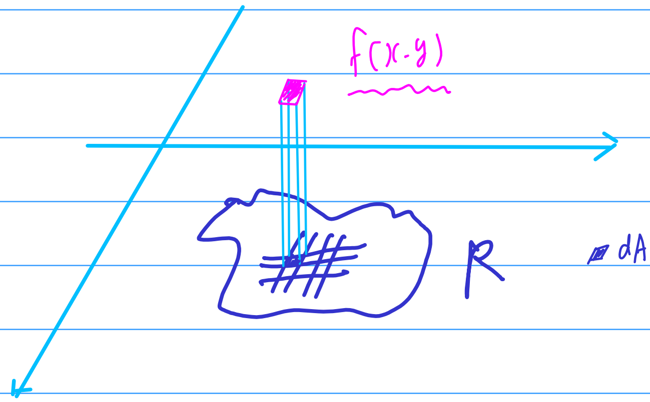
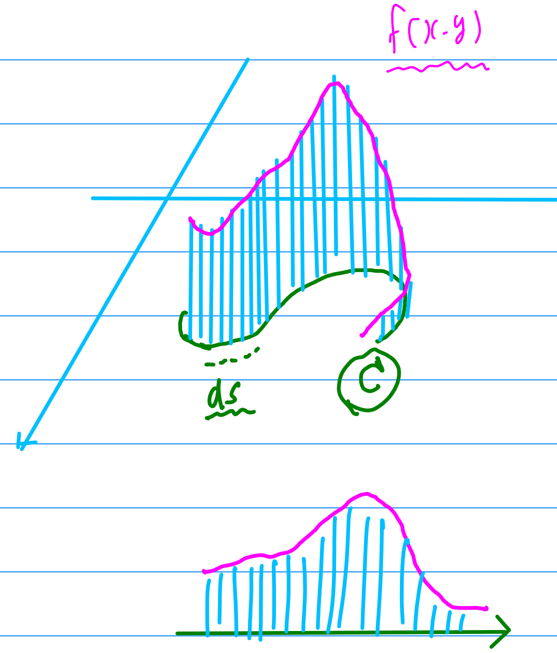
Vector Calculus (H.1)
Line Integrals

20160213

$$\iint_R f(x,y)\,dA \qquad\qquad \int_C f(x,y)\,ds$$





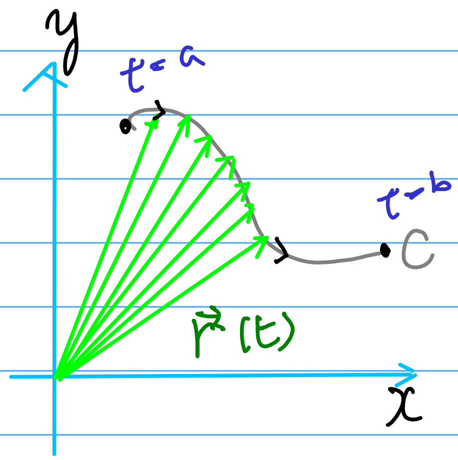
$dA = dx\,dy$

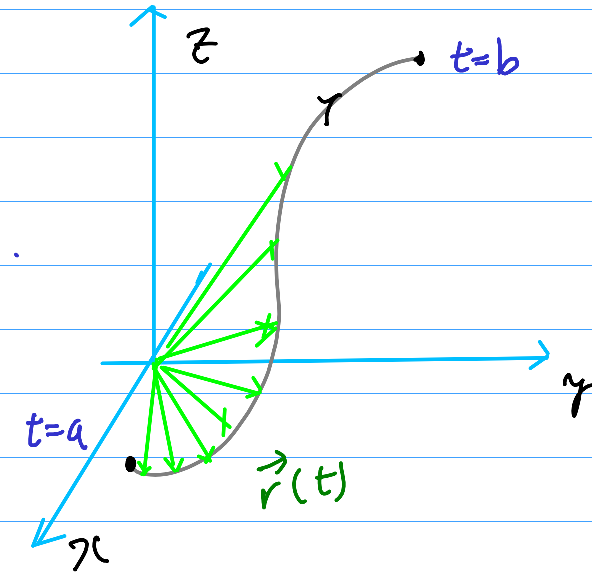
Fubini

$$\int \left[\int \, dx \right] dy$$

$\mathbb{R}^2$ $\int_C f(x,y)\,ds$

$\mathbb{R}^3$ $\int_C f(x,y,z)\,ds$





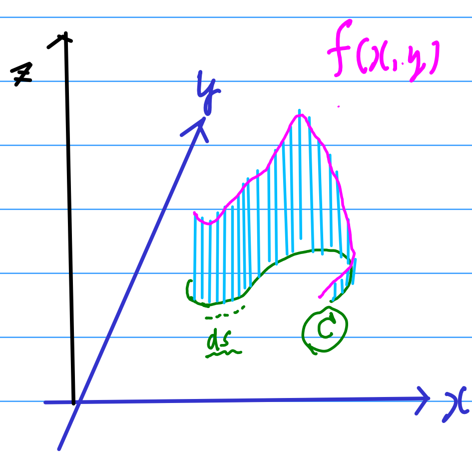
$\vec{r}(t) = \langle m(t), n(t) \rangle$

$\vec{r}(t) = \langle m(t), n(t), k(t) \rangle$

$x = m(t)$
$y = n(t)$

t: parameter (time)

