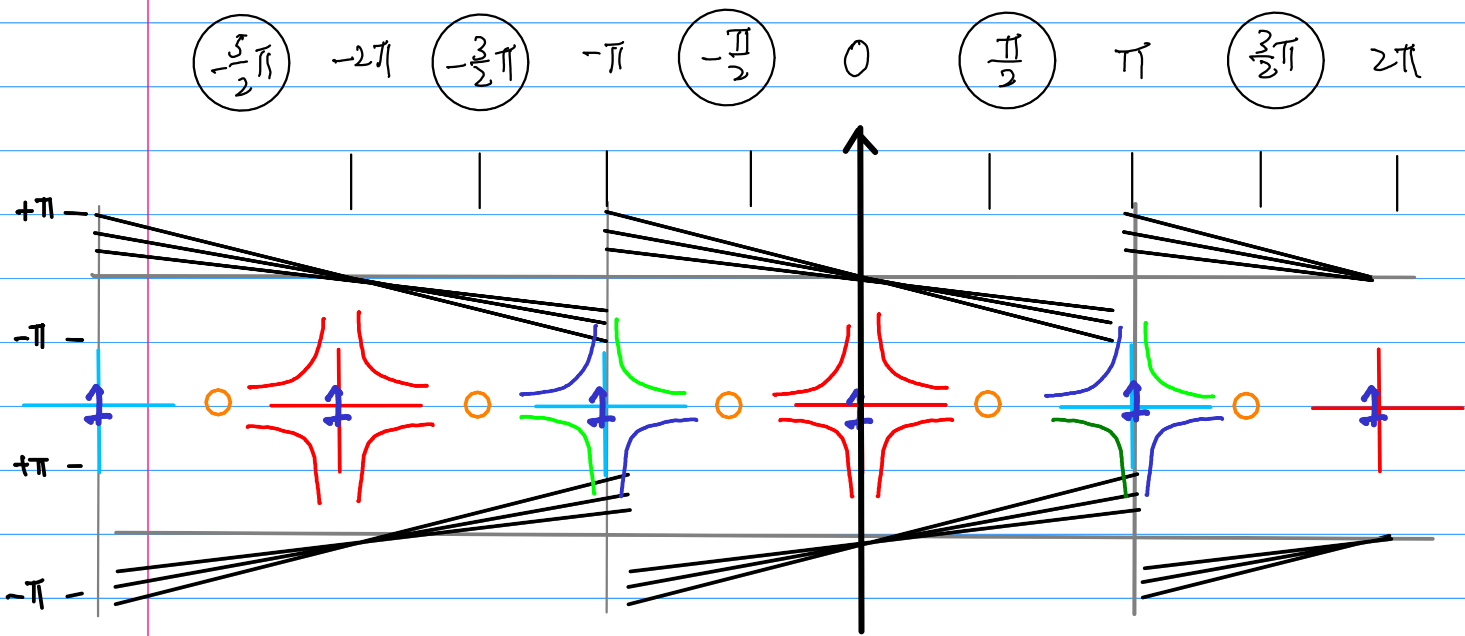


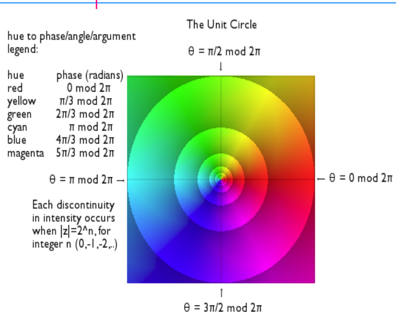
make the slope smaller

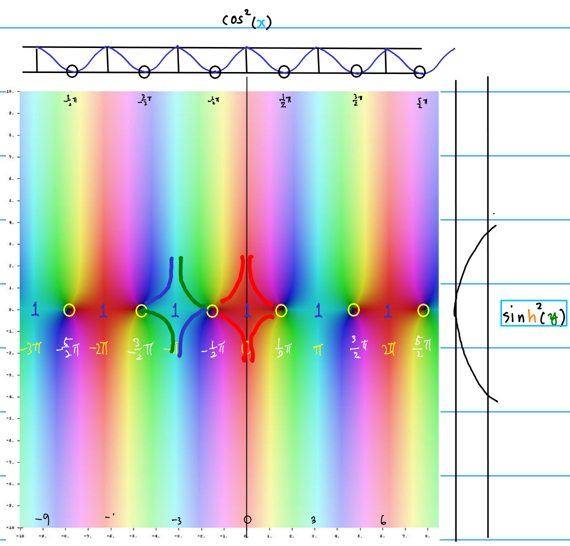
$$\left.\begin{array}{l} 1 > y_1 > y_2 > y_3 > 0 \\ 1 > \tanh(y_1) > \tanh(y_2) > \tanh(y_3) > 0 \end{array}\right\}$$

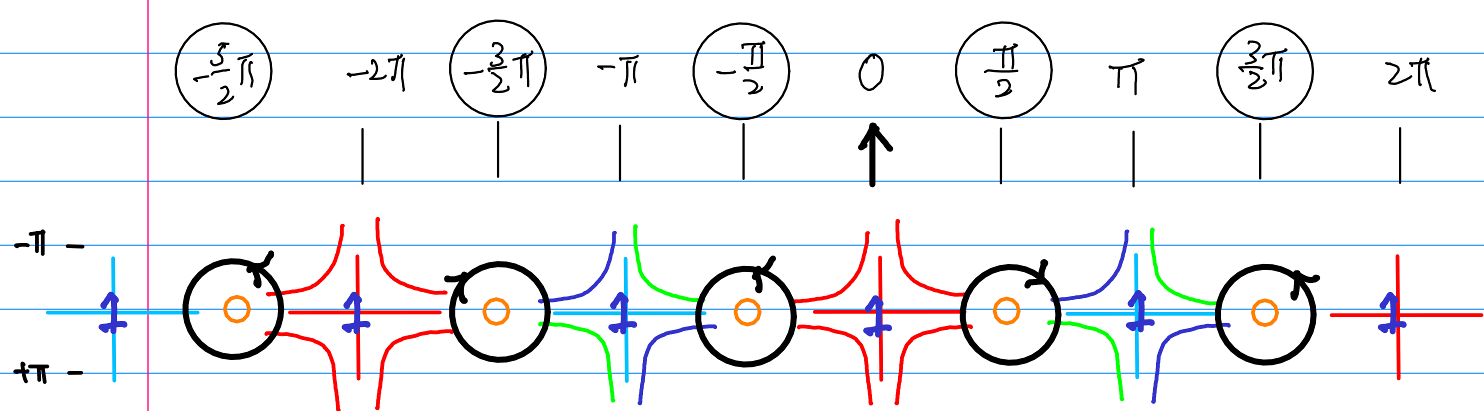
② arg(cos z)   CCW / CW at zeros

6





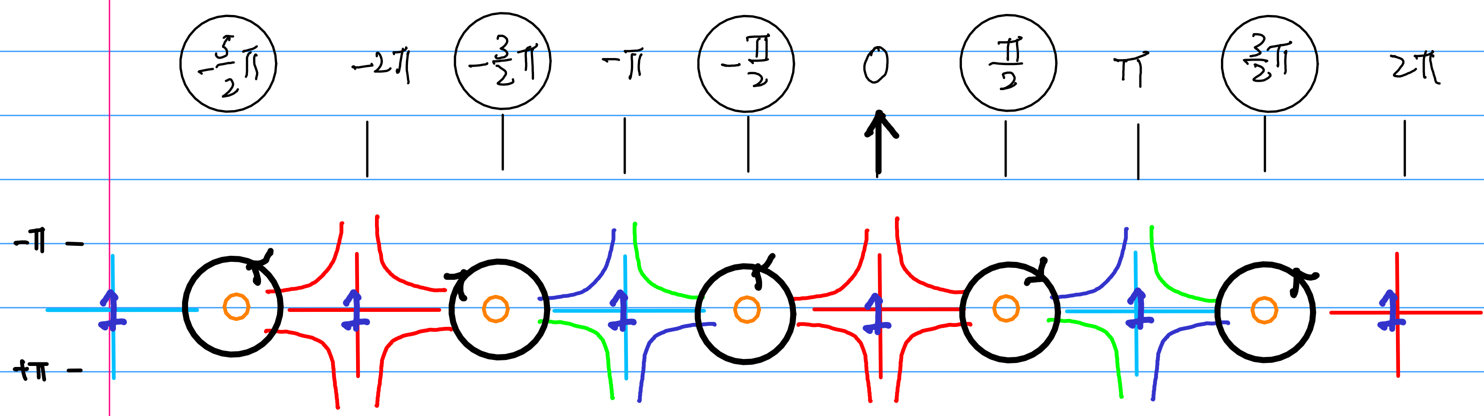




① $\arg(\cos z)$

7

$$\begin{cases} \text{CCW} & \left(x = -\frac{\pi}{2} + 2n\pi\right) \\ \text{CW} & \left(x = +\frac{\pi}{2} + 2n\pi\right) \end{cases}$$



zeros	$-\frac{5}{2}\pi$	-2π	$-\frac{3}{2}\pi$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π
CCW	$-\frac{5}{2}\pi$	-2π	$-\frac{3}{2}\pi$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π
CW	$-\frac{5}{2}\pi$	-2π	$-\frac{3}{2}\pi$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π

* arrow toward near one 1

② arg(cos z)

https://en.wikipedia.org/wiki/Trigonometric_functions

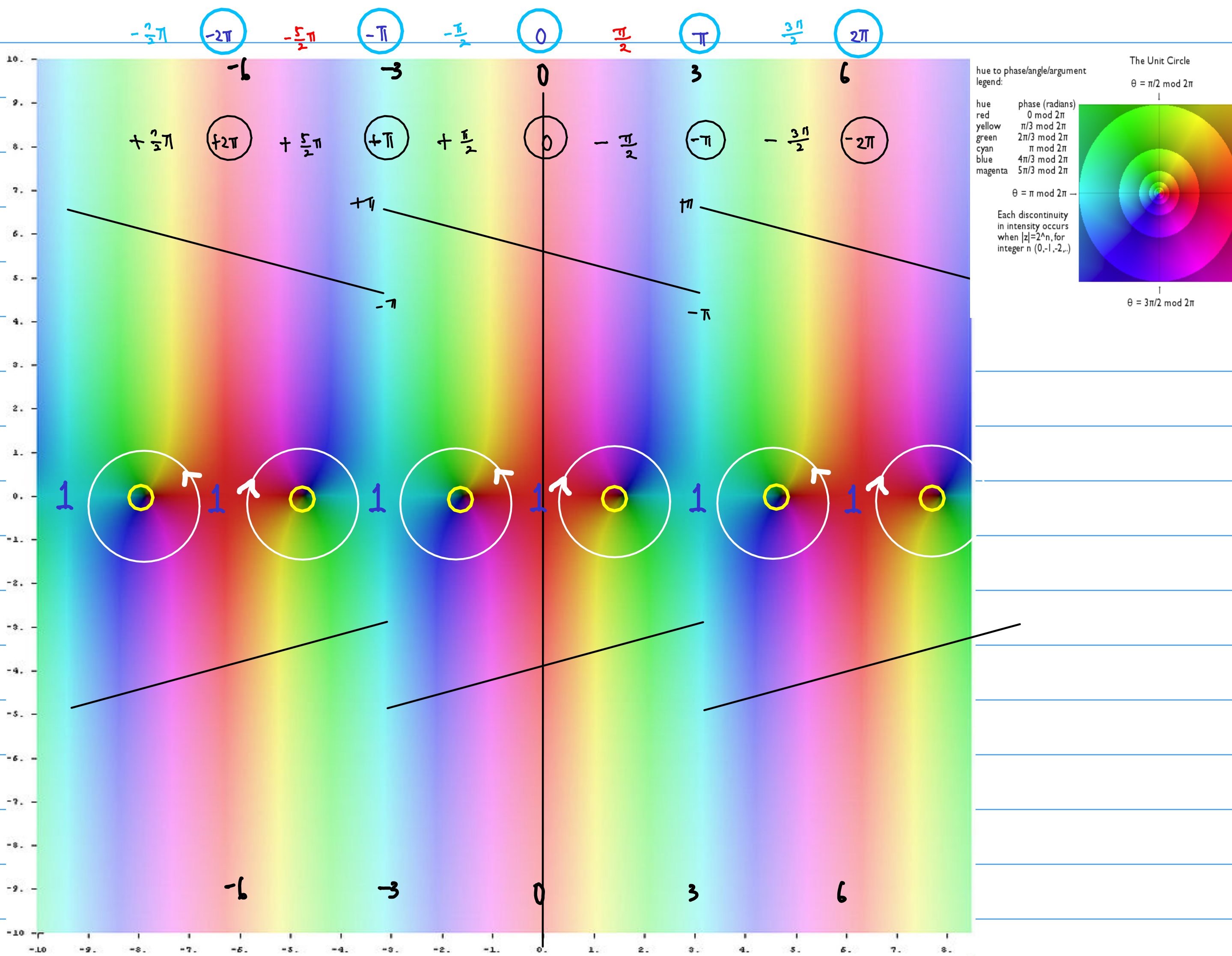
$$\tan\theta = -\tan(x)\,\tanh(y)$$

$\tan(x) = \pm\infty$ $\quad x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \cdots$

$\tan\theta = \mp\infty$ $\quad \boxed{\theta = 0, \pm\pi}$

$\tan(x) = 0$ $\quad x = 0, \pm\pi, \pm 2\pi, \cdots$

$\tan\theta = 0$ $\quad \boxed{\theta = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi}$



