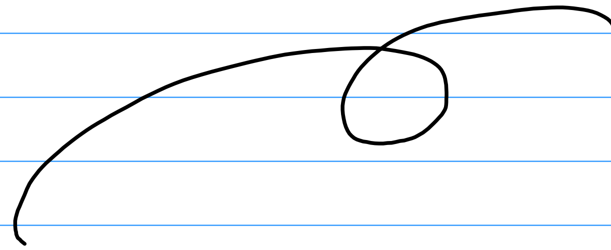
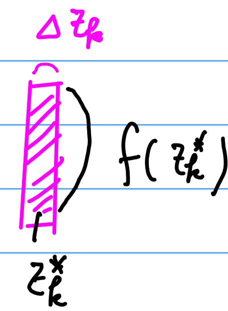
Complex Integration (H.1)

20160112

18. line integration

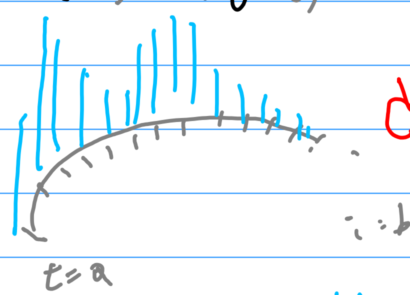


$$\int_C f(z)\,dz = \lim_{|P|\to 0}\sum_{k=1}^{n} f(z_k^*)\,\Delta z_k$$

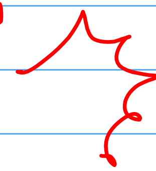


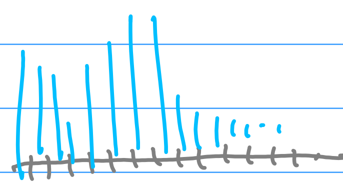
$z = x + iy = x(t) + iy(t)$ ..... t parametric