$x = m(t)$
$y = n(t)$
$z = k(t)$

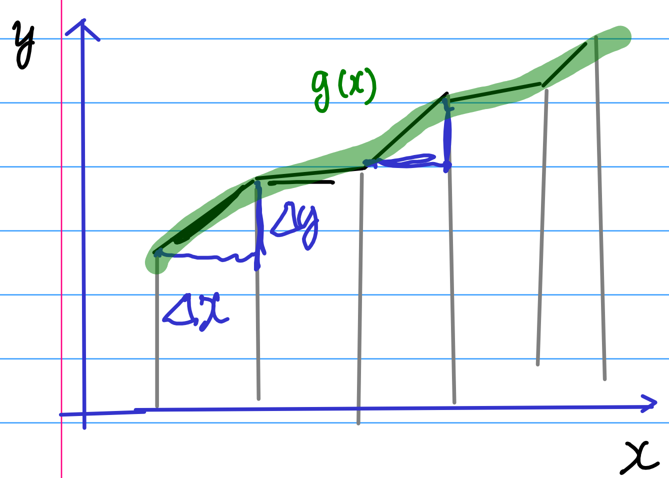


$f(x,y,z)$ defined on (x,y,z) points on the contour C

* 3-variable function

4-dimensional plot √

arc length

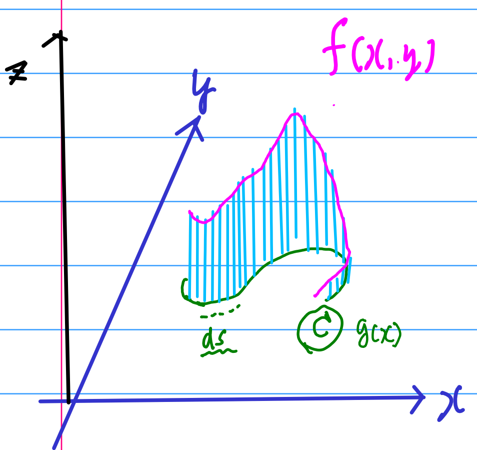


contour $y = g(x)$

$$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \Delta s$$

$$\Delta y = g'(x)\ \Delta x$$

$$dy = g'(x)\ dx$$



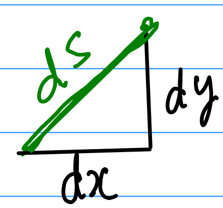
$$\sqrt{(dx)^2 + [g'(x)]^2 (dx)^2}$$

$$\sqrt{1 + [g'(x)]^2}\ dx = ds$$

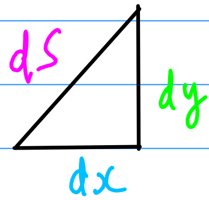
$$L = \int_c \sqrt{1+[g'(x)]^2}\ dx$$

$$= \int_c \sqrt{1+\left[\frac{dy}{dx}\right]^2}\ dx$$

$$= \int_c ds$$



$$\boxed{ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx \Leftrightarrow \sqrt{(dx)^2 + (dy)^2}}$$



$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \boxed{y = g(x)} \qquad \sqrt{1 + g'(x)} \cdot dx$$

$$= \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad \boxed{x = h(y)} \qquad \sqrt{1 + h'(y)} \cdot dy$$

$$ds = \sqrt{\left(\frac{dm}{dt}\right)^2 + \left(\frac{dn}{dt}\right)^2}\, dt \qquad \boxed{\begin{array}{l} x = m(t) \\ y = n(t) \end{array}}$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \boxed{\begin{array}{l} x = x(t) \\ y = y(t) \end{array}}$$

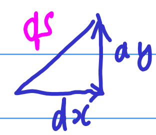
Parameterized x & y

$$\begin{cases} x = m(t) \\ y = n(t) \end{cases}$$

$$dx = \frac{dm}{dt}\,dt$$

$$dy = \frac{dn}{dt}\,dt$$

$$ds = \sqrt{(dx)^2 + (dy)^2}$$



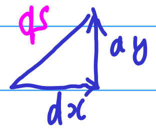
$$\sqrt{\left(\frac{dm}{dt}\,dt\right)^2 + \left(\frac{dn}{dt}\,dt\right)^2} = \sqrt{\left(\frac{dm}{dt}\right)^2 + \left(\frac{dn}{dt}\right)^2}\,dt = ds$$

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

$$dx = \frac{dx}{dt}\,dt$$

$$dy = \frac{dy}{dt}\,dt$$

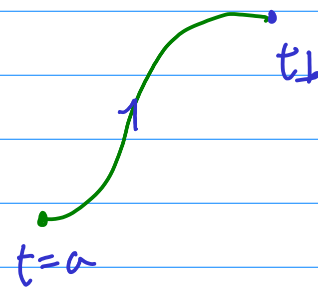
$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = ds$$



$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = ds$$

Arc Length L

$$L = \int_C ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$



Line Integral

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt}$$

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

$$\boxed{\int_{-C} f(x,y)\, ds = \int_{C} f(x,y)\, ds}$$

$$\boxed{\begin{aligned} \int_{-C} f(x,y)\, dx &= -\int_{C} f(x,y)\, dx \\ \int_{-C} f(x,y)\, dy &= -\int_{C} f(x,y)\, dy \end{aligned}}$$

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_a^b f(x(t), y(t))\, \|\vec{r}\,'(t)\|^2\, dt$$

$$= \int_a^b f(\vec{r}(t))\, \|\vec{r}\,'(t)\|^2\, dt$$

Curve defined parametrically

$$\int_C G(x,y)\,\boxed{dx} = \int_a^b G(f(t), g(t))\, f'(t)\,\boxed{dt}$$

$$\int_C G(x,y)\,\boxed{dy} = \int_a^b G(f(t), g(t))\, g'(t)\,\boxed{dt}$$

$$\int_C G(x,y)\,\boxed{ds} = \int_a^b G(f(t), g(t))\sqrt{f'(t) + g'(t)}\,\boxed{dt}$$

Curve defined by an explicit function

$$\int_C G(x,y)\,\boxed{dx} = \int_C G(x, f(x))\,\boxed{dx}$$

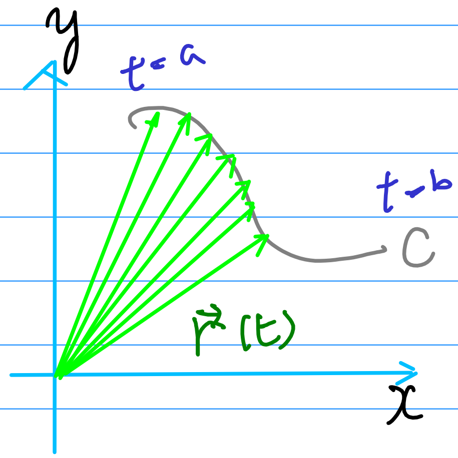
$$\int_C G(x,y)\,\boxed{dy} = \int_C G(x, f(x))\, f'(x)\,\boxed{dx}$$

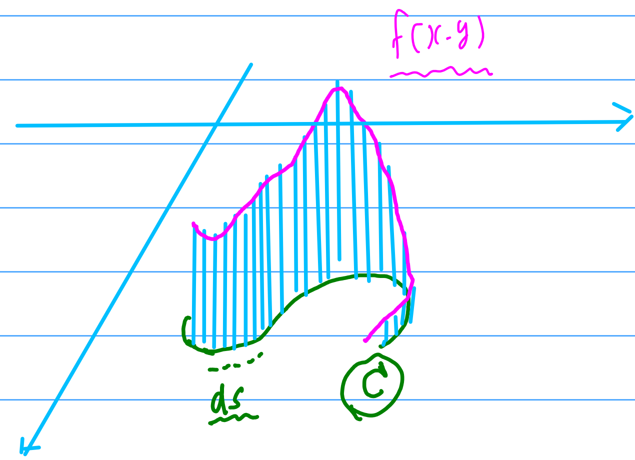
$$\int_C G(x,y)\,\boxed{ds} = \int_C G(x, f(x))\sqrt{1 + [f'(x)]^2}\,\boxed{dx}$$