3

$$\sin(x+iy) = \sin(x)\boxed{\cosh(y)} + i\cos(x)\boxed{\sinh(y)}$$

$$\cos(x+iy) = \cos(x)\boxed{\cosh(y)} - i\sin(x)\boxed{\sinh(y)}$$

$$\tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)} = \frac{\sin(x)\boxed{\cosh(y)} + i\cos(x)\boxed{\sinh(y)}}{\cos(x)\boxed{\cosh(y)} - i\sin(x)\boxed{\sinh(y)}}$$

$$|\tan z|^2 = \frac{|\sin z|^2}{|\cos z|^2} = \frac{\sin^2(x) + \boxed{\sinh^2(y)}}{\cos^2(x) + \boxed{\sinh^2(y)}}$$

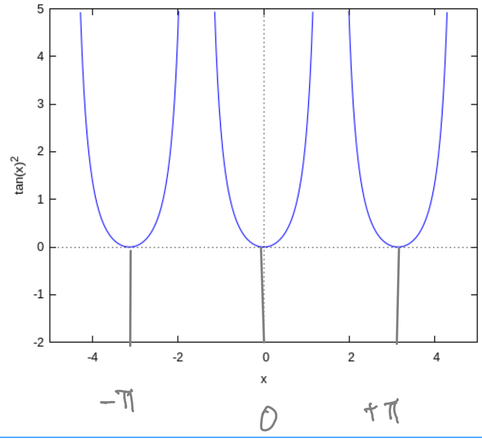
zeros $\quad y=0$ & $x = 0, \pm\pi, \pm 2\pi, \cdots \quad \sinh(0) = 0$

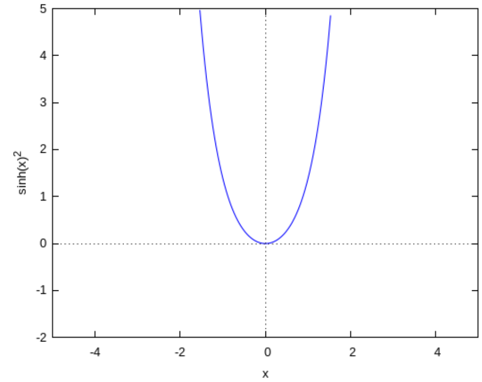
$\infty \quad y=0$ & $x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \cdots \quad \sinh(0) = 0$

zeros : $|\sin z| = 0 \quad x = 0, \pm\pi, \pm 2\pi, \cdots \quad |\cos z| = 1$

∞ : $|\cos z| = 0 \quad x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \cdots \quad |\sin z| = 1$

$\tan^2(x)$ $\sinh^2(y)$



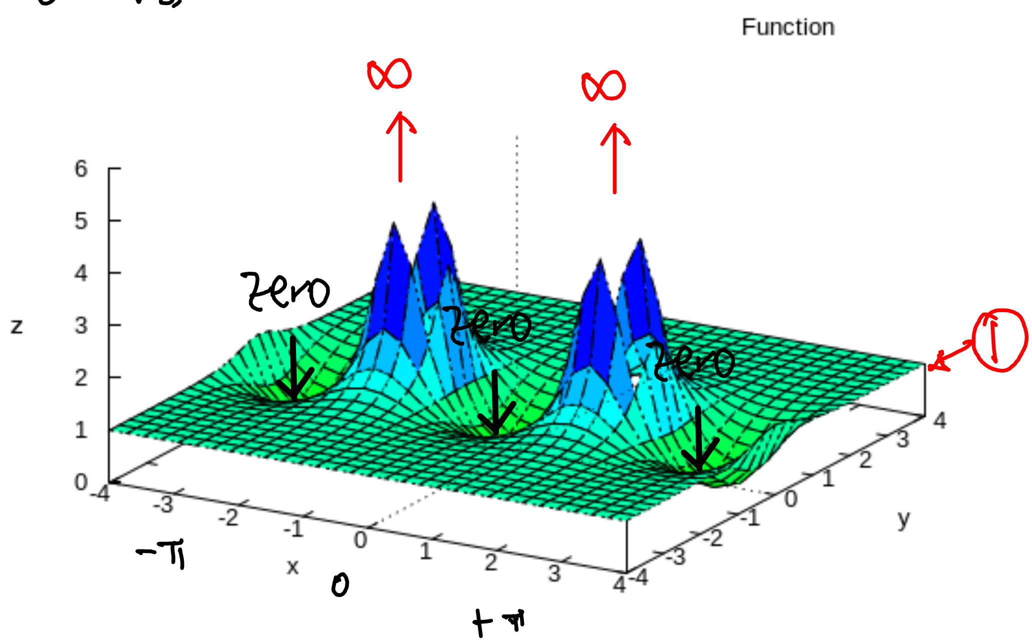


$$|\tan z|^2 = \frac{|\sin z|^2}{|\cos z|^2} = \frac{\sin^2(x) + \boxed{\sinh^2(y)}}{\cos^2(x) + \boxed{\sinh^2(y)}}$$

$$0 \leq \sin^2(x) \leq 1$$
$$0 \leq \cos^2(x) \leq 1$$

$$\boxed{\sinh^2(y)} \gg 1 \rightarrow |\tan z|^2 = 1$$

$\tan^2(z$



③

https://en.wikipedia.org/wiki/Trigonometric_functions

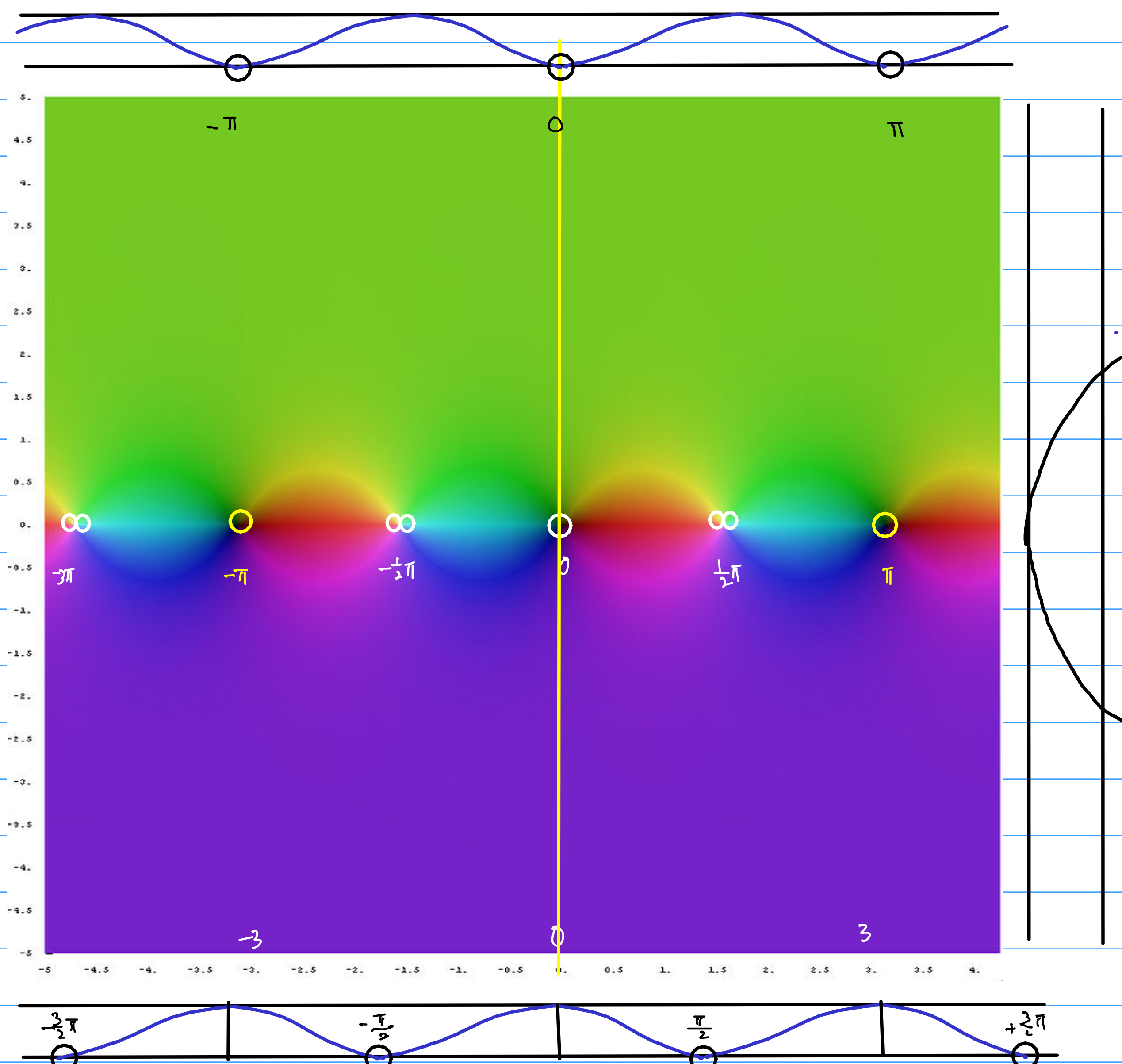
$|\tan z|$ brightness

$$|\tan z|^2 = \frac{|\sin z|^2}{|\cos z|^2} = \frac{\sin^2(x) + \boxed{\sinh^2(y)}}{\cos^2(x) + \boxed{\sinh^2(y)}}$$

zeros $\leftarrow$ $y=0$ & $x = 0, \pm\pi, \pm 2\pi, \cdots$ $\sinh(0) = 0$

poles $\leftarrow$ $y=0$ & $x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \cdots$ $\sinh(0) = 0$

