Vector Functions (1A)

- Vector Functions
- Motion
- Curvature

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Vector Valued Functions

Set of points $(x,\ y,\ z)$

Parametric functions

$$\left\{\begin{aligned} x &= f(t) \\ y &= g(t) \\ z &= h(t) \end{aligned}\right. \Rightarrow (f(t),\ g(t),\ h(t))$$

parameter t

Vector Valued Function

a given t ➡ A point in 3-d space → Position Vector

$$\mathbf{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$$

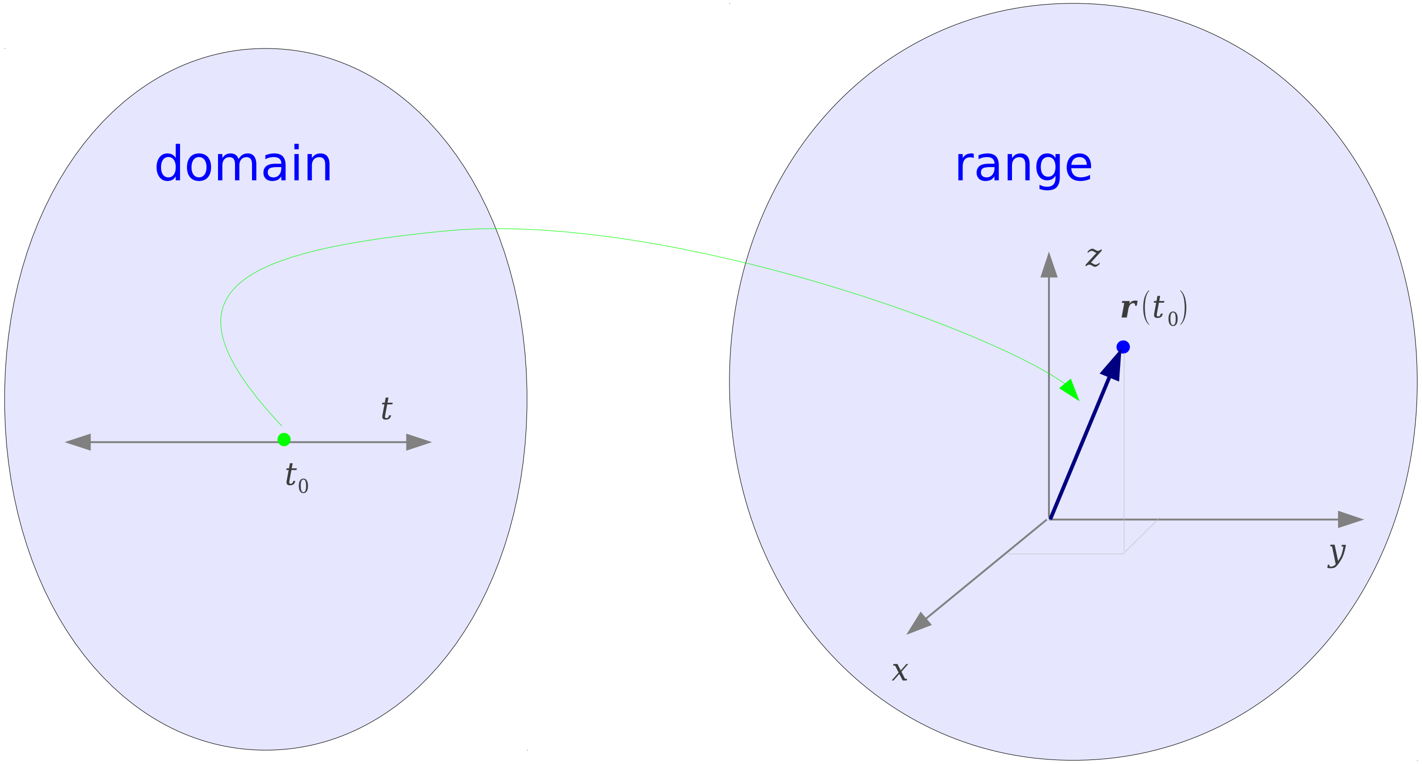
$$= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

Vector Valued Functions (1)