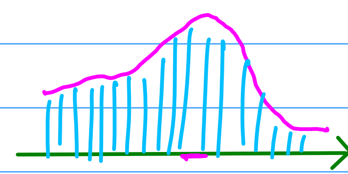
$$\int_C f(x,y)\,ds$$

$$\vec{r}(t) = \langle m(t), n(t) \rangle$$

R^2



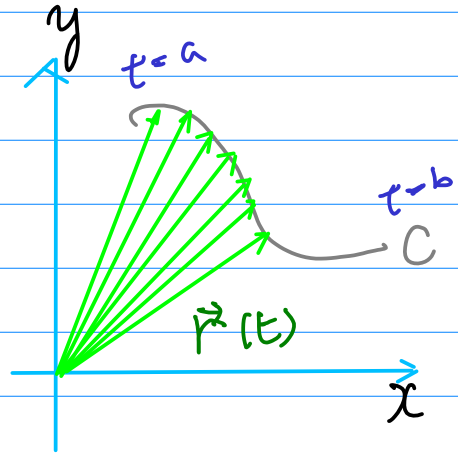


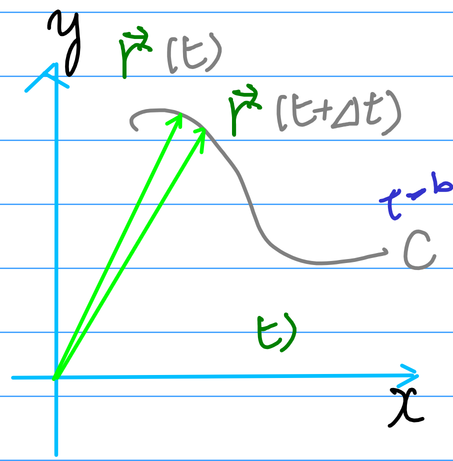


$$\int_C f(x,y)\,ds$$

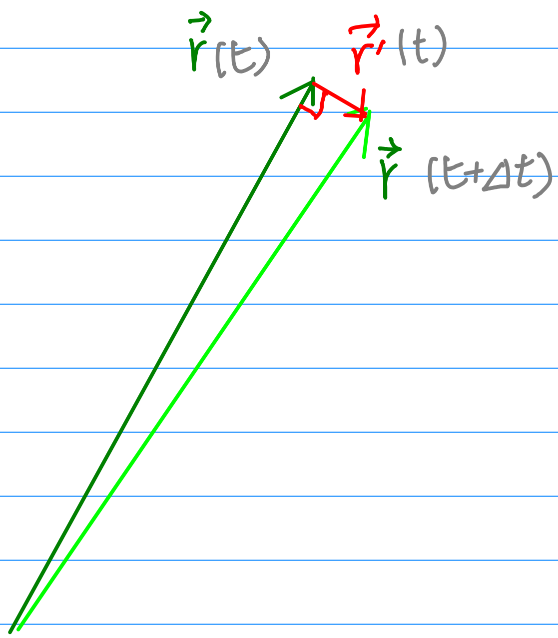
$$= \int_C f(m(t), n(t))\,ds$$

$$= \int_a^b f(m(t), n(t)) \sqrt{\left(\frac{dm}{dt}\right)^2 + \left(\frac{dn}{dt}\right)^2}\,dt$$





$\vec{r}(t) = \langle m(t), n(t) \rangle$ $\vec{r}(t) = \langle m(t), n(t) \rangle$

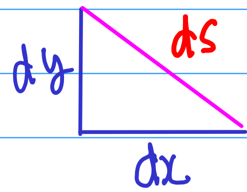


$$\boxed{\vec{r}(t+\Delta t) - \vec{r}(t)}$$

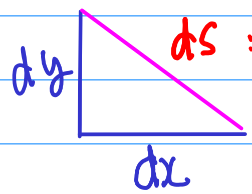
$$\lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \vec{r}'(t)$$

$$\vec{r}'(t) \perp \vec{r}(t)$$

a vector orthogonal to $\underline{\vec{r}(t)}$



$ds \Rightarrow \|\vec{r}'(t)\|$



$\Rightarrow \|\vec{r}\,'(t)\|$

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

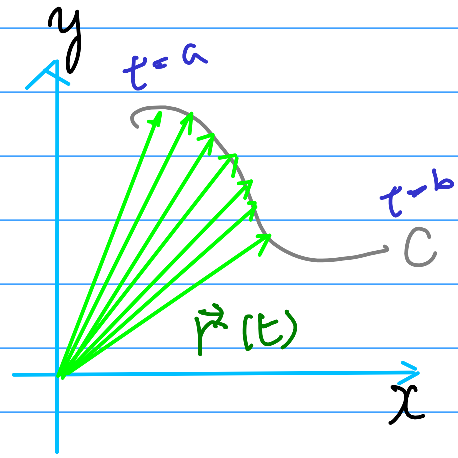
$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

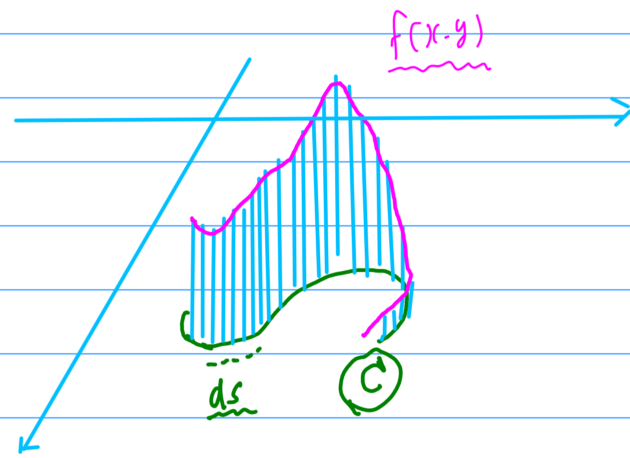
$$= \|\vec{r}\,'(t)\|\, dt$$

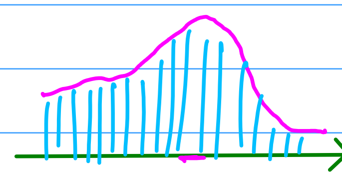
$$= \left\|\frac{d\vec{r}}{dt}\right\| dt$$

$$\int \|\vec{r}\,'(t)\|\, dt = L$$

$$\int_C f(x,y)\,ds$$





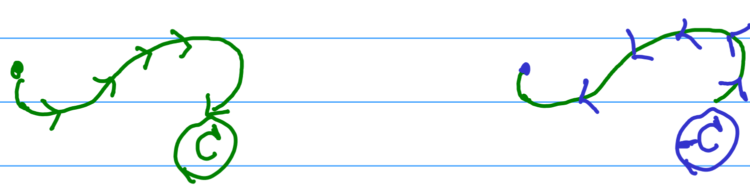


$$\int_C f(x,y)\,ds$$

$$= \int_C f(m(t), n(t))\,ds$$

$$= \int_a^b f(m(t), n(t))\sqrt{\left(\frac{dm}{dt}\right)^2 + \left(\frac{dn}{dt}\right)^2}\,dt$$