$\sin^2(x)$

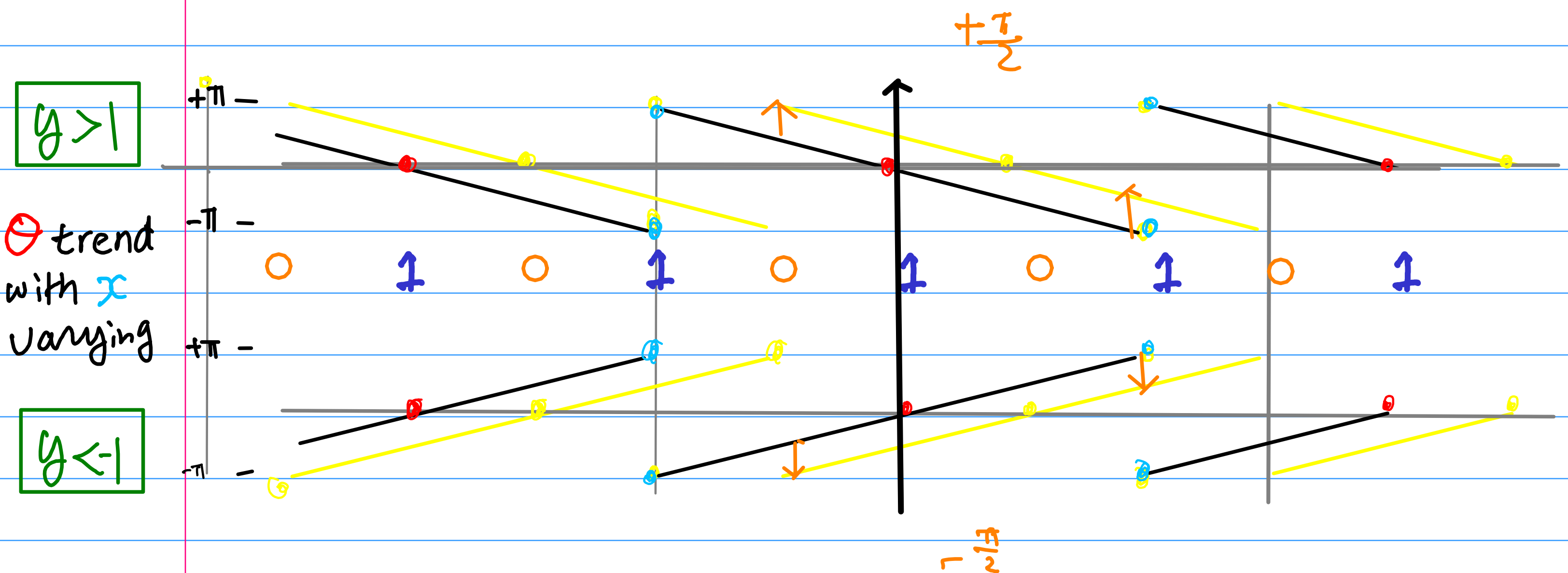


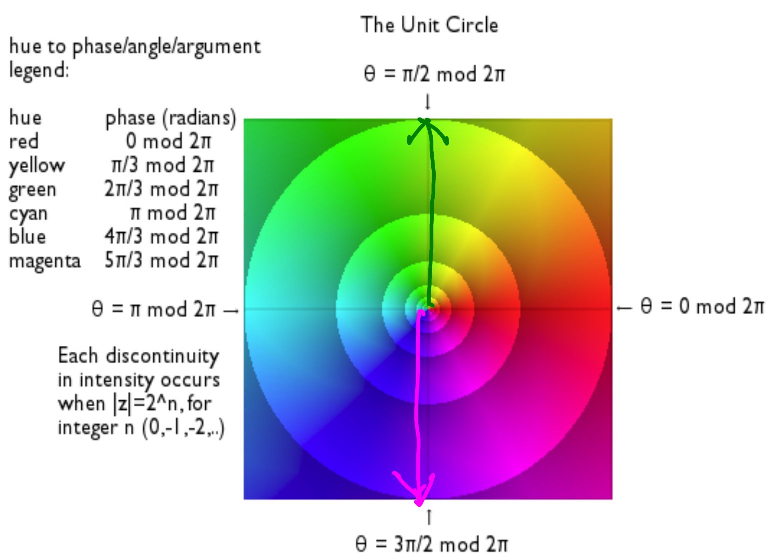
$\cos^2(x)$

$\sinh^2($

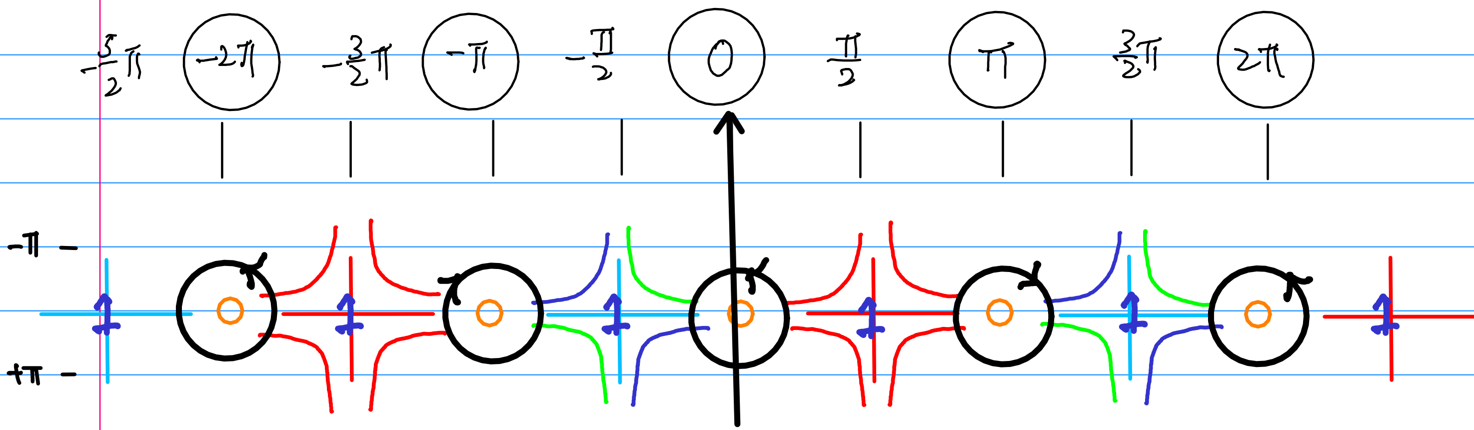
$$\tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)}$$

$$\arg\{\tan(x+iy)\} = \arg\{\sin(x+iy)\} - \arg\{\cos(x+iy)\}$$

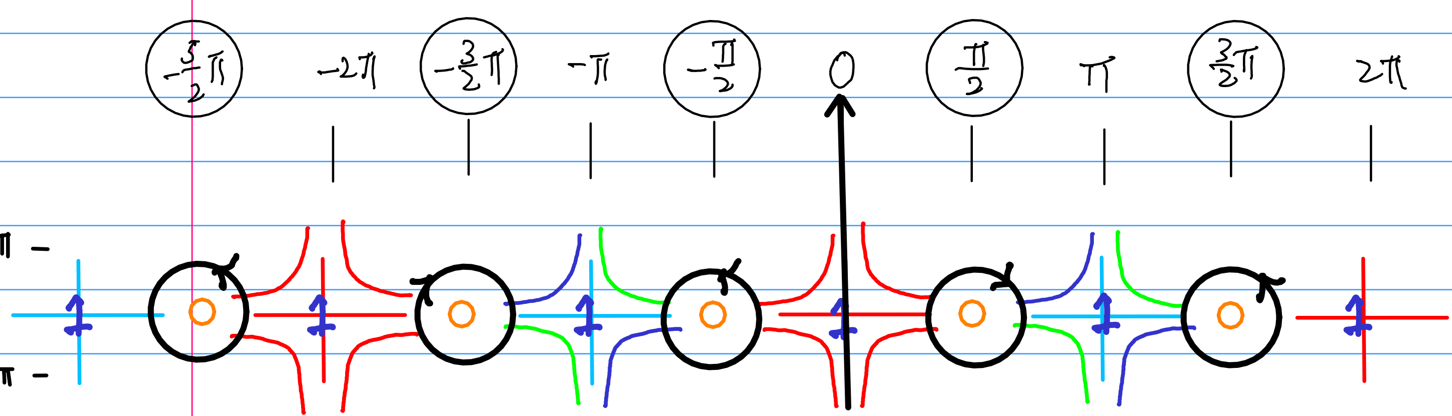


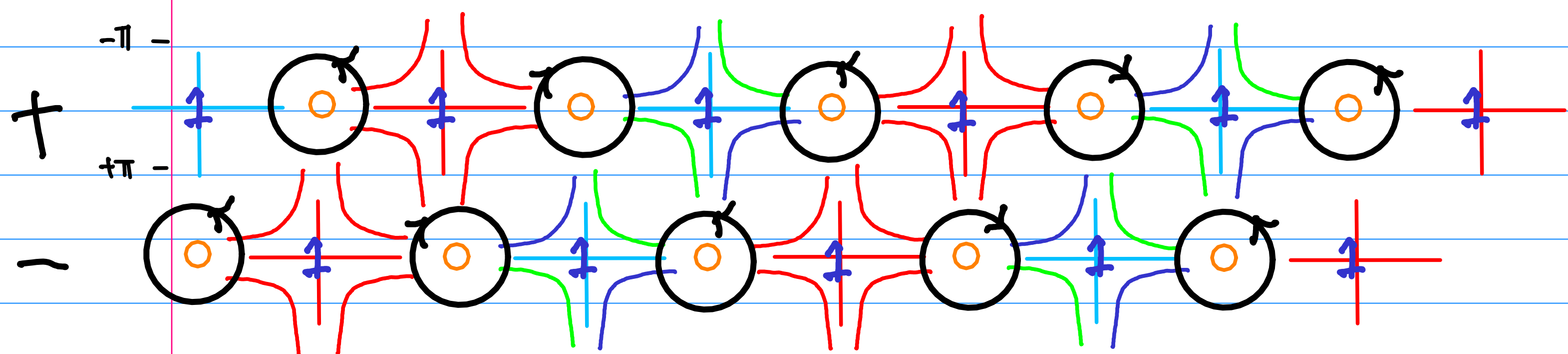


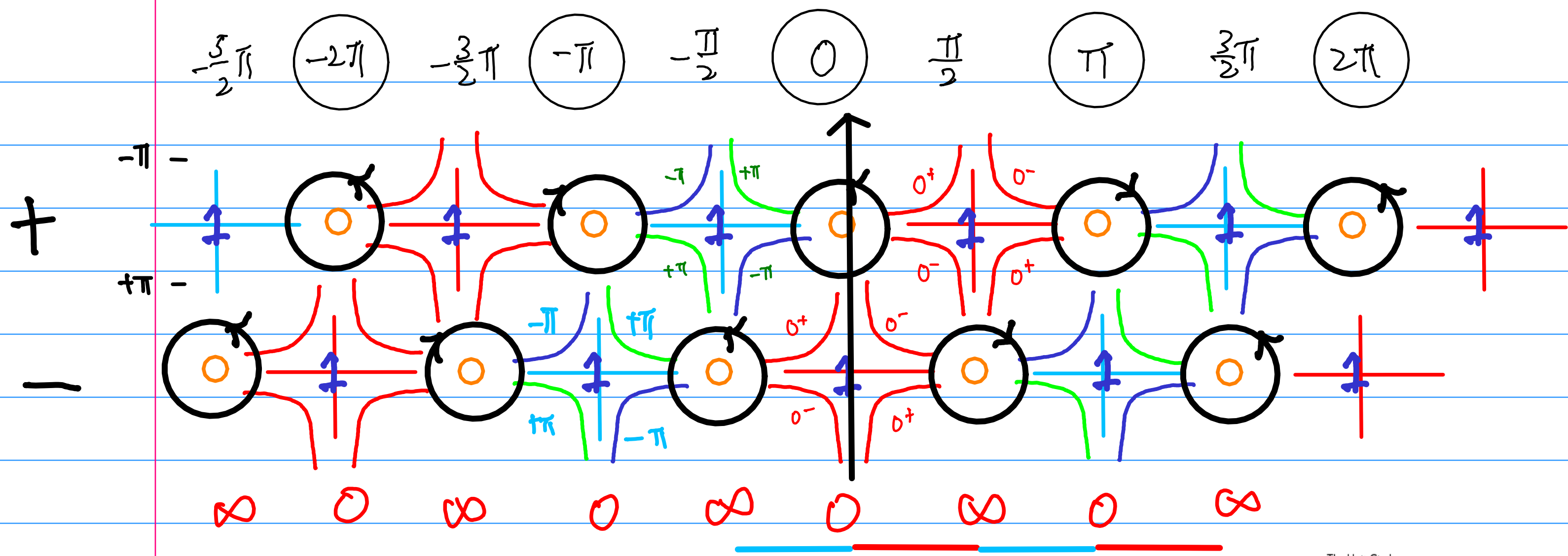
$\arg(\sin(z))$

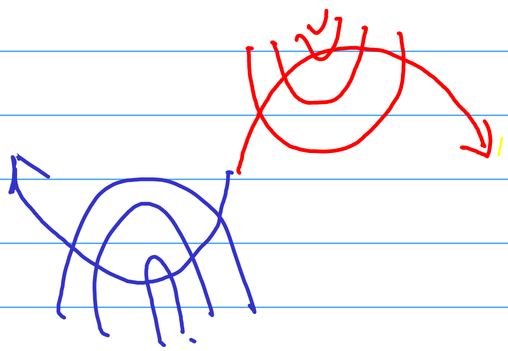


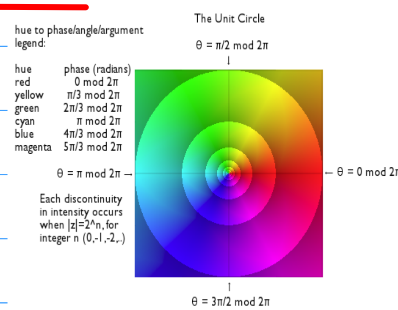
$\arg(\cos(z))$

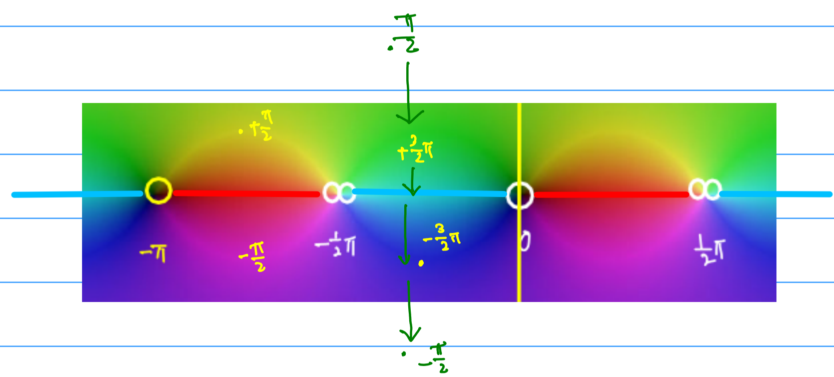


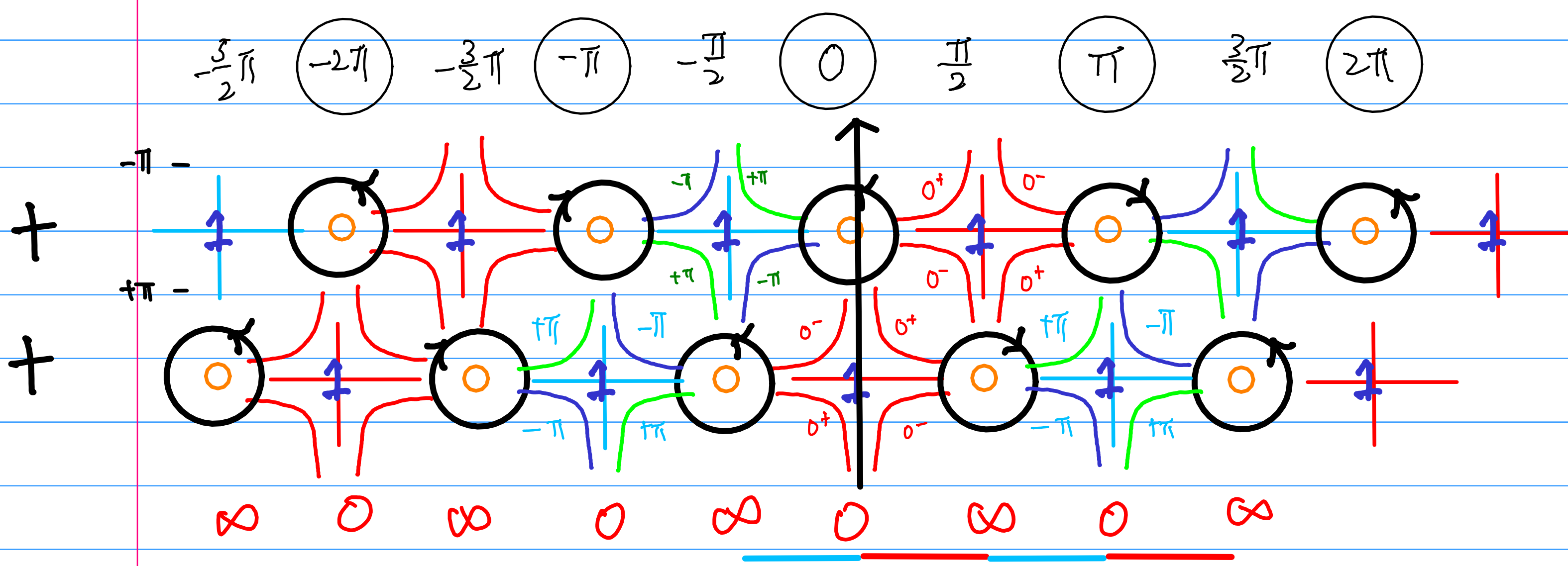



−5/2 π
−2π
−3/2 π
−π
−π/2
0
π/2
π
3/2 π
2π
+
−
−π
+π
0+
0−
∞
0

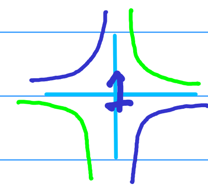


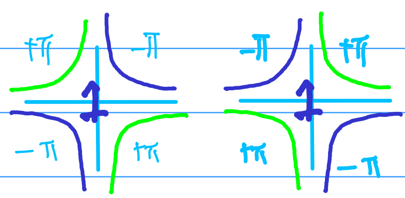

hue to phase/angle/argument legend:
hue phase (radians)
red 0 mod 2π
yellow π/3 mod 2π
green 2π/3 mod 2π
cyan π mod 2π
blue 4π/3 mod 2π
magenta 5π/3 mod 2π
Each discontinuity in intensity occurs when |z|=2^n, for integer n (0,-1,-2,..)
The Unit Circle
θ = π/2 mod 2π
θ = π mod 2π
θ = 0 mod 2π
θ = 3π/2 mod 2π