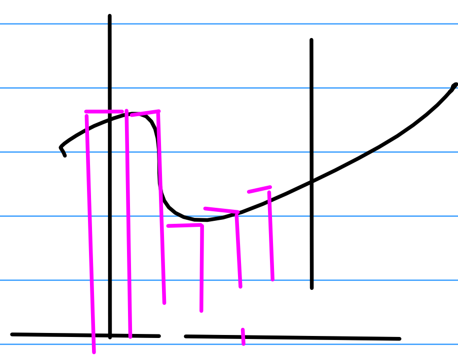
$a \leq t \leq b$



C : smooth





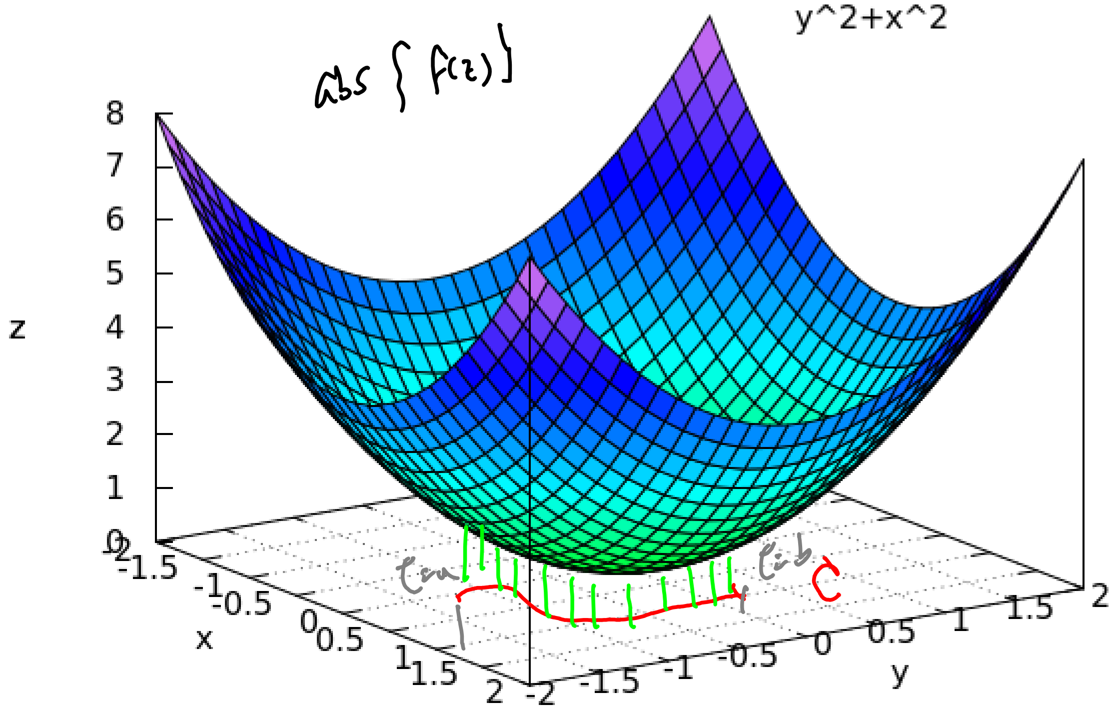


$$\int_C f(z)\,dz$$

$f(z) = z^2$

$\arg\{f(z)\}$

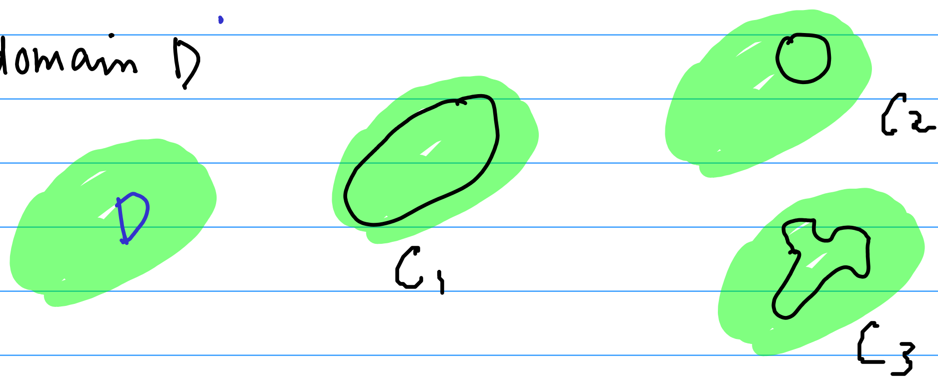
$\text{abs}\{f(z)\}$



Contour

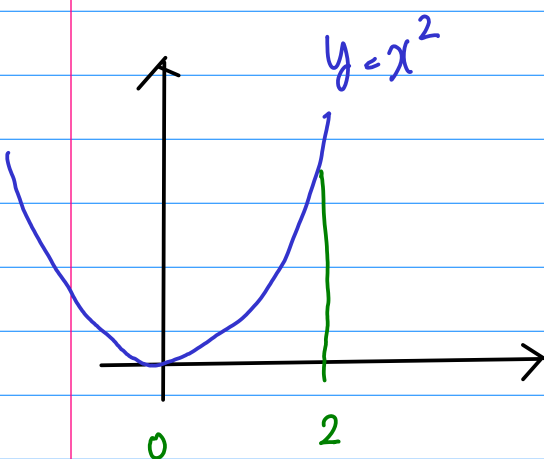
Simple closed contour C

domain D



$$\int_C f(z)\,dz = 0$$

closed = $\boxed{\text{start} = \text{end}}$



$$\int_0^2 f(x)\,dx = \int_0^2 x^2\,dx$$

$$= \left[\frac{1}{3}x^3\right]_0^2$$