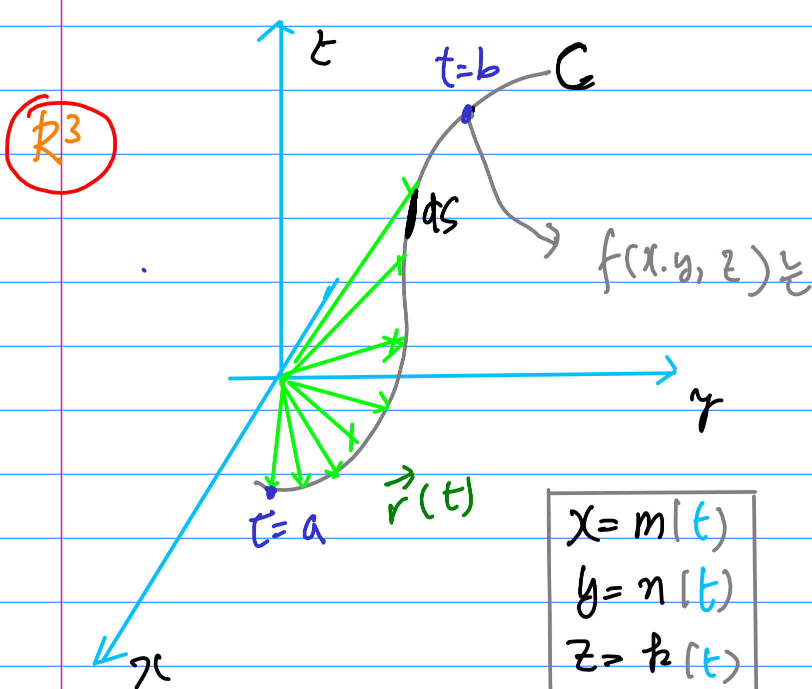
$$= \int_a^b f(m(t), n(t))\left\|\frac{d\vec{r}}{dt}\right\|\,dt$$



$$\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds$$

$$\int_C f(x,y,z)\, ds$$

$$\vec{r}(t) = \langle m(t),\, n(t),\, k(t) \rangle$$

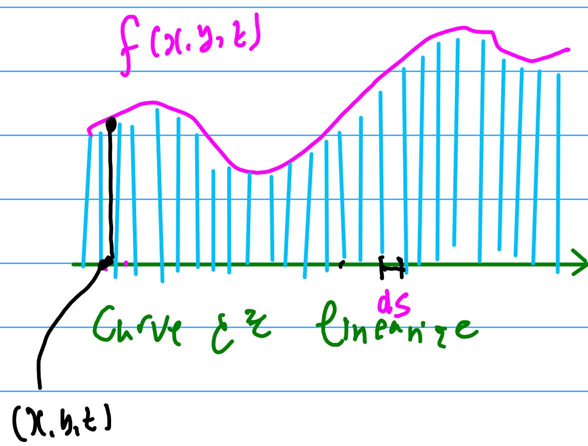


$x = m(t)$
$y = n(t)$
$z = k(t)$

f(x.y, z) defined on (x, y, z) points on the contour C

* 3-variable function

4-dimensional plot √



$$\int_C f(x.y,z)\, ds$$

$$= \int_C f(m(t), n(t), k(t))\, ds$$

$$= \int_a^b f(m(t), n(t), k(t)) \sqrt{\left(\frac{dm}{dt}\right)^2 + \left(\frac{dn}{dt}\right)^2 + \left(\frac{dk}{dt}\right)^2}\, dt$$

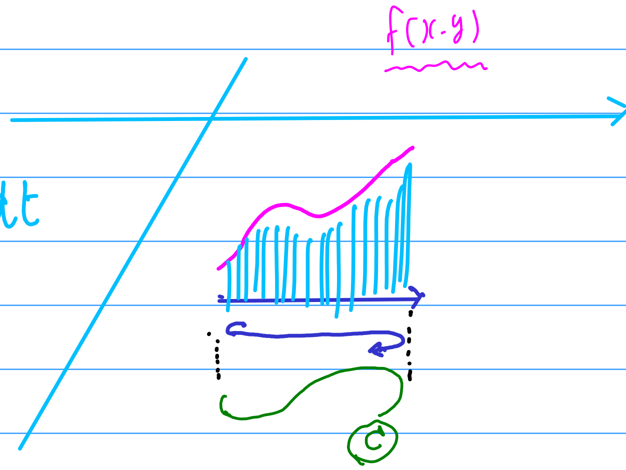
$$= \int_a^b f(m(t), n(t), k(t)) \left\| \frac{d\vec{r}}{dt} \right\| dt$$

$$\int_C f(x,y)\,dx \qquad \int_C f(x,y)\,dy$$

$$\int_a^b f(m(t), n(t)) \sqrt{\left(\frac{dm}{dt}\right)^2 + \cancel{\left(\frac{dn}{dt}\right)^2}^{\,0}}\, dt$$

$$\int_a^b f(m(t), n(t)) \sqrt{\left(\frac{dm}{dt}\right)^2}\, dt$$

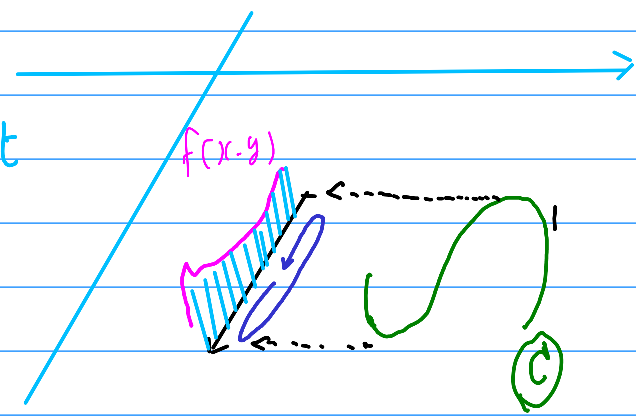
$$\Rightarrow \int_a^b f(m(t), n(t))\, m'(t)\, dt \overset{\Delta}{=} \int_C f(x,y)\, dx$$



$$\int_a^b f(m(t), n(t)) \sqrt{\cancel{\left(\frac{dm}{dt}\right)^2}^{\,0} + \left(\frac{dn}{dt}\right)^2}\, dt$$