π/2
+π/2
+3/2 π
−π
−π/2
−1/2 π
−3/2 π
0
1/2 π
−π/2

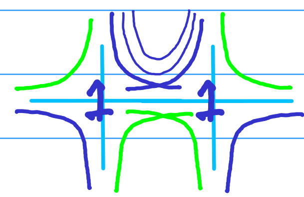


$-\frac{5}{2}\pi$ -2π $-\frac{3}{2}\pi$ $-\pi$ $-\frac{\pi}{2}$ 0 $\frac{\pi}{2}$ π $\frac{3}{2}\pi$ 2π
$-\pi$
$+\pi$
∞ 0 ∞ 0 ∞ 0 ∞ 0 ∞





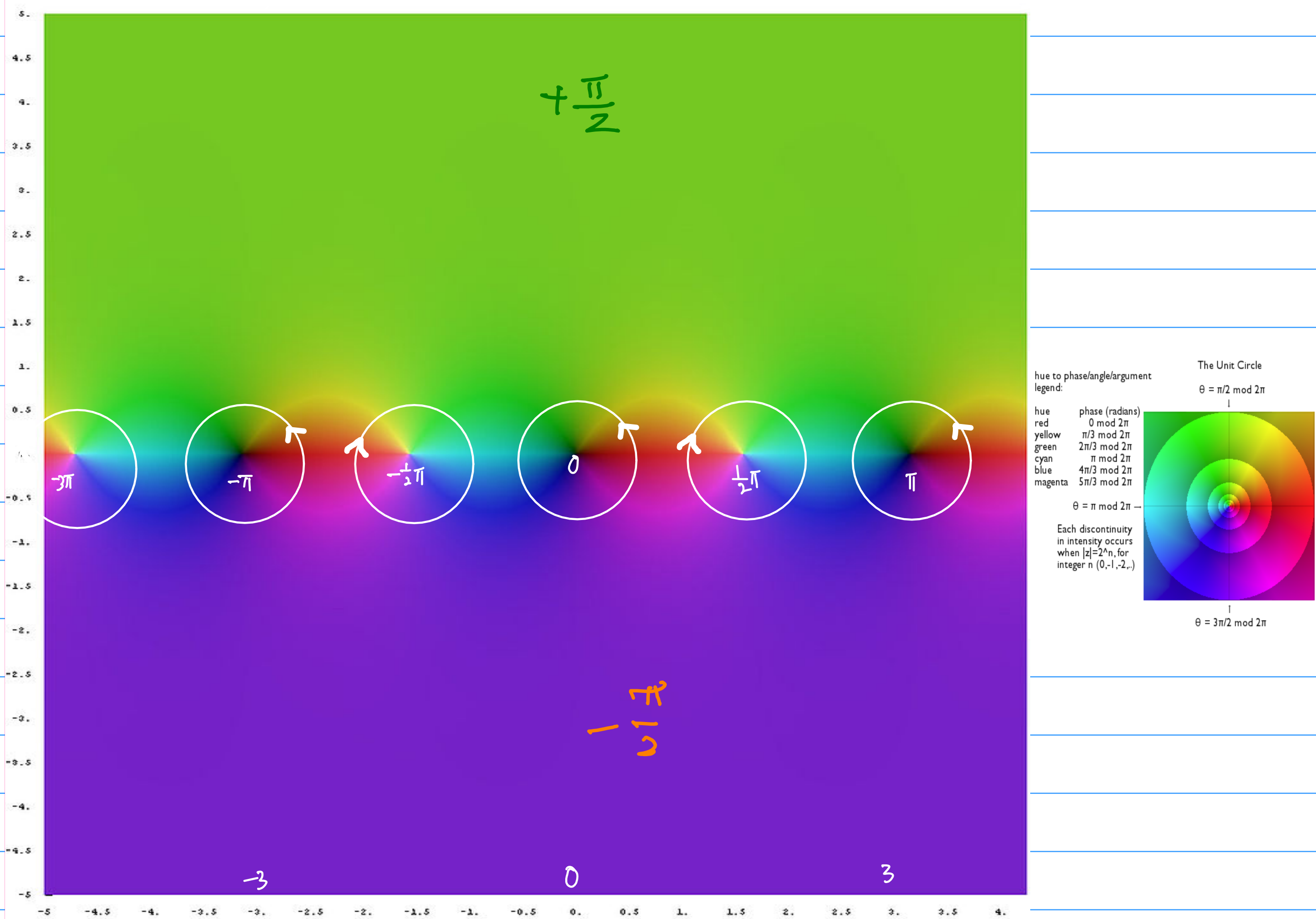
$+\pi$ $-\pi$ $-\pi$ $+\pi$
$-\pi$ $+\pi$ $+\pi$ $-\pi$

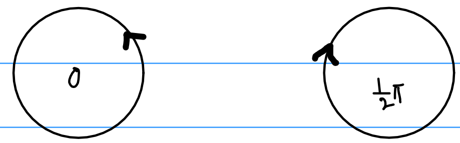


$-\pi - \pi = -2\pi \to 0^+$

https://en.wikipedia.org/wiki/Trigonometric_functions

③ arg (tan z)





$[0 \sim \pi]$	$[0 \sim \pi]$
$\rightarrow [0 \sim \pi]$	$\rightarrow [-\pi \sim 0]$
$[-\pi \sim 0]$	$[-\pi \sim 0]$
$\rightarrow [-\pi \sim 0]$	$\rightarrow [0 \sim \pi]$

4

$$\sin(x+iy) = \sin(x)\boxed{\cosh(y)} + i\cos(x)\boxed{\sinh(y)}$$

$$\cos(x+iy) = \cos(x)\boxed{\cosh(y)} - i\sin(x)\boxed{\sinh(y)}$$

$$\cot(x+iy) = \frac{\cos(x+iy)}{\sin(x+iy)} = \frac{\cos(x)\boxed{\cosh(y)} - i\sin(x)\boxed{\sinh(y)}}{\sin(x)\boxed{\cosh(y)} + i\cos(x)\boxed{\sinh(y)}}$$

$$|\cot z|^2 = \frac{|\cos z|^2}{|\sin z|^2} = \frac{\cos^2(x) + \boxed{\sinh^2(y)}}{\sin^2(x) + \boxed{\sinh^2(y)}}$$

zeros $y=0$ & $x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \cdots$ $\sinh(0)=0$