Vector Valued Function

$$\mathbf{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$$

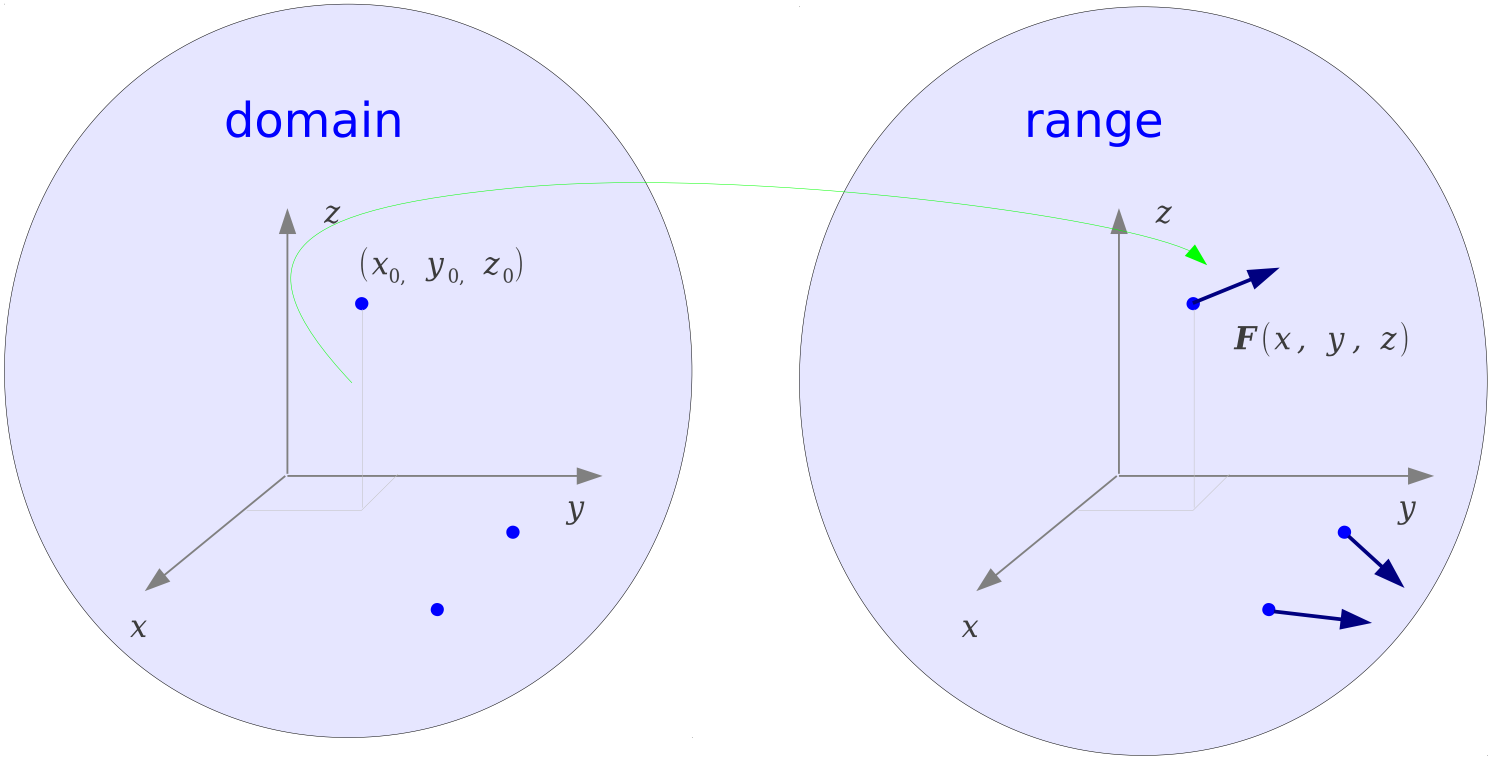
$$= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$



a given t_0 ➡ A position vector

Vector Valued Functions (2)

Vector Field (x, y, z) $\boldsymbol{F}(x, y, z)$



a given point in a 3-d space (x_0, y_0, z_0) ➡ A vector (u_0, v_0, w_0)

Limit of a Vector Function

Vector Valued Function $\boldsymbol{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$

Limit of a Vector Valued Function

$$\lim_{t \to a} \boldsymbol{r}(t) = \left\langle \lim_{t \to a} f(t),\ \lim_{t \to a} g(t),\ \lim_{t \to a} h(t) \right\rangle$$

Limit of a Vector Valued Function

$\boldsymbol{r}(a)$ is defined

$\lim_{t \to a} \boldsymbol{r}(t)$ exists

$$\boldsymbol{r}(a) = \lim_{t \to a} \boldsymbol{r}(t)$$

Derivative of a Vector Function

Vector Valued Function $\boldsymbol{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$

Derivative of a Vector Valued Function

$$\boldsymbol{r}'(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)}{\Delta t}$$

$\boldsymbol{r}(t)$	Position Vector	$\boldsymbol{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$
$\boldsymbol{r}'(t)$	Velocity Vector Tangent Vector, also	$\boldsymbol{r}'(t) = \langle f'(t),\ g'(t),\ h'(t) \rangle$

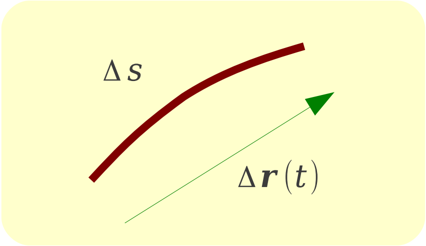
Arc Length (1)