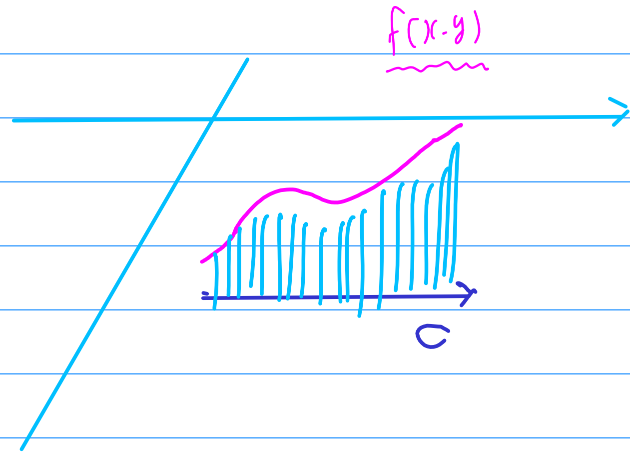
$$\int_a^b f(m(t), n(t)) \sqrt{\left(\frac{dn}{dt}\right)^2}\, dt$$

$$\Rightarrow \int_a^b f(m(t), n(t))\, n'(t)\, dt \overset{\Delta}{=} \int_C f(x,y)\, dy$$



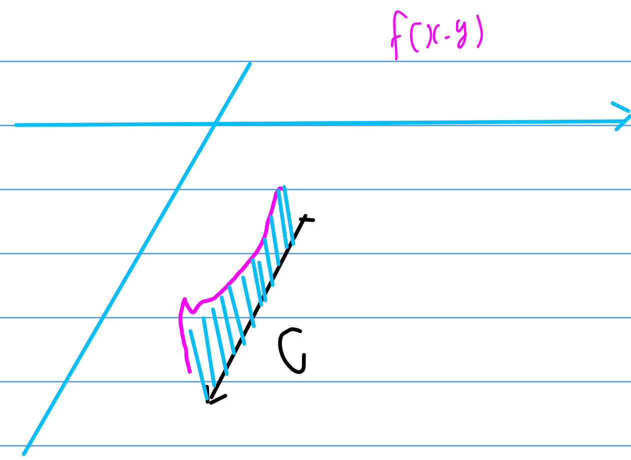
$$\int_C \boxed{f(x,y)}\, dx$$

$$\overset{\triangle}{=} \int_a^b f(m(t), n(t))\, m'(t)\, dt$$



$$\int_C \boxed{f(x,y)}\, dy$$

$$\overset{\triangle}{=} \int_a^b f(m(t), n(t))\, n'(t)\, dt$$

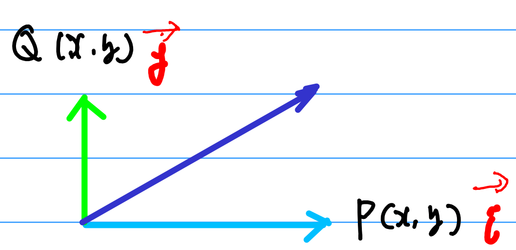


$$\int_C P(x,y)\,dx = \int_C P\,dx$$

$$\int_C Q(x,y)\,dy = \int_C Q\,dy$$

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy \equiv \int_C P\,dx + Q\,dy$$

Generally P & Q are the vector components of a vector field



$$\int_C f(x,y)\,dx = -\int_{-C} f(x,y)\,dx$$

$$\int_C f(x,y)\,dy = -\int_{-C} f(x,y)\,dy$$

$$\int_C P\,dx + Q\,dy = -\int_{-C} P\,dx + Q\,dy$$

$$\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds$$

R^2 **Line Integral with a Vector Field** $\vec{F}(x, y)$

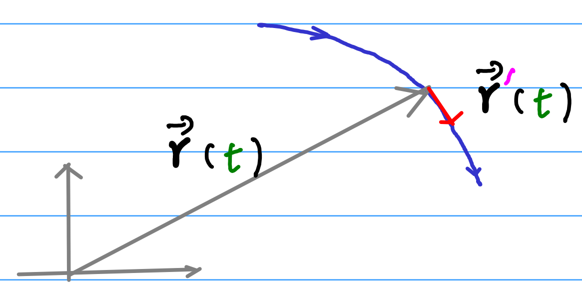
$$\vec{F}(x, y) = P(x, y)\,\vec{i} + Q(x, y)\,\vec{j}$$

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$$

$$\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t))$$
$$= P(x, y)\,\vec{i} + Q(x, y)\,\vec{j}$$

$$\int \vec{F}\cdot d\vec{r} = \int \vec{F}\cdot\vec{T}\,ds$$
$$= \int \vec{F}\cdot\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}\,ds$$
$$= \int \vec{F}\cdot\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}\,\|\vec{r}\,'(t)\|\,dt$$
$$= \int \vec{F}\cdot\vec{r}\,'(t)\,dt$$
$$= \int \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$\boxed{\vec{T} = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}}$$



Vector Field $\vec{F}(x, y)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F} \cdot \vec{r}'(t)\, dt$$