∞ $y=0$ & $x = 0, \pm\pi, \pm 2\pi, \cdots$ $\sinh(0)=0$

https://en.wikipedia.org/wiki/Trigonometric_functions

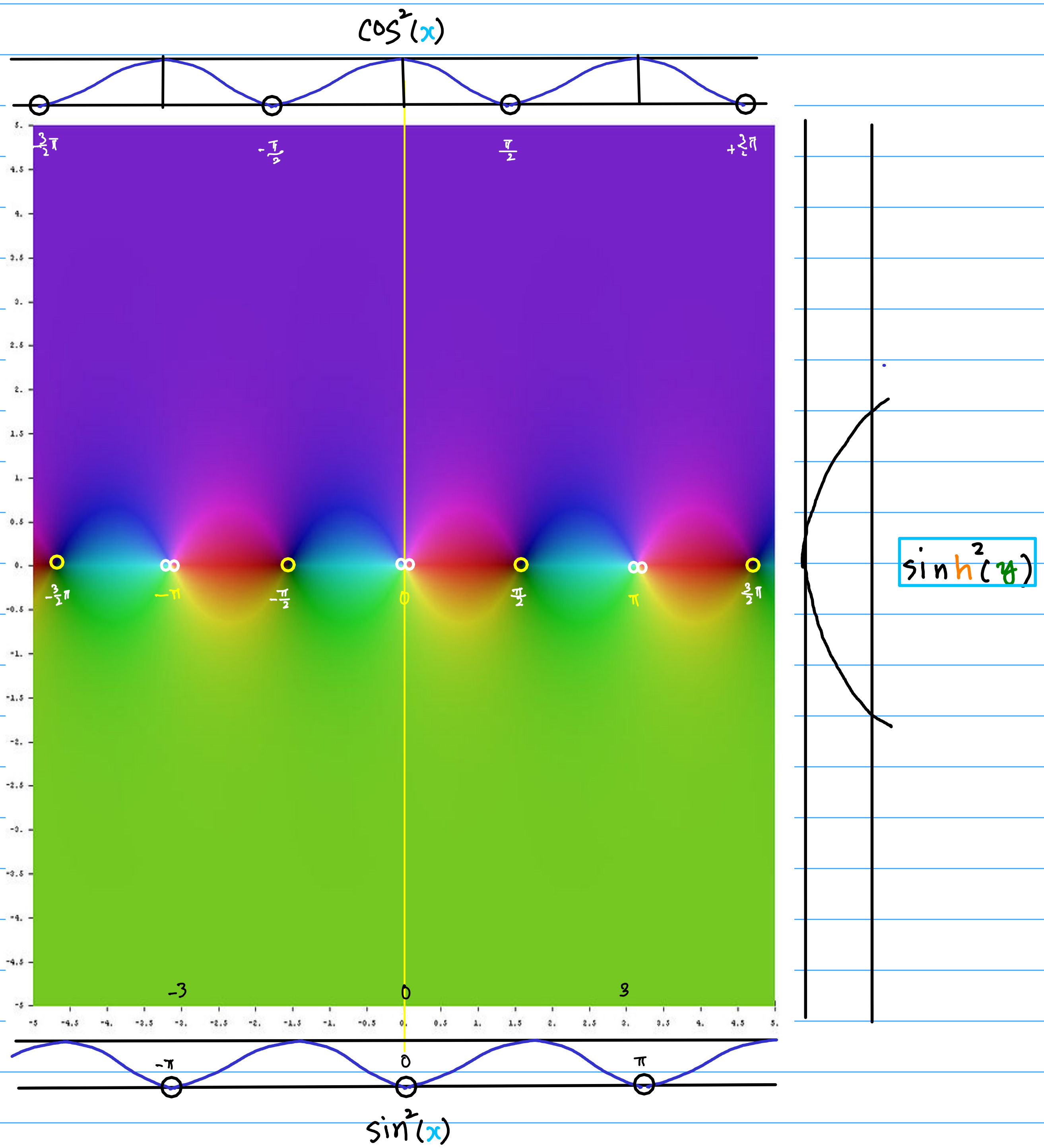
④

$|\cot z|$ brightness

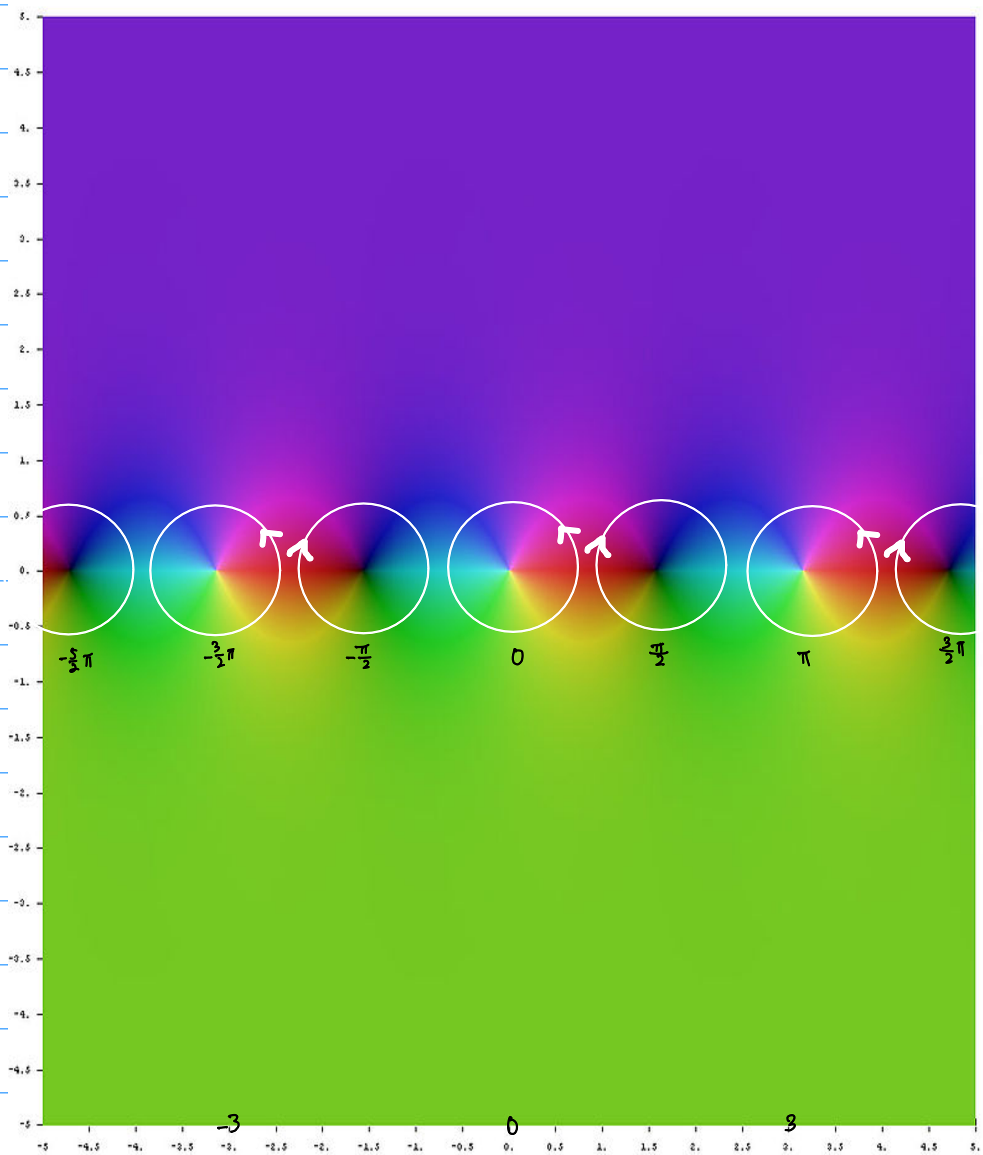
$$|\cot z|^2 = \frac{|\cos z|^2}{|\sin z|^2} = \frac{\cos^2(x) + \boxed{\sinh^2(y)}}{\sin^2(x) + \boxed{\sinh^2(y)}}$$

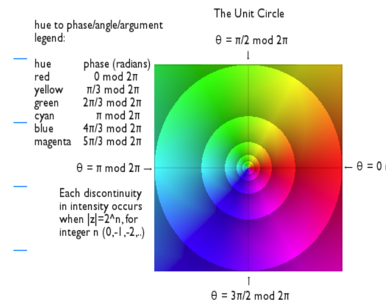
zero $\longleftarrow$ $y=0$ & $x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \cdots$ $\quad \sinh(0) = 0$

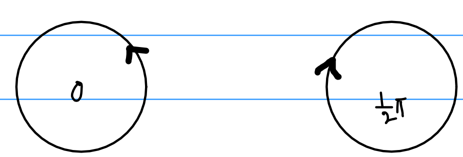
pole $\longleftarrow$ $y=0$ & $x = 0, \pm\pi, \pm 2\pi, \cdots$ $\quad \sinh(0) = 0$



④ ang(Cot z)







$[0 \sim \pi]$	$[0 \sim \pi]$
$\rightarrow [0 \sim \pi]$	$\rightarrow [-\pi \sim 0]$
$[-\pi \sim 0]$	$[-\pi \sim 0]$
$\rightarrow [-\pi \sim 0]$	$\rightarrow [0 \sim \pi]$

5

$$|\cos z|^2 = \cos^2(x) + \boxed{\sinh^2(y)}$$

Zero $\longleftarrow$ $y=0$ & $x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \cdots$ $\sinh(0)=0$

1 $\longleftarrow$ $y=0$ & $x = 0, \pm\pi, \pm 2\pi, \cdots$ $\sinh(0)=0$

$$|\sec z|^2 = \frac{1}{\cos^2(x) + \boxed{\sinh^2(y)}} = \frac{1}{|\cos z|^2}$$

Pole $\longleftarrow$ $y=0$ & $x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \cdots$ $\sinh(0)=0$

1 $\longleftarrow$ $y=0$ & $x = 0, \pm\pi, \pm 2\pi, \cdots$ $\sinh(0)=0$

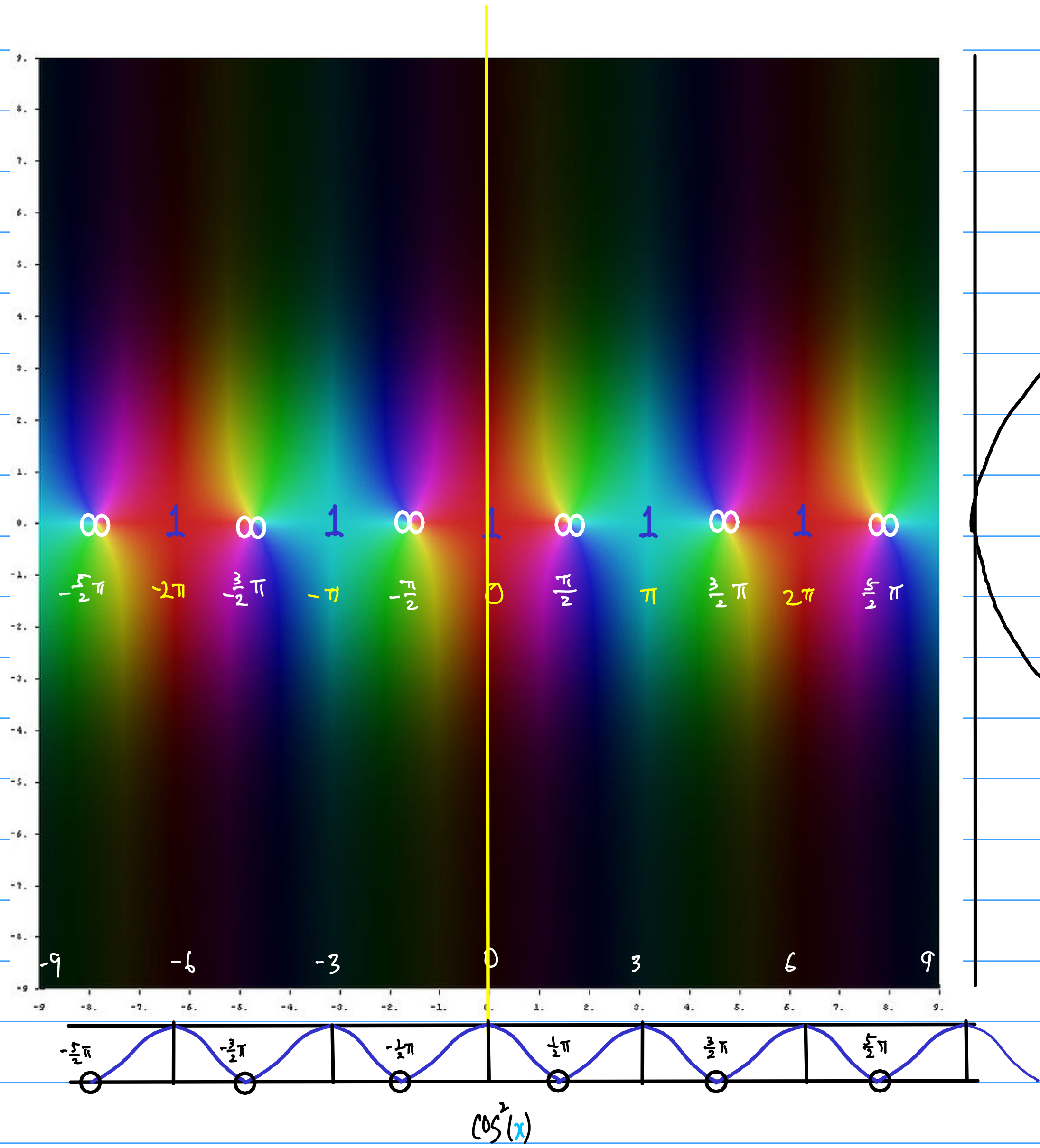
https://en.wikipedia.org/wiki/Trigonometric_functions

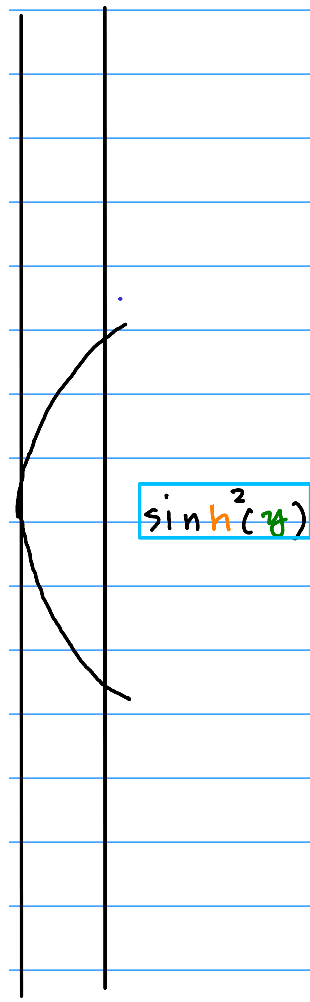
⑤ $|\sec z|$ brightness

$$|\sec z|^2 = \frac{1}{|\cos z|^2} = \frac{1}{\cos^2(x) + \boxed{\sinh^2(y)}}$$

pole $\leftarrow$ $y=0$ & $x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \cdots$ $\quad \sinh(0) = 0$

1 $\leftarrow$ $y=0$ & $x = 0, \pm\pi, \pm 2\pi, \cdots$ $\quad \sinh(0) = 0$

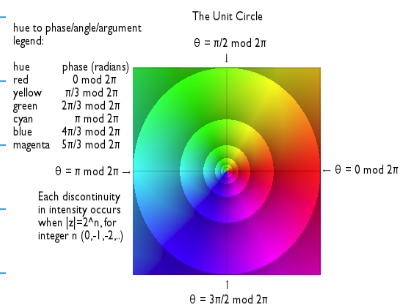


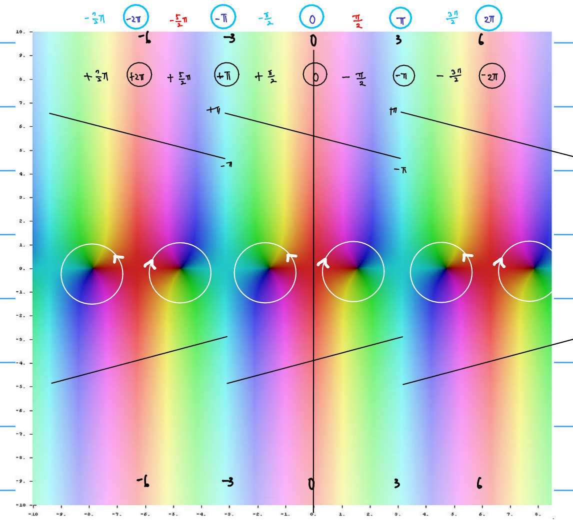


https://en.wikipedia.org/wiki/Trigonometric_functions

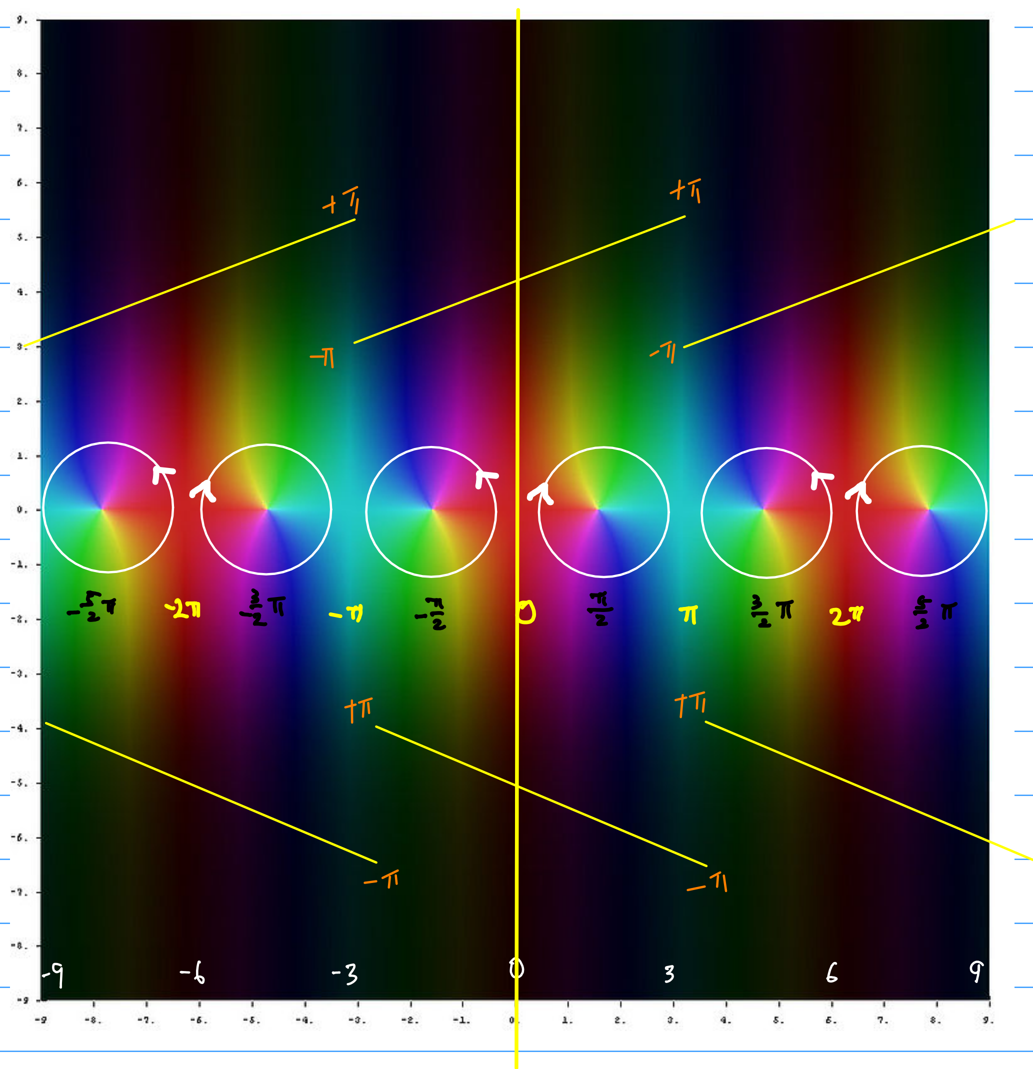
5 arg(sec z)

arg(cos z)





zeros



poles

6

$$|\sin z|^2 = \sin^2(x) + \boxed{\sinh^2(y)}$$

Zero $\longleftarrow$ $y=0$ & $x = 0, \pm\pi, \pm 2\pi, \cdots$ $\quad \sinh(0) = 0$