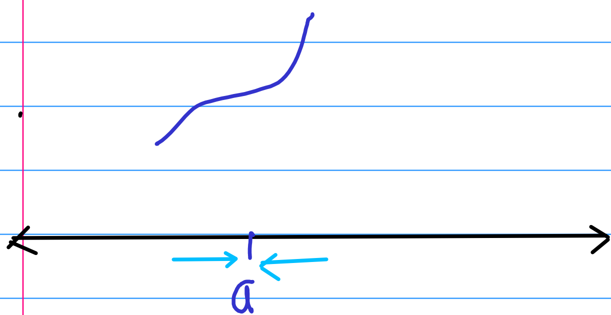
$$= \frac{8}{3}$$

$$\int_2^0 f(x)\,dx = -\frac{8}{3}$$

$$\int_0^2 f(x)\,dx + \int_2^0 f(x)\,dx = 0$$

Real Derivative Analogy

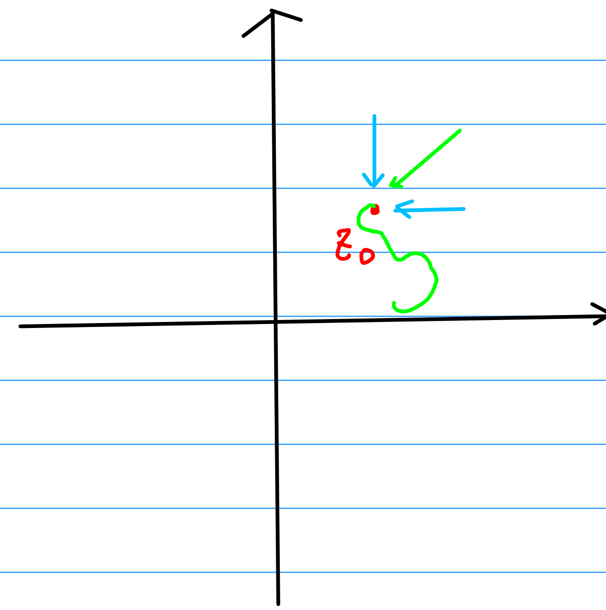
real



$f'(x)$

only two ways
of approaching

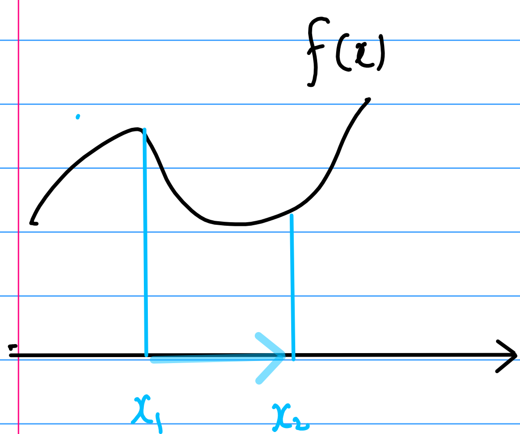
complex

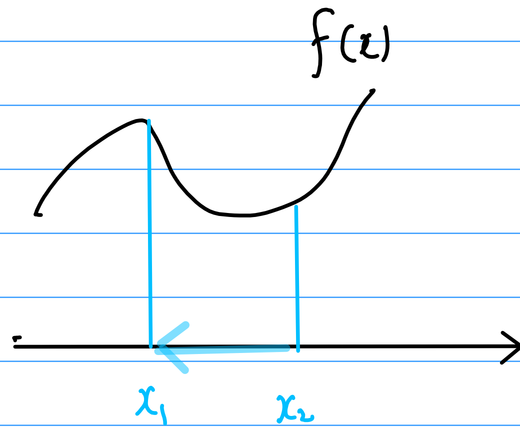


$f'(z)$

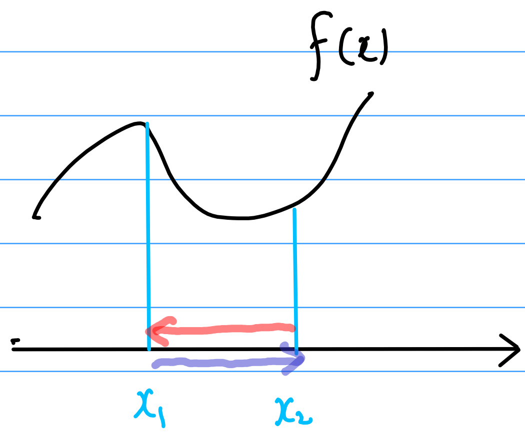
infintely many ways
of approaching

Real Integration Analogy