$$= \int_a^b (P\vec{i} + Q\vec{j}) \cdot (x'\vec{i} + y'\vec{j})\, dt$$

$$= \int_a^b P x'\, dt + \int_a^b Q y'\, dt$$

$$= \int_C P\, dx + \int_C Q\, dy$$

P, Q can be x, y component of a **gradient** vector field

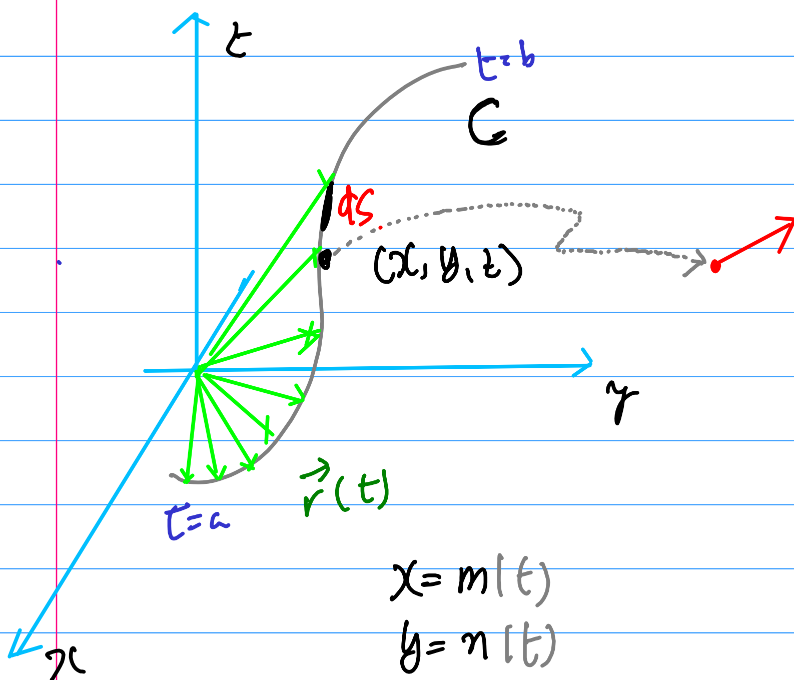
$\Rightarrow$ conservative field

any closed contour C

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \int_C P\, dx + Q\, dy = \int_C \nabla f \cdot d\vec{r} = 0}$$

$$\boxed{\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t)) = P(x, y)\vec{i} + Q(x, y)\vec{j}}$$

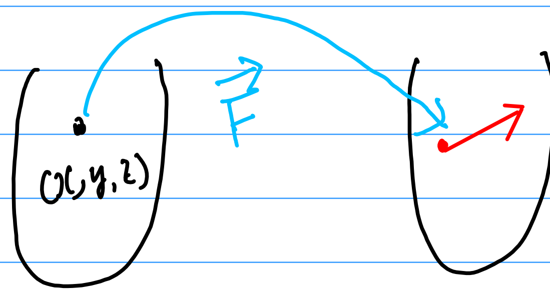
R^3 Line Integral with a Vector Field $\vec{F}(x, y, z)$



$$\int_C f(x, y, z)\, ds$$

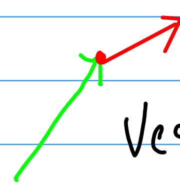
$x = m(t)$
$y = n(t)$
$z = k(t)$

· (x, y, z) point 에서
함수 값으로 Vector는 한 방



vector valued function

$\vec{r}(t) = \langle x, y, z \rangle$
$= \langle m(t), n(t), k(t) \rangle$



Vector field

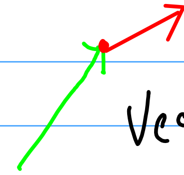
$\vec{F}(x, y, z) =$ $= \langle P, Q, R \rangle$

$= \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$

$= P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$

$\langle x, y, z \rangle \longrightarrow \langle P, Q, R \rangle$

$$\vec{r}(t) = \langle x, y, z \rangle$$
$$= \langle m(t), n(t), k(t) \rangle$$



Vector field

$$\vec{F}(x,y,z) = \quad = \langle P, Q, R \rangle$$

$$\vec{F}(\vec{r}(t)) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

$$\int_C f(x,y,z)\, ds$$

$$= \int_C f(m(t), n(t), k(t))\, ds$$

$$= \int_a^b f(m(t), n(t), k(t)) \sqrt{\left(\frac{dm}{dt}\right)^2 + \left(\frac{dn}{dt}\right)^2 + \left(\frac{dk}{dt}\right)^2}\, dt$$

$$= \int_a^b f(m(t), n(t), k(t)) \left\| \frac{d\vec{r}}{dt} \right\| dt$$

Line Integrals of vector fields

Vector field $\langle x, y, z\rangle \longrightarrow \langle P, Q, R\rangle$

$P(x, y, z)$
$Q(x, y, z)$
$R(x, y, z)$

$$\vec{F}(x, y, z) = P(x, y, z)\,\vec{i} + Q(x, y, z)\,\vec{j} + R(x, y, z)\,\vec{k}$$

curve C

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \vec{r}\,'(t)\,dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \langle P, Q, R \rangle \cdot \langle x', y', z' \rangle\, dt$$

$$= \int_a^b Px' + Qy' + Rz' \, dt$$

$$= \int_a^b Px' \, dt + \int_a^b Qy' \, dt + \int_a^b Rz' \, dt$$

$$= \int_C P\, dx + \int_C Q\, dy + \int_C R\, dz$$

$$= \int_C P\, dx + Q\, dy + R\, dz$$

$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$$

Fundamental Theorem

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$= \int_a^b \nabla f(\vec{r}(t)) \cdot d\vec{r}'(t)\,dt$$

$$= \int_a^b \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$$

$$= \int_a^b \left[\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right] dt$$

$$= \int_a^b \frac{d}{dt}\left[f(\vec{r}(t)) \right] dt$$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

$\vec{F}$: a continuous vector field

1. $\vec{F}$: a conservative vector field
   if there exists a function s.t. $\vec{F} = \nabla f$
   f : a potential function for the vector field $\vec{F}$

$$\nabla \triangleq \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}$$

Gradient Vector Field

$$\nabla f(x,y) \triangleq \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

2. $\int_C \vec{F} \cdot d\vec{r}$: independent of path

   if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$

   for any two paths C_1 and C_2
   with the same initial and final points

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \langle P, Q, R \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$$

$$= \int_a^b P(x,y,z)\frac{dx}{dt} + Q(x,y,z)\frac{dy}{dt} + R(x,y,z)\frac{dz}{dt}\, dt$$

$$= \int_a^b \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$$

$$= \int_a^b \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\, dt$$

$$= \int_a^b \frac{df}{dt}\, dt$$

3. a closed path: its initial and final points are the same point

4. a simple path: it doesn't cross itself

5. an open region: not include any of its boundary points

6. a connected region: can connect any two points with a path that lies completely in D

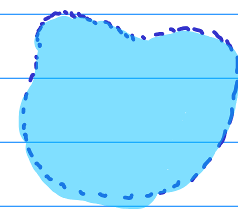
7. a simply-connected region: connected and containing no holes

1. $\int_C \nabla f \cdot d\vec{r}$ independent of path

2. $\vec{F}$: a conservative vector field

$\Rightarrow \int_C \vec{F} \cdot d\vec{r}$. independent of path

3. $\vec{F}$: a continuous vector field
on an open connected region D



and $\int_C \vec{F} \cdot d\vec{r}$: independent of path

$\Rightarrow \vec{F}$: a conservative vector field $\boxed{\vec{F} = \nabla f}$

4. $\int_C \vec{F} \cdot d\vec{r}$: independent of path

$\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = 0$ for every closed path