1 $\longleftarrow$ $y=0$ & $x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \cdots$ $\quad \sinh(0) = 0$

$$|\csc z|^2 = \frac{1}{\sin^2(x) + \boxed{\sinh^2(y)}} = \frac{1}{|\sin z|^2}$$

Pole $\longleftarrow$ $y=0$ & $x = 0, \pm\pi, \pm 2\pi, \cdots$ $\quad \sinh(0) = 0$

1 $\longleftarrow$ $y=0$ & $x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \cdots$ $\quad \sinh(0) = 0$

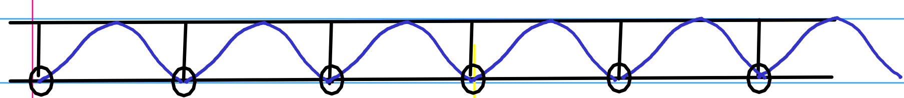
⑥ $|\csc z|$ brightness

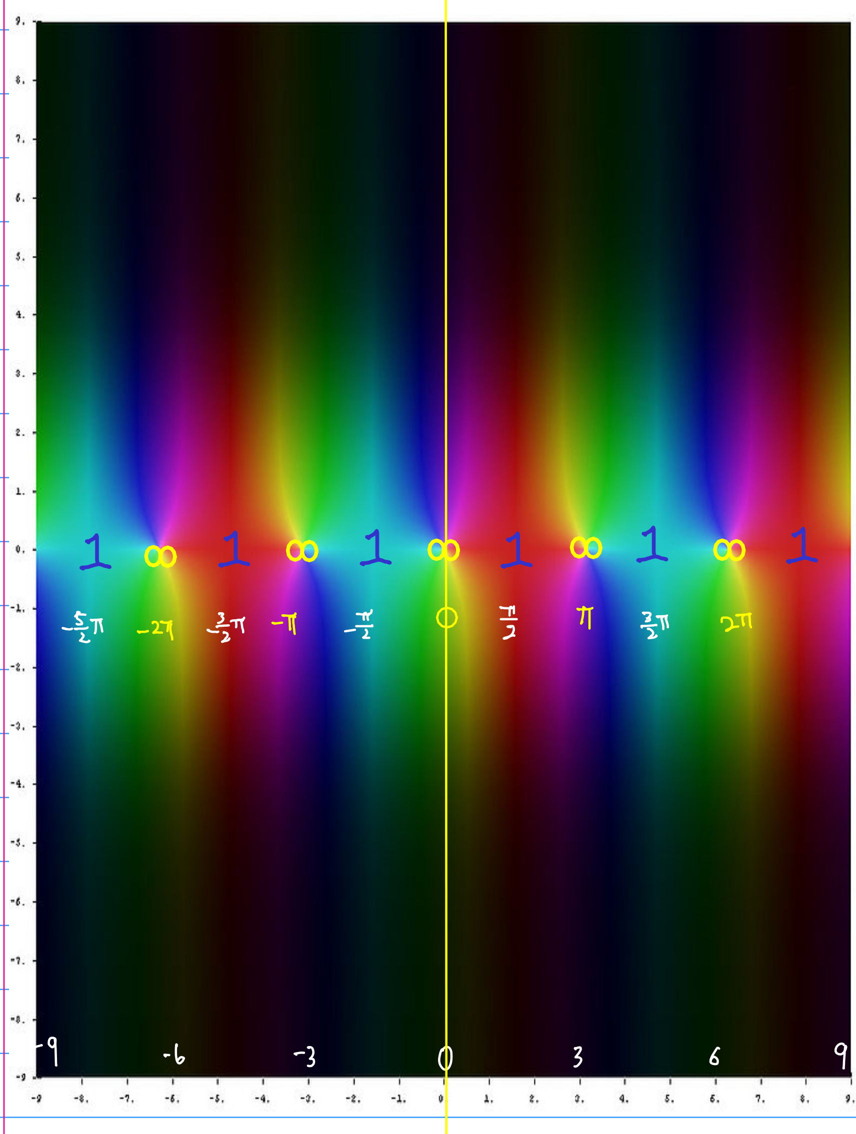
$$|\csc z|^2 = \frac{1}{|\sin z|^2} = \frac{1}{\sin^2(x) + \boxed{\sinh^2(y)}}$$

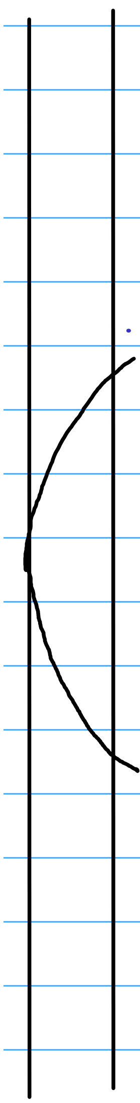
pole ← $y=0$ & $x = 0, \pm\pi, \pm 2\pi, \cdots$ $\quad \sinh(0) = 0$

1 ← $y=0$ & $x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \cdots$ $\quad \sinh(0) = 0$

$\sin^2(x)$



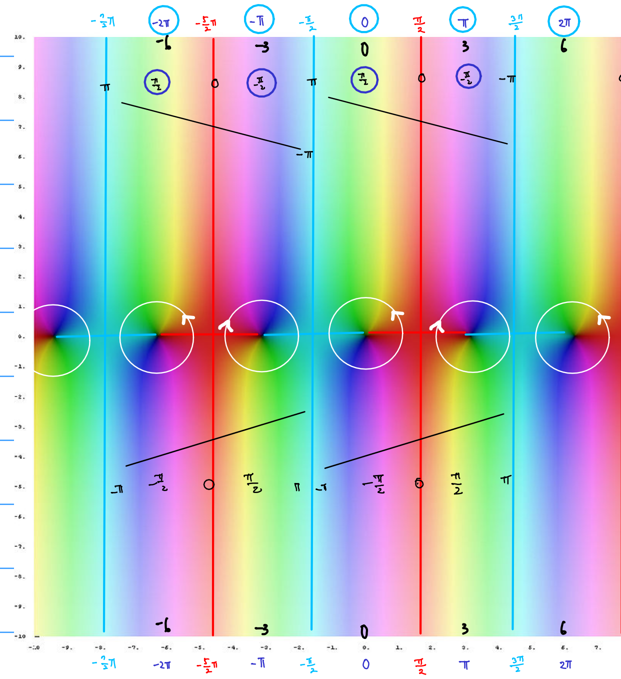


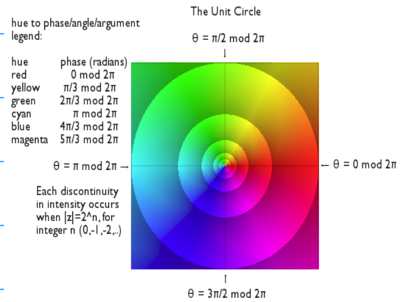


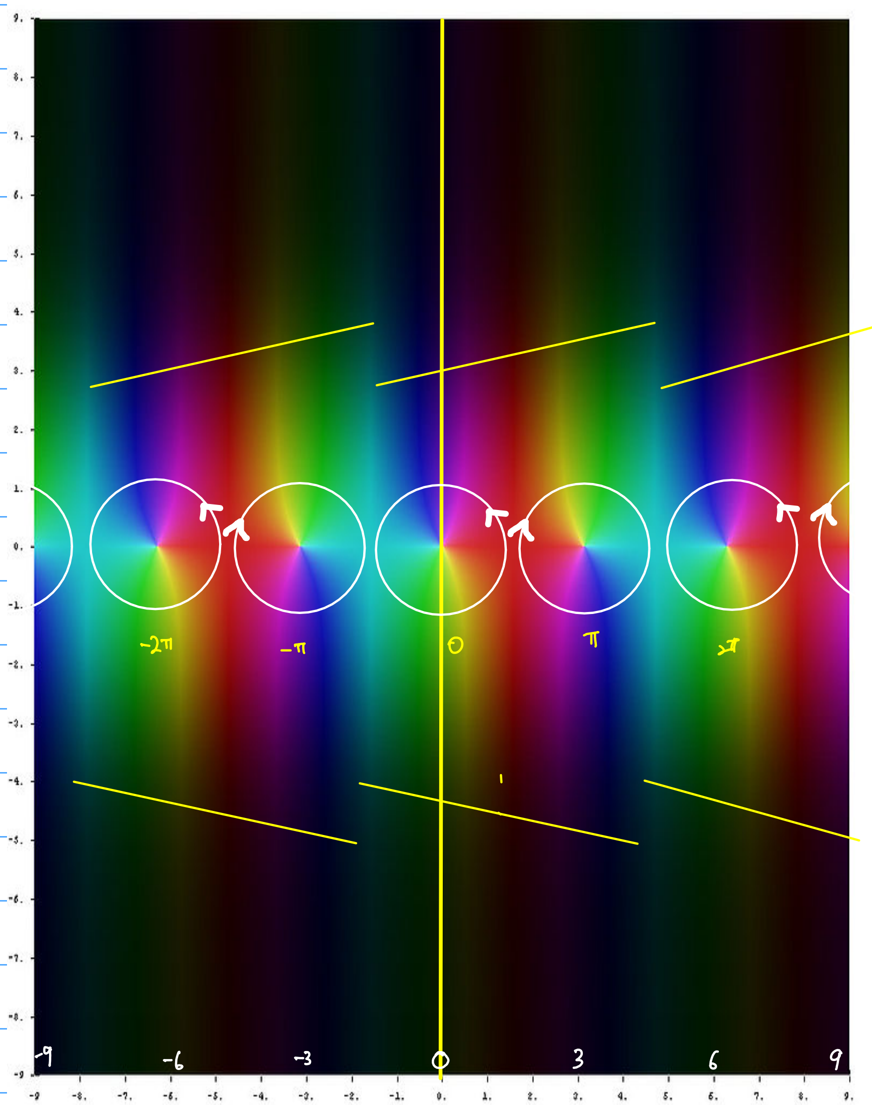
$\boxed{\sinh^2(y)}$

https://en.wikipedia.org/wiki/Trigonometric_functions

⑥ ang(csc z)