Vector Valued Function

$$\boldsymbol{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$$

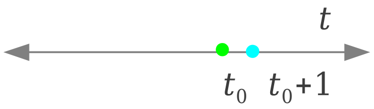
Derivative of a Vector Valued Function

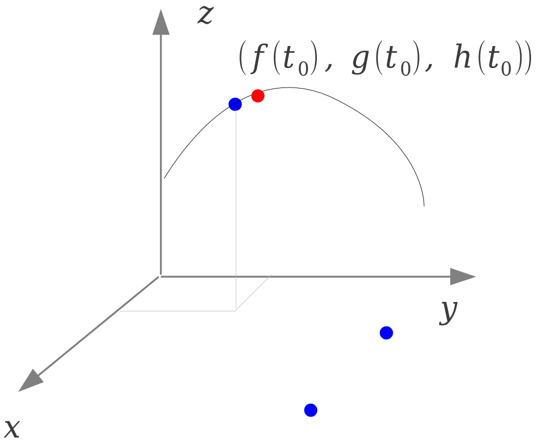
$$\boldsymbol{r}'(t) = \langle f'(t),\ g'(t),\ h'(t) \rangle$$

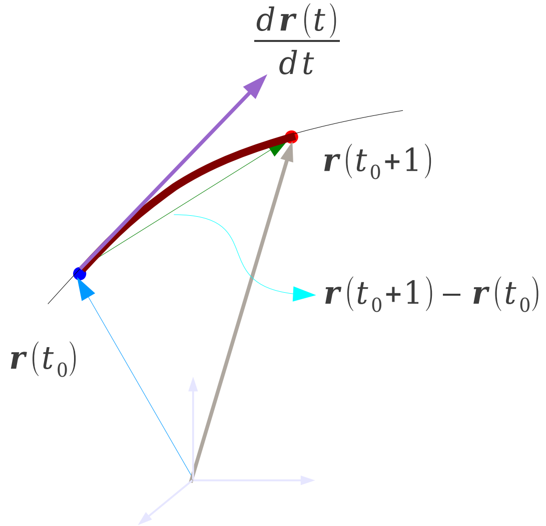


$$\boldsymbol{r}(t_0) = (f(t_0),\ g(t_0),\ h(t_0))$$

$$\boldsymbol{r}(t_0+1) = (f(t_0+1),\ g(t_0+1),\ h(t_0+1))$$







Arc Length (2)