5. $\oint_C \vec{F} \cdot d\vec{r} = 0$ for every closed path

$\Rightarrow \int_C \vec{F} \cdot d\vec{r}$: independent of path

1. $\int_C \nabla f \cdot d\vec{r}$ independent of path

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$$

$$= \int_a^b \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\, dt$$

$$= \int_a^b \frac{df}{dt}\, dt$$

2. $\vec{F}$: a conservative vector field

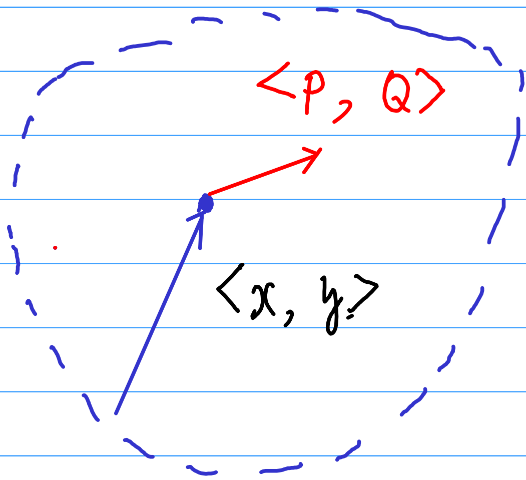
$\Rightarrow \int_C \vec{F} \cdot d\vec{r}$. independent of path

$\vec{F}$: a conservative vector field $\Rightarrow$ there exists $\boxed{\vec{F} = \nabla f}$

$$\int_C \vec{F} \cdot d\vec{r} = \underbrace{\int_C \nabla f \cdot d\vec{r}}$$ $\Rightarrow$ independent of path

In vector calculus a **conservative vector field** is a vector field that is the gradient of some function, known in this context as a scalar potential.[1] Conservative vector fields have the property that the line integral is path independent, i.e. the choice of integration path between any point and another does not change the result. Path independence of a line integral is equivalent to the vector field being conservative. A conservative vector field is also irrotational; in three dimensions this means that it has vanishing curl. An irrotational vector field is necessarily conservative provided that a certain condition on the geometry of the domain holds, i.e. the domain is simply connected.

Conservative vector fields appear naturally in mechanics: they are vector fields representing forces of physical systems in which energy is conserved.[2] For a conservative system, the work done in moving along a path in configuration space depends only on the endpoints of the path, so it is possible to define a potential energy independently of the path taken.



Open and simply connected region D

Vector field

$\vec{F} = P\,\vec{i} + Q\,\vec{j}$

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}} \Rightarrow \vec{F} : \text{conservative vector field}$$

$$\Rightarrow \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = P\,\vec{i} + Q\,\vec{j} = \vec{F}$$

$$\frac{\partial f}{\partial x} = P \qquad \frac{\partial f}{\partial y} = Q$$

$$f(x, y) = \int P(x, y)\,dx + g(y) \qquad f(x, y) = \int Q(x, y)\,dy + h(x)$$

Conservative Field

A vector field $\vec{F}$ is said to be $\boxed{\text{conservative}}$ if there exists a scalar field f such that $\vec{F} = \nabla f$

If $\vec{F}$ is a conservative field, then

⇒ Path Independence

$$\int_P \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

⇒ Irrotational (Curl-free) Vector Field

$$\nabla \times \vec{F} = \vec{0}$$

Conservative $\Longrightarrow$ Irrotational

$$\text{Curl}\,\vec{F} = \nabla \times \vec{F}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$\vec{F}$: Conservative

$$\boxed{P = \frac{\partial f}{\partial x} \qquad Q = \frac{\partial f}{\partial y} \qquad R = \frac{\partial f}{\partial z}}$$

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \Rightarrow \frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y} = 0$$

$$\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) \Rightarrow \frac{\partial}{\partial z}\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z} = 0$$

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \Rightarrow \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = 0$$

Schwarz Theorem continuous 2nd order partial derivatives : symmetric

$\vec{F}$: Irrotational

$$\text{Curl}\left(\vec{F}\right) = \nabla \times \nabla f = \vec{0}$$

$$||$$

$$\text{Curl}\left(\nabla f\right) = \nabla \times \nabla f = \vec{0}$$

$$\vec{F} = \nabla f$$

conservative field $\rightarrow$ $\vec{F} = \nabla f$

$$\text{Curl}(\nabla f) = \nabla \times \nabla f = \vec{0}$$

$$\text{Curl}(\vec{F}) = \nabla \times \nabla f = \vec{0}$$ $\rightarrow$ irrotational

① $f(x, y, z)$ has a continuous partial derivatives

$$\Rightarrow \quad \text{Curl}(\nabla f) = \nabla \times \nabla f = \vec{0}$$

② $\vec{F}$ is a conservative vector field

$$\Rightarrow \quad \text{Curl}(\vec{F}) = \nabla \times \nabla f = \vec{0}$$

③ $\vec{F}$ defined on all of $\mathbb{R}^3$,
each component has
a continuous 1st order partial derivative

$$\text{Curl}(\vec{F}) = \vec{0}$$

$\Longrightarrow$ $\vec{F}$ is a conservative vector field $\vec{F} = \nabla f$

Test for a Conservative Field

Suppose $\vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$
conservative vector field
in an open region R
continuous P & Q
continuous 1st partial derivatives

then $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ for all (x, y) in R.

$$\boxed{\begin{array}{l} \vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j} \;:\; \text{conservative in } R \\ \rightleftarrows \quad \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} \quad \text{for all } (x, y) \text{ in } R. \end{array}}$$

Fundamental Theorem

C : a path in an open region R

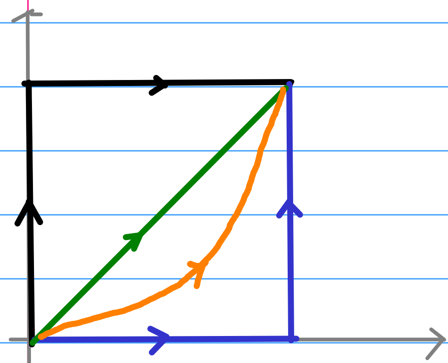
$r(t) = x(t)\vec{i} + y(t)\vec{j}$

$\vec{F}$: a conservative vector field in R

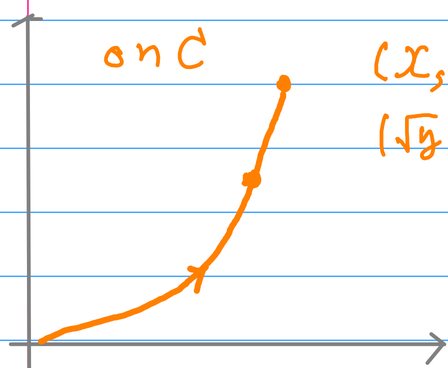
$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = \phi(B) - \phi(A)$$