⑥ $\arg(\csc z)$

$$|\csc z|^2 = \frac{1}{|\sin z|^2} = \frac{1}{\sin^2(x) + \boxed{\sinh^2(y)}}$$

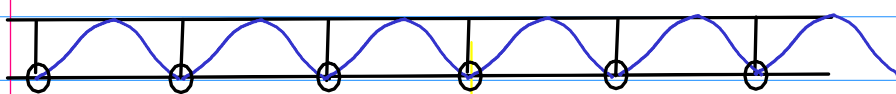
$\sinh(0) = 0$

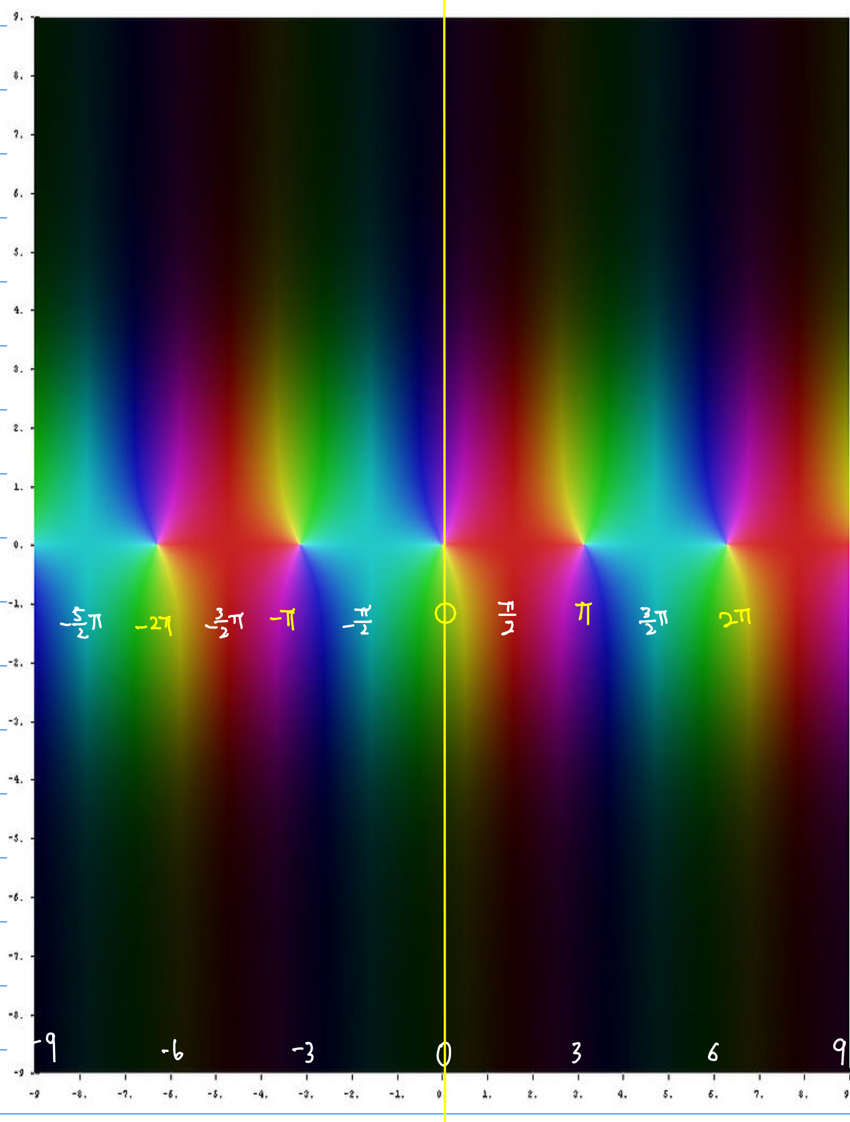
pole: 1

$x = 0, \pm\pi, \pm 2\pi, \cdots$

$x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \cdots$

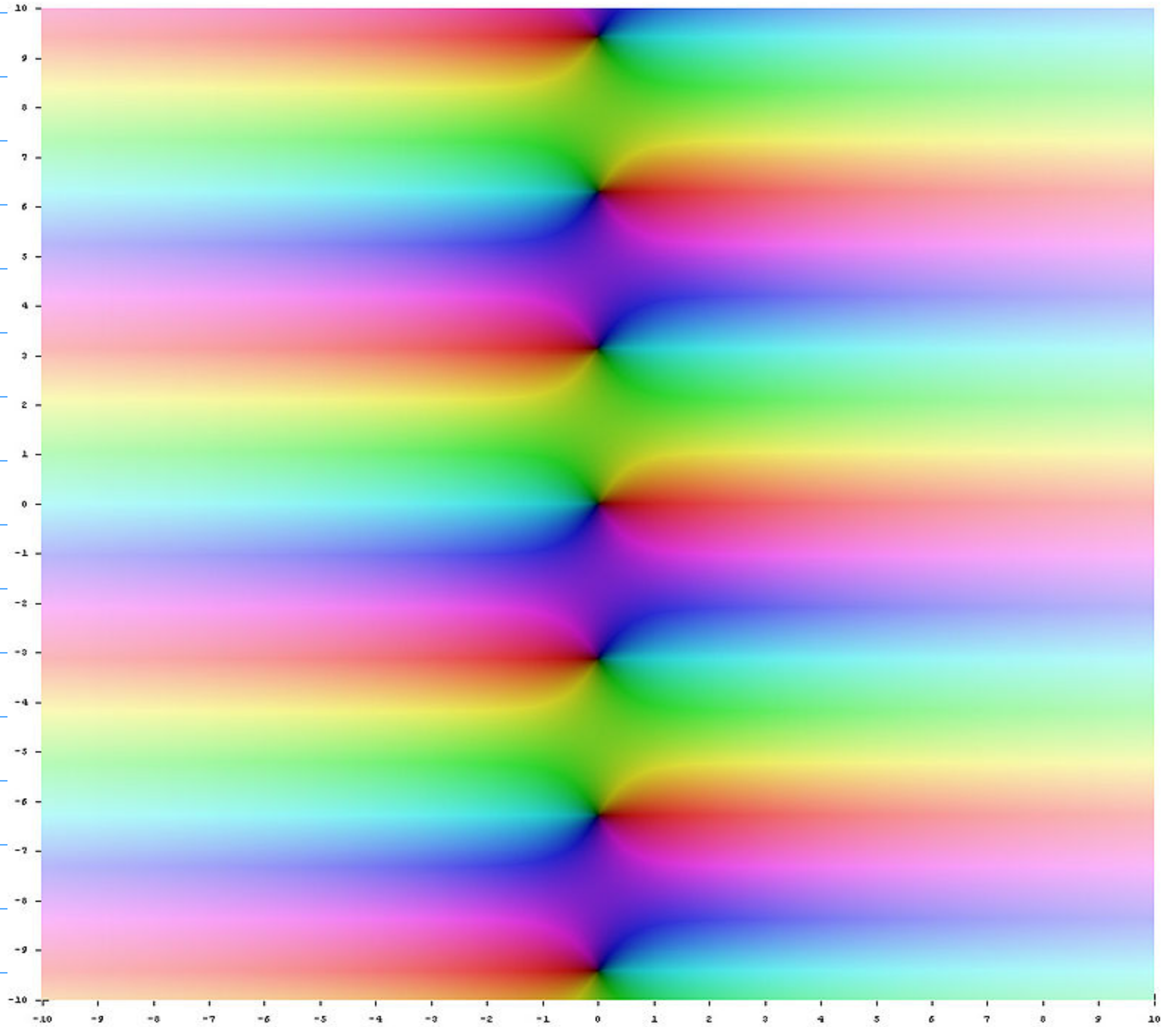
$\sin^2(x)$





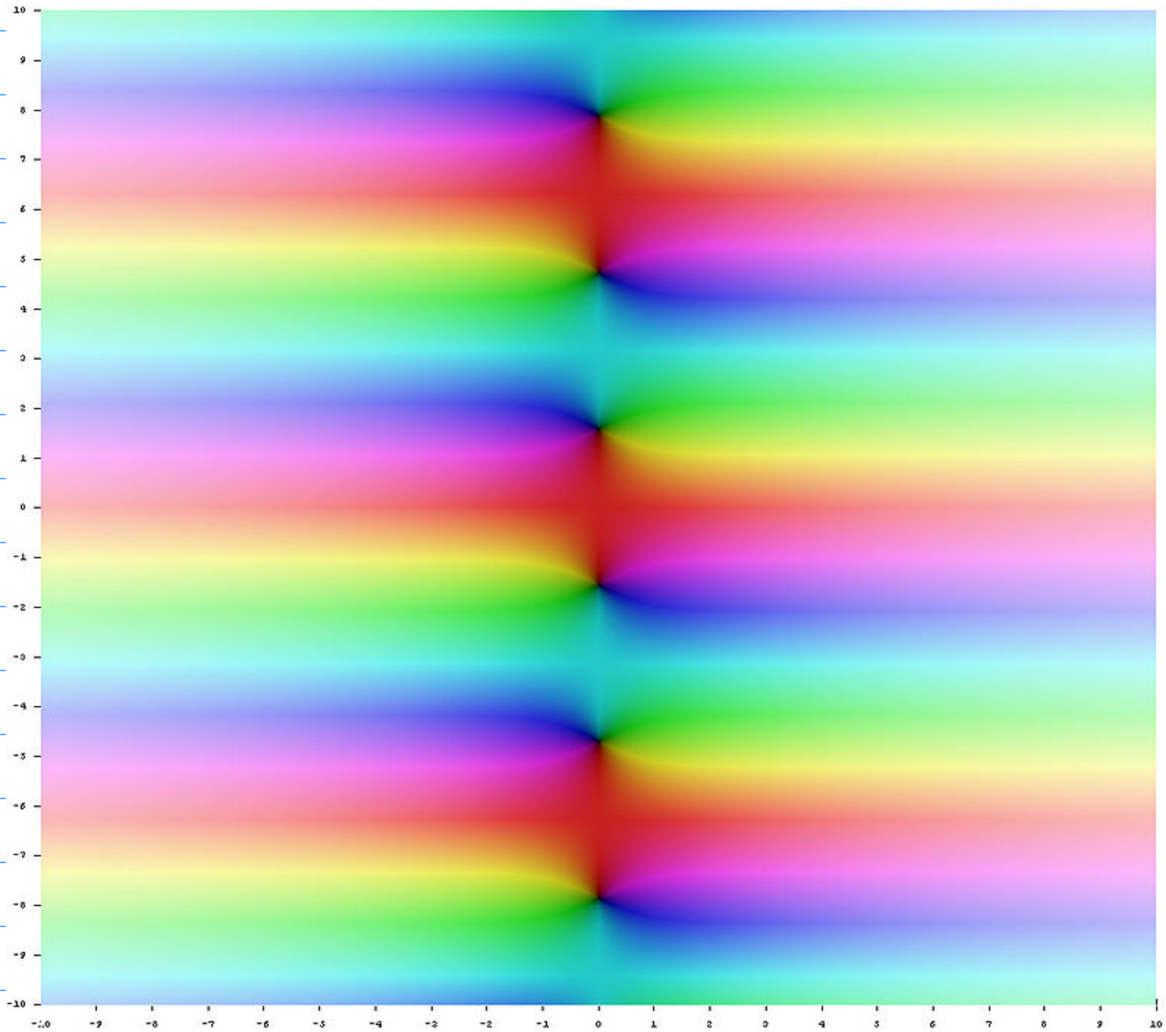
sinh z

https://en.wikipedia.org/wiki/Hyperbolic_function



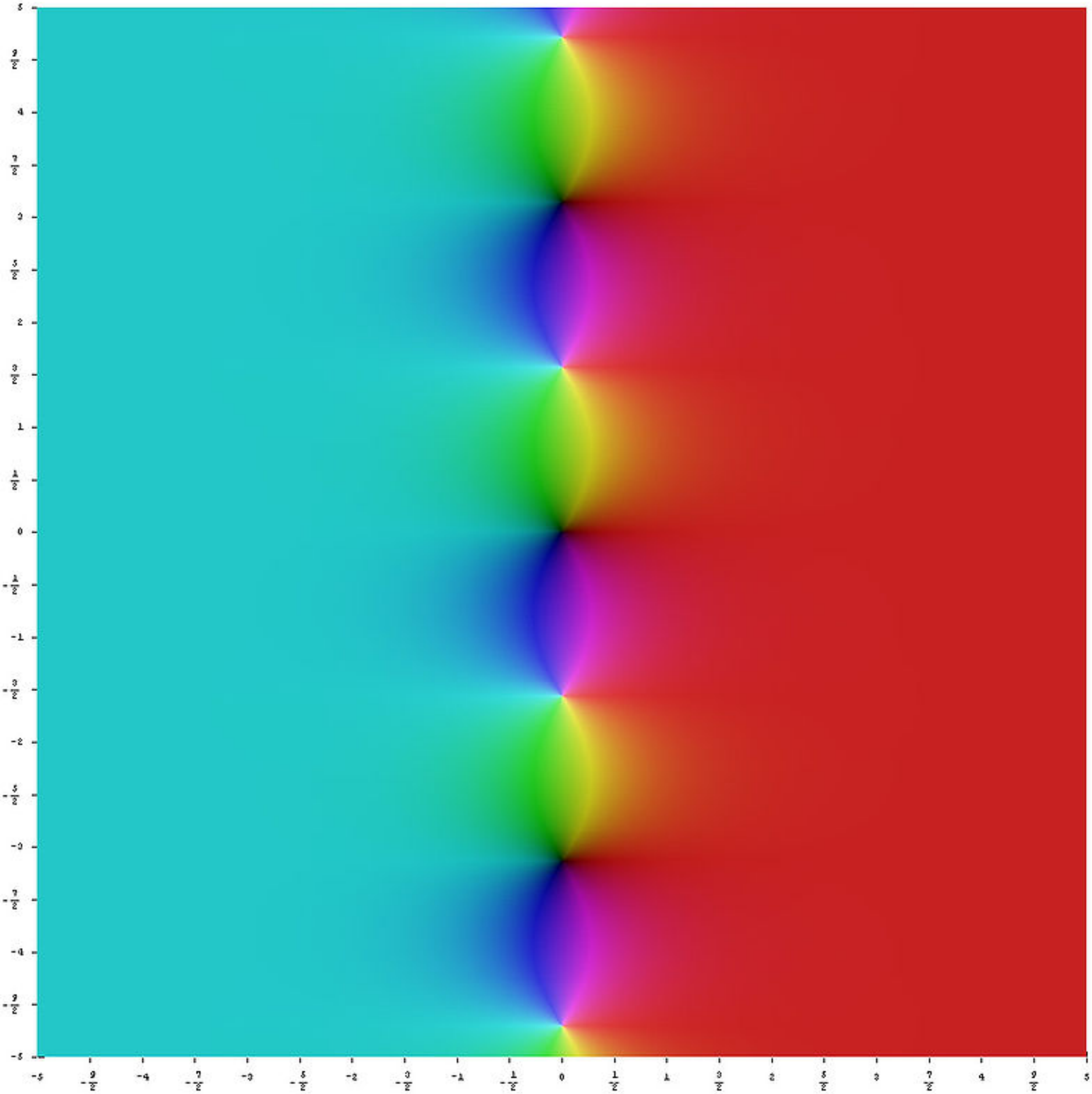
cosh z

https://en.wikipedia.org/wiki/Hyperbolic_function



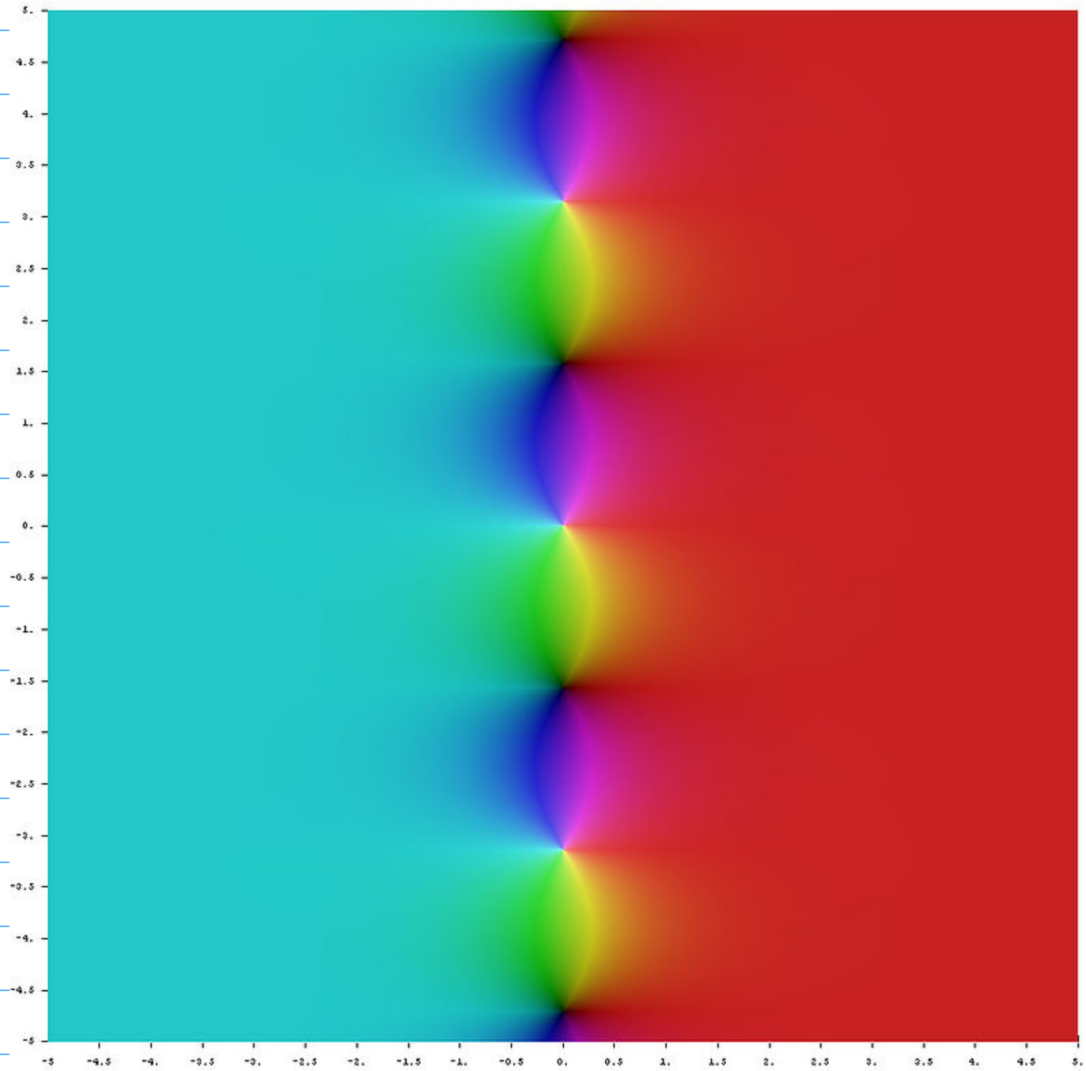
tanh z