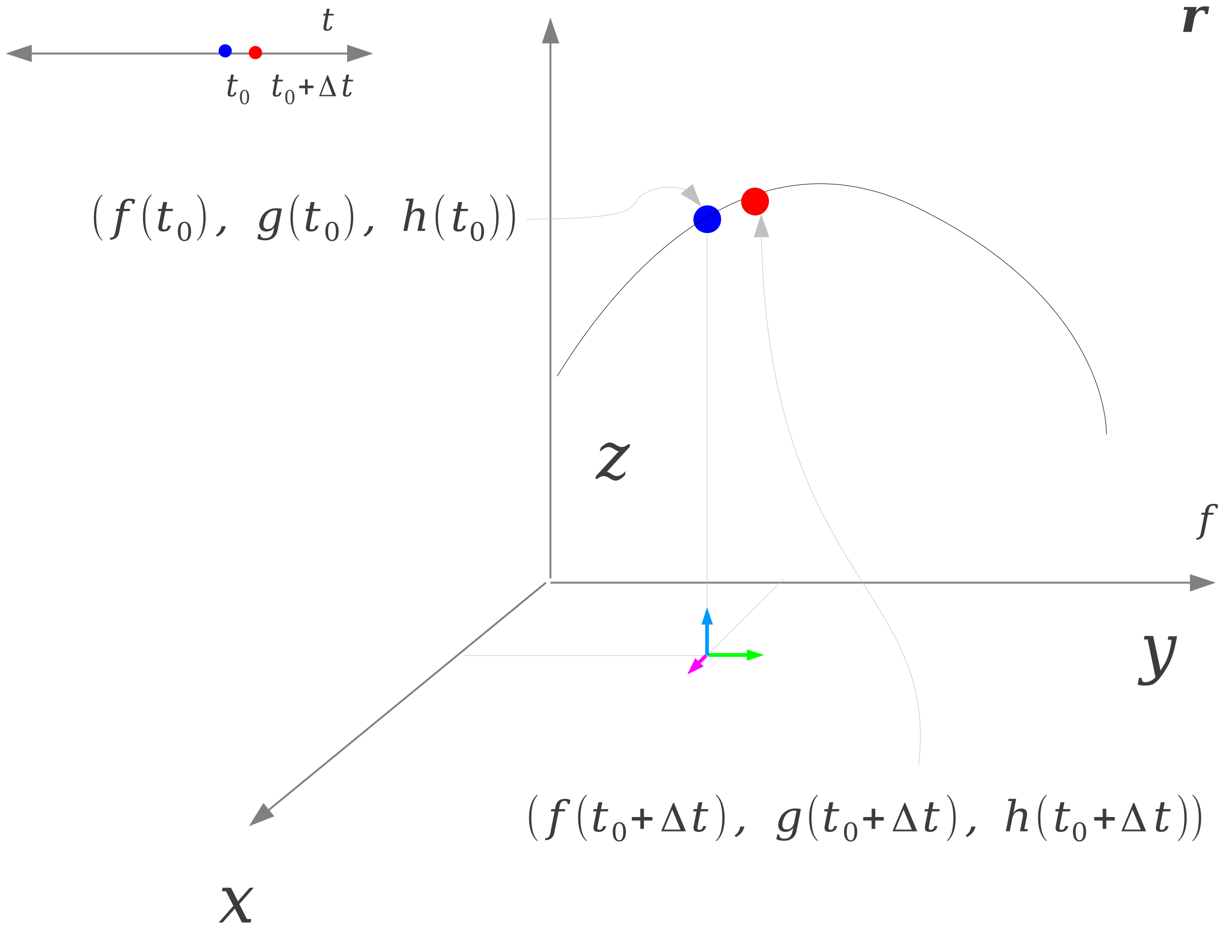
Vector Valued Function

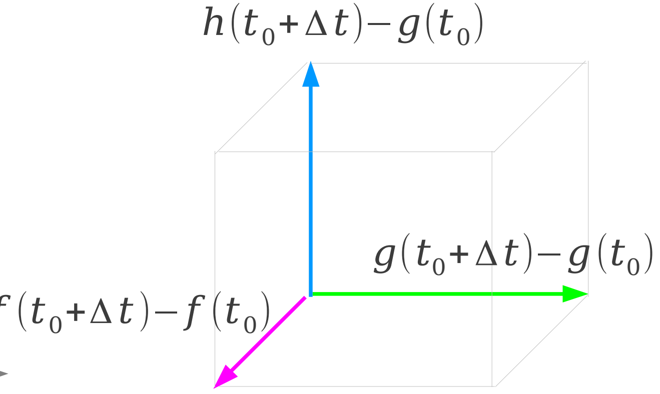
$$\mathbf{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$$

Derivative of a Vector Valued Function

$$\mathbf{r}'(t) = \langle f'(t),\ g'(t),\ h'(t) \rangle$$

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$





Arc Length (3)

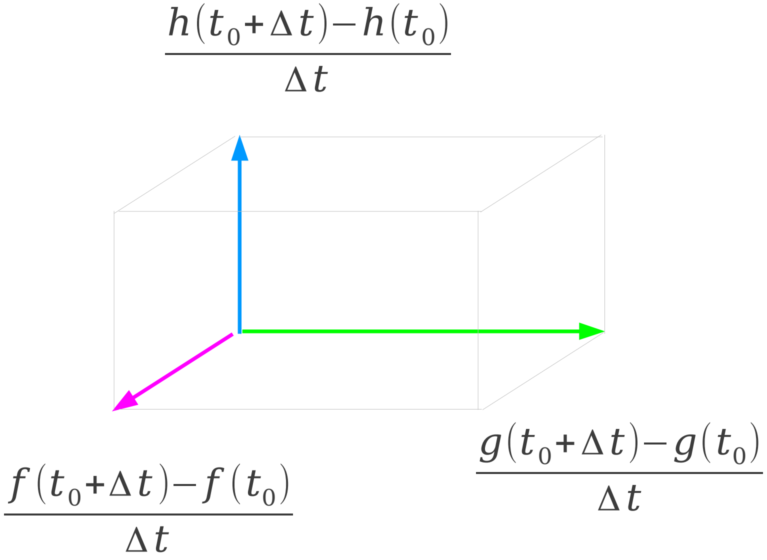
Vector Valued Function

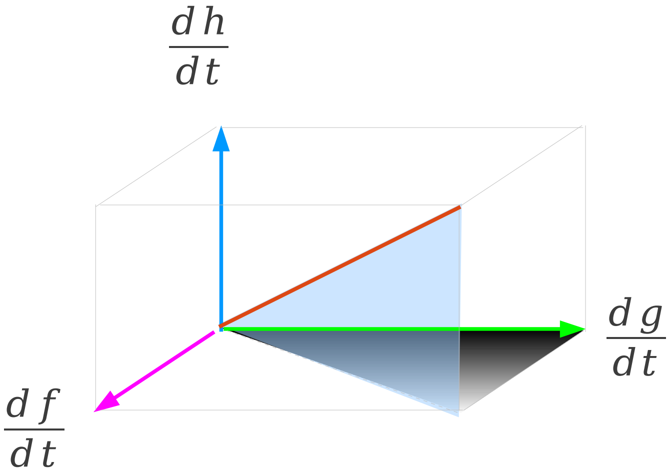
$$\boldsymbol{r}(t) = \langle f(t),\ g(t),\ h(t)\rangle$$

Derivative of a Vector Valued Function

$$\boldsymbol{r}'(t) = \langle f'(t),\ g'(t),\ h'(t)\rangle$$

$$\boldsymbol{r}'(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)}{\Delta t}$$





$$\sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} = \|\boldsymbol{r}'(t)\|$$

Arc Length (4)

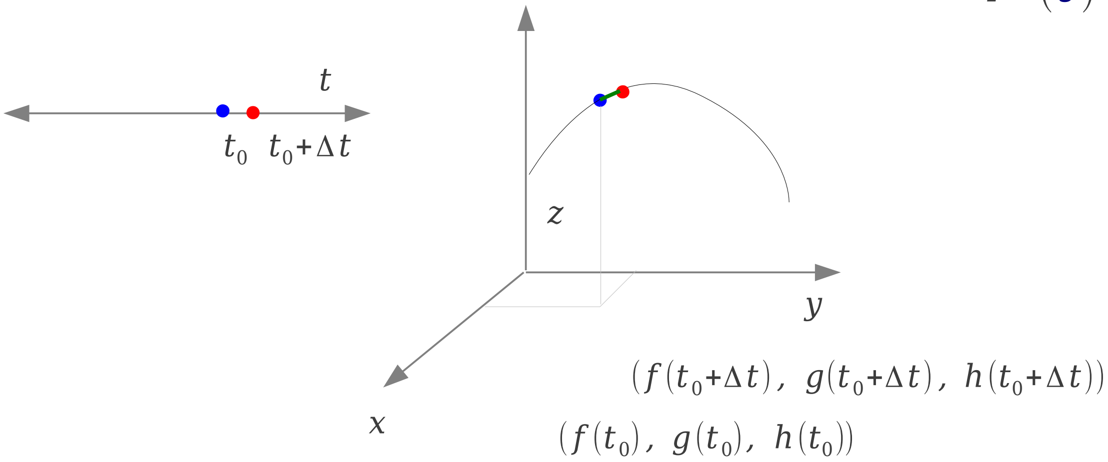
Vector Valued Function

$$\boldsymbol{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$$

Derivative of a Vector Valued Function

$$\boldsymbol{r}'(t) = \langle f'(t),\ g'(t),\ h'(t) \rangle$$

$$\boldsymbol{r}'(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)}{\Delta t}$$



Length of a Smooth Curve

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

$$= \int_a^b \|\boldsymbol{r}'(t)\|\, dt$$

Arc Length as a Parameter

Length of a Smooth Curve

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h\ (t)]^2}\, dt$$

$$= \int_a^b \|\mathbf{r}'(t)\|\, dt$$

Directed Distance from $P(t_0)$

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)|\, d\tau$$

s increases in the direction of increasing t

➡ Arc Length Parameter

$$= \int_{t_0}^{t} \|\mathbf{r}'(\tau)\|\, d\tau$$

$$= \int_{t_0}^{t} \sqrt{[f'(\tau)]^2 + [g'(\tau)]^2 + [h'(\tau)]^2}\, d\tau$$

Speed, Velocity, Unit Tangent Vector

Arc Length Parameter

$$s(t) = \int_{t_0}^{t} |\boldsymbol{v}(\tau)|\, d\tau$$

Speed is the absolute value of $\boldsymbol{v}(t)$

$$\frac{ds}{dt} = |\boldsymbol{v}(t)|$$

$$\frac{d}{dt} s(t) = \frac{d}{dt} \int_{t_0}^{t} |\boldsymbol{v}(\tau)|\, d\tau$$

Velocity

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v}(t)$$

Unit Tangent Vector

$$\boldsymbol{T} = \frac{\boldsymbol{v}(t)}{|\boldsymbol{v}(t)|}$$

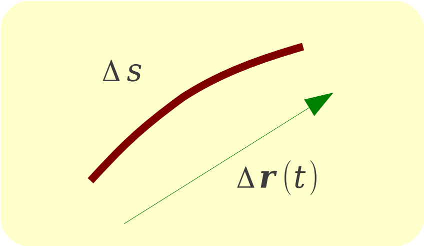
Unit Tangent Vector

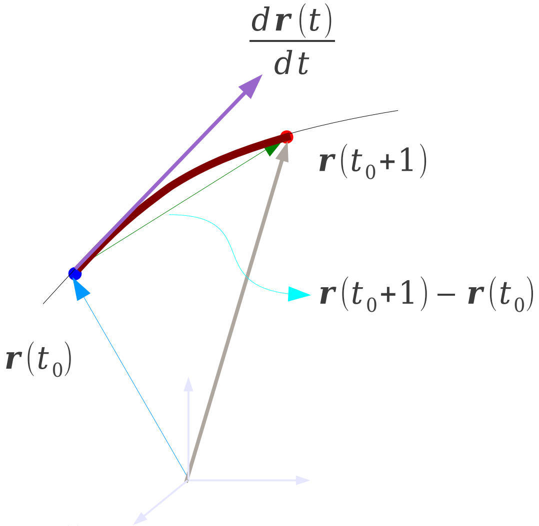
$$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)|\, d\tau \qquad \frac{ds}{dt} = |\mathbf{v}(t)| \qquad \frac{d\mathbf{r}}{dt} = \mathbf{v}(t) \qquad \mathbf{T} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$$

$$\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{1}{|\mathbf{v}(t)|}$$

Unit Tangent Vector

$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = \mathbf{v}(t)\frac{1}{|\mathbf{v}(t)|} = \mathbf{T}$$

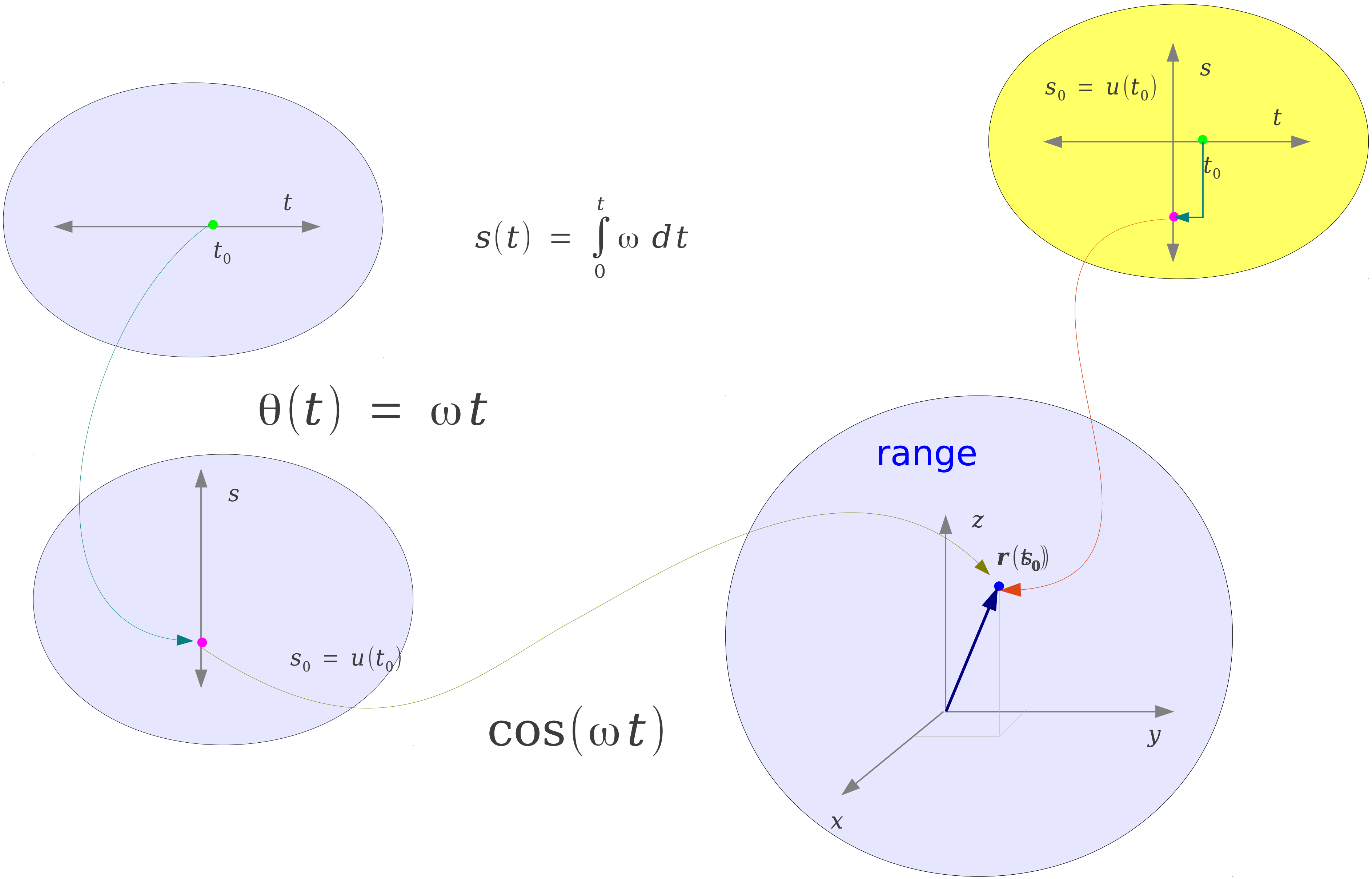




$$\mathbf{r}(t_0) = (f(t_0),\ g(t_0),\ h(t_0))$$

$$\mathbf{r}(t_0+1) = (f(t_0+1),\ g(t_0+1),\ h(t_0+1))$$

Composite Function



$$s(t) = \int_0^t \omega \, dt$$

$$\theta(t) = \omega t$$

$$\cos(\omega t)$$

range

Chain Rule of a Vector Function (1)

Derivative of a Vector Valued Function

$$s = u(t)$$

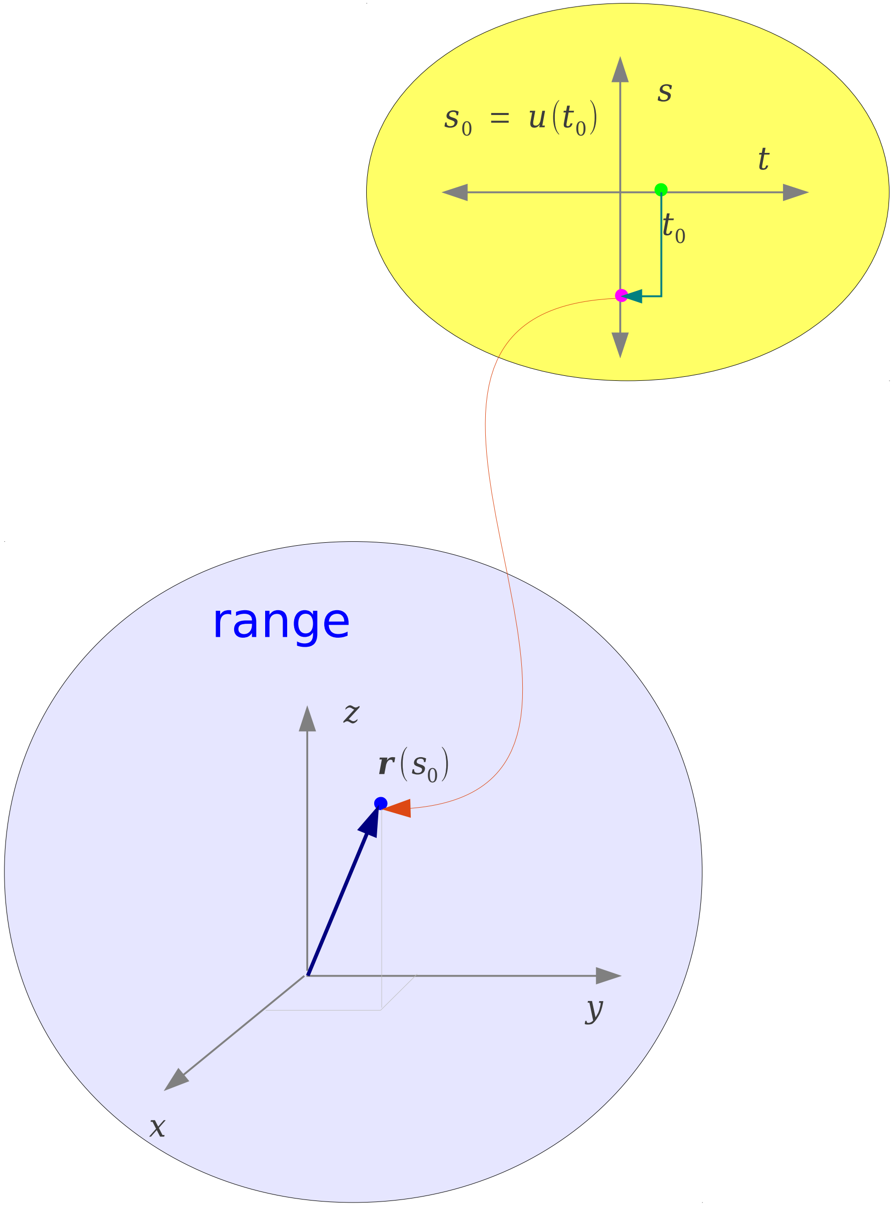
$$\frac{ds}{dt} = \frac{du(t)}{dt} \Rightarrow u'(t)$$

$$\frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}}{ds}\frac{ds}{dt} = \boldsymbol{r}'(s)\,u'(t)$$

$$s(t) = \int_{t_0}^{t} |\boldsymbol{v}(\tau)|\, d\tau = \int_{t_0}^{t} \|\boldsymbol{r}'(\tau)\|\, d\tau$$

$$= \int_{t_0}^{t} \sqrt{[f'(\tau)]^2 + [g'(\tau)]^2 + [h'(\tau)]^2}\, d\tau$$

$$\frac{ds}{dt} = |\boldsymbol{v}(t)| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$



Chain Rule of a Vector Function (2)

Vector Valued Function $\quad \boldsymbol{r}(s) = \langle f(s),\ g(s),\ h(s) \rangle$

Scalar Function $\quad s = u(t)$

$$\boldsymbol{r}(u(t)) = \langle f(u(t)),\ g(u(t)),\ h(u(t)) \rangle$$

Derivative of a Vector Valued Function

$$s = u(t)$$

$$\frac{ds}{dt} = \frac{du(t)}{dt} \quad \Rightarrow \quad u'(t)$$

$$\frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}}{ds}\frac{ds}{dt} = \boldsymbol{r}'(s)u'(t)$$

Integration of a Vector Function

Vector Valued Function

$$\boldsymbol{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$$

$$= f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$$

Limit of a Vector Valued Function

$$\int \boldsymbol{r}(t)\,dt = \left\langle \int f(t)\,dt,\ \int g(t)\,dt,\ \int h(t)\,dt \right\rangle$$

$$= \int f(t)\,dt\,\boldsymbol{i} + \int g(t)\,dt\,\boldsymbol{j} + \int h(t)\,dt\,\boldsymbol{k}$$

Displacement, Velocity, Acceleration

Displacement

$$\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$$

Velocity

$$\boldsymbol{v}(t) = \boldsymbol{r}'(t) = f'(t)\boldsymbol{i} + g'(t)\boldsymbol{j} + h'(t)\boldsymbol{k}$$

Acceleration

$$\boldsymbol{a}(t) = \boldsymbol{v}'(t) = \boldsymbol{r}''(t) = f''(t)\boldsymbol{i} + g''(t)\boldsymbol{j} + h''(t)\boldsymbol{k}$$

Speed

$$\|\boldsymbol{v}(t)\| = \left\|\frac{\boldsymbol{r}(t)}{dt}\right\| = \|f'(t)\boldsymbol{i} + g'(t)\boldsymbol{j} + h'(t)\boldsymbol{k}\|$$

$$= \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Unit Tangent of a Vector Function (1)

Displacement

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

Velocity

$$\mathbf{v}(t) = \mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

Unit Tangent

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

$$\mathbf{r}'(t) \quad \Rightarrow \quad \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt}$$

Arc length

s

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h(t)]^2}\, dt = \int_a^b \|\mathbf{r}'(t)\|\, dt$$

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\|$$

Unit Tangent of a Vector Function (2)

Displacement $$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

Velocity $$\mathbf{v}(t) = \mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

velocity

- speed
- direction

Unit Tangent

direction $$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

$$\frac{d\mathbf{r}}{ds} = \frac{\frac{d\mathbf{r}}{dt}}{\frac{ds}{dt}} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \mathbf{T}(t)$$

speed

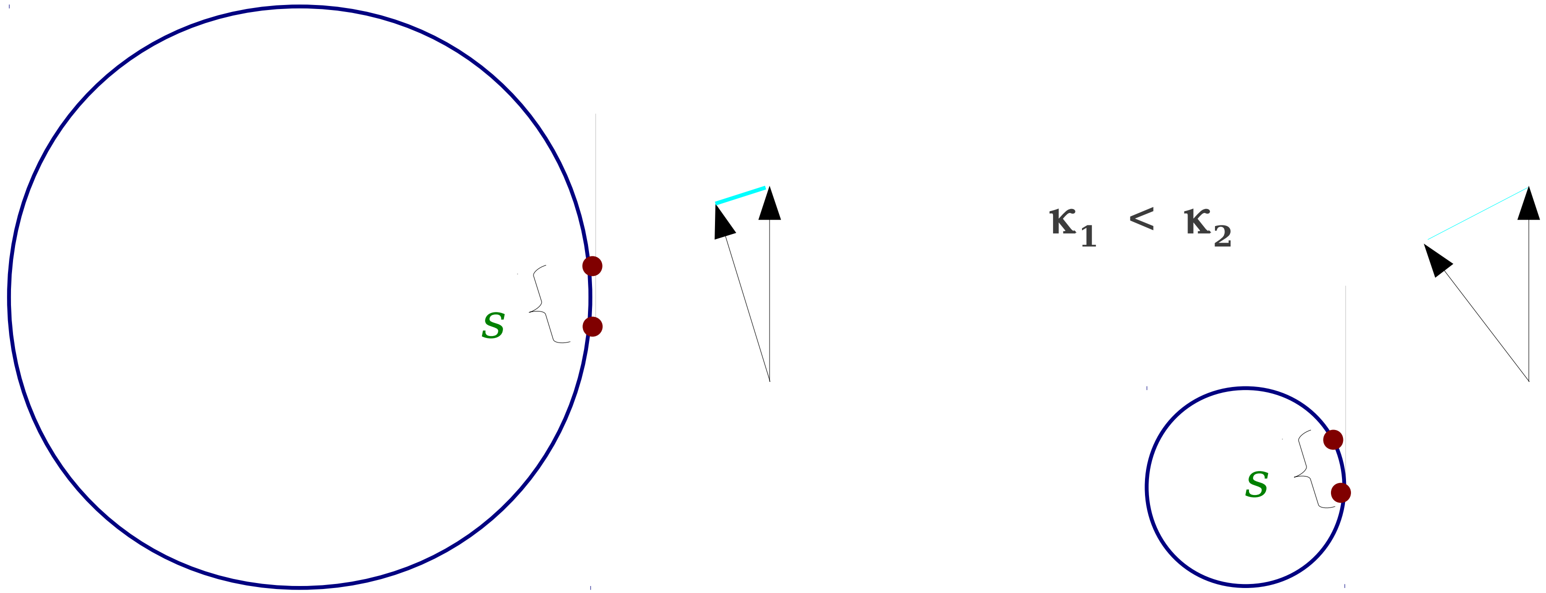
$$\mathbf{r}'(t) \Rightarrow \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} \Leftarrow \frac{ds}{dt} = \|\mathbf{r}'(t)\|$$

Curvature of a Vector Function (1)

Vector Valued Function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$

Unit Tangent $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \dfrac{d\mathbf{r}}{ds}$

Curvature $\kappa = \left\|\dfrac{d\mathbf{T}}{ds}\right\|$



$\kappa_1 < \kappa_2$

Curvature of a Vector Function (2)

Vector Valued Function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$

Unit Tangent $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \dfrac{d\mathbf{r}}{ds}$

Curvature $\kappa = \left\| \dfrac{d\mathbf{T}}{ds} \right\|$

Arc length s

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} \qquad \frac{ds}{dt} = \|\mathbf{r}'(t)\|$$

$$\frac{d\mathbf{T}}{ds} = \frac{\dfrac{d\mathbf{T}}{dt}}{\dfrac{ds}{dt}} = \frac{\mathbf{T}'(t)}{\|\mathbf{r}'(t)\|} = \kappa(t)$$

Line Equations (2)

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[4] D.G. Zill, “Advanced Engineering Mathematics”