$B = (x(b), y(b))$
$A = (x(a), y(a))$

Path Independence Example



$$\int_C y\,dx + x\,dy = 1$$



$x = \sqrt{y}$

$$\int_C y\,dx + x\,dy$$

$$= \int_0^1 x^2\,dx + \int_0^1 \sqrt{y}\,dy$$

$$= \left[\tfrac{1}{3}x^3\right]_0^1 + \left[\tfrac{2}{3}y^{\frac{3}{2}}\right]_0^1$$

$$= 1$$

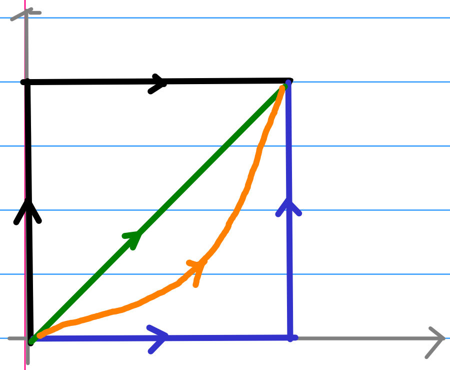
$y = x^2 \qquad dy = 2x\,dx$

$$\int_C y\,dx + x\,dy$$

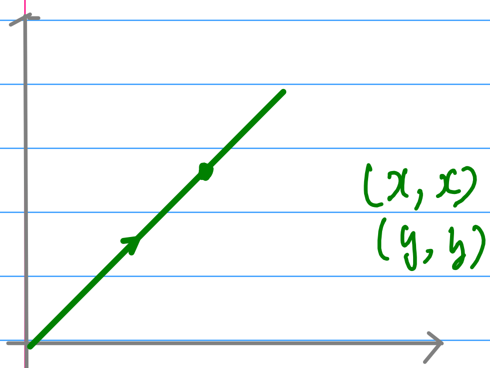
$$= \int_0^1 x^2\,dx + \int_0^1 x \cdot 2x\,dx$$

$$= \left[\tfrac{1}{3}x^3\right]_0^1 + \left[\tfrac{2}{3}x^3\right]_0^1$$

$$= 1$$



$$\int_C y\,dx + x\,dy = 1$$

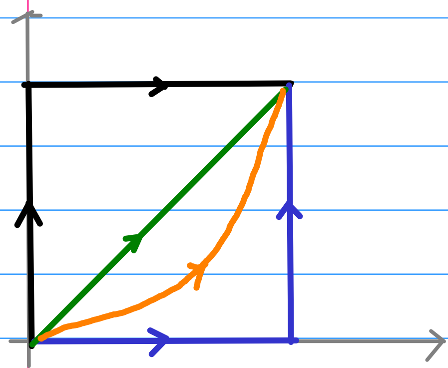


$y = x$

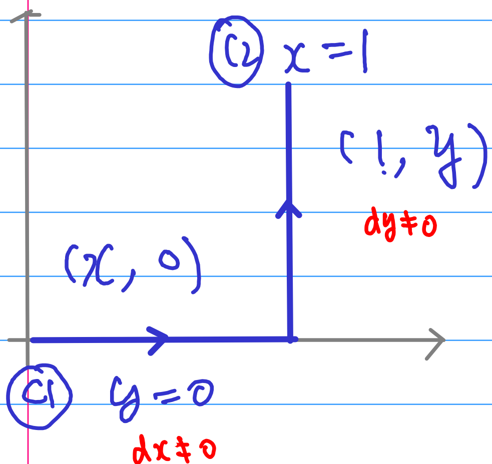
$$\int_C y\,dx + x\,dy$$
$$= \int_0^1 x\,dx + \int_0^1 y\,dy$$
$$= \left[\tfrac{1}{2}x^2\right]_0^1 + \left[\tfrac{1}{2}y^2\right]_0^1$$
$$= 1$$

$y = x \qquad dy = dx$

$$\int_C y\,dx + x\,dy$$
$$= \int_0^1 x\,dx + \int_0^1 x \cdot dx$$
$$= \left[\tfrac{1}{2}x^2\right]_0^1 + \left[\tfrac{1}{2}x^2\right]_0^1$$
$$= 1$$



$$\int_C y\,dx + x\,dy = 1$$



$$\int_{C1} \overset{y=0}{y}\,dx + \int_{C2} \overset{x=1}{x}\,dy$$

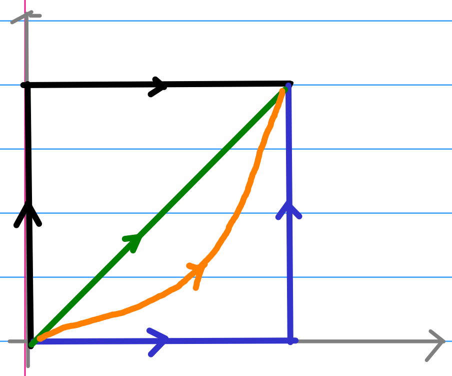
$$= \int_0^1 0 \cdot dx + \int_0^1 1 \cdot dy$$

$$= \Big[\,0\,\Big]_0^1 + \Big[\,y\,\Big]_0^1$$

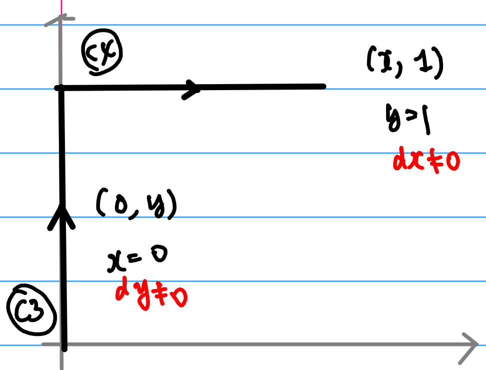
$$= 1$$

$$\int_C y\,dx + x\,dy$$

$$= \int_{C1} y\,dx + \int_{C2} x\,dy$$



$$\int_C y\,dx + x\,dy = 1$$



$$y=1 \qquad\qquad x=0$$

$$\int_{C4} y\,dx + \int_{C3} x\,dy$$

$$= \int_0^1 1\cdot dx + \int_0^1 0\cdot dy$$

$$= \Big[x\Big]_0^1 + \Big[0\Big]_0^1$$

$$= 1$$

$$\int_C y\,dx + x\,dy$$

$$= \int_{C4} y\,dx + \int_{C3} x\,dy$$