https://en.wikipedia.org/wiki/Hyperbolic_function



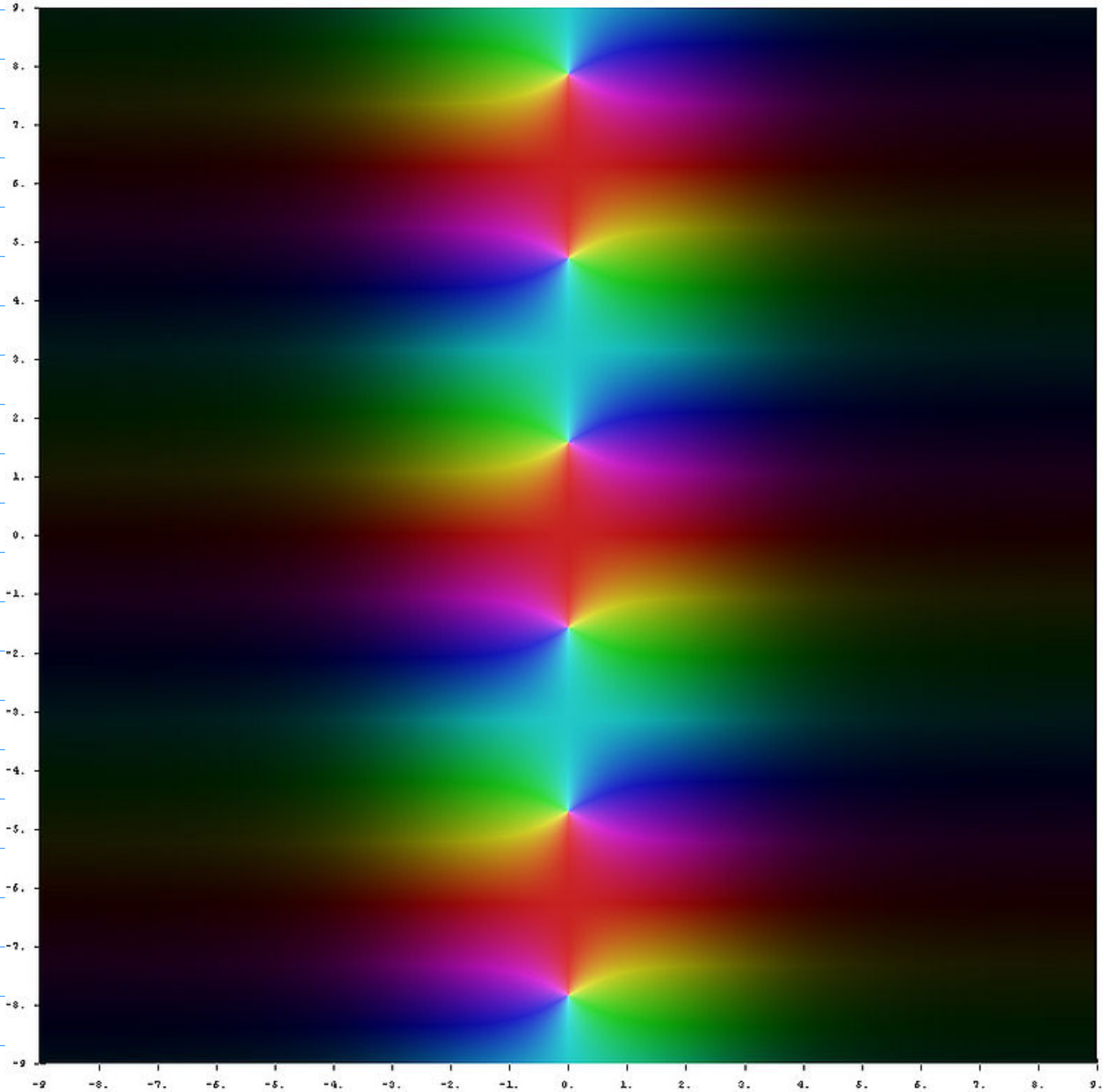
coth z

https://en.wikipedia.org/wiki/Hyperbolic_function



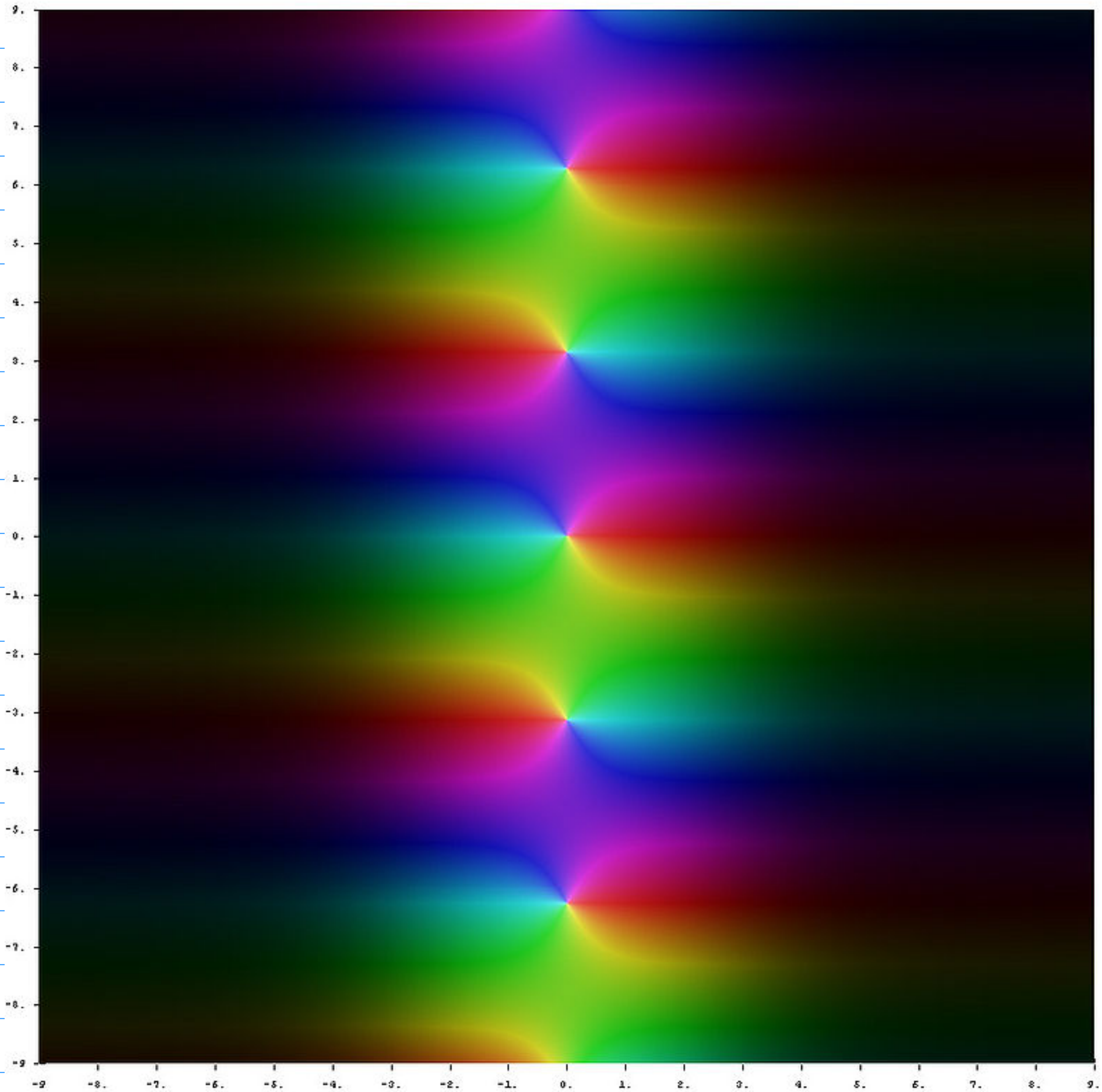
Sech z

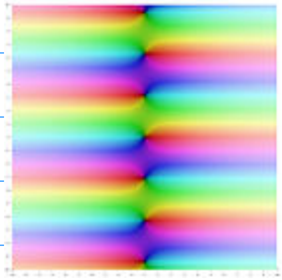
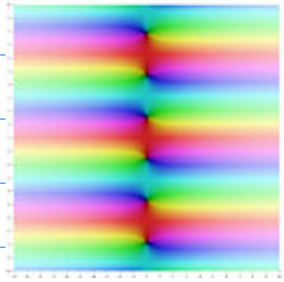
https://en.wikipedia.org/wiki/Hyperbolic_function

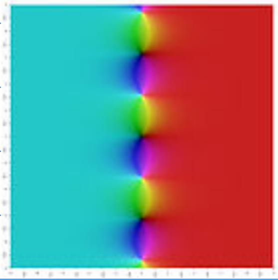
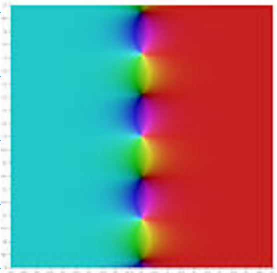


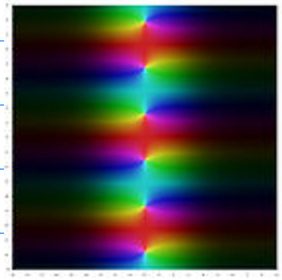
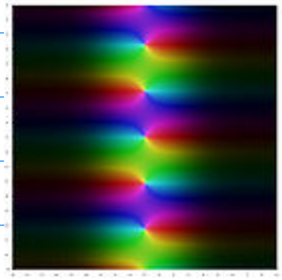
csch z

https://en.wikipedia.org/wiki/Hyperbolic_function



https://en.wikipedia.org/wiki/Hyperbolic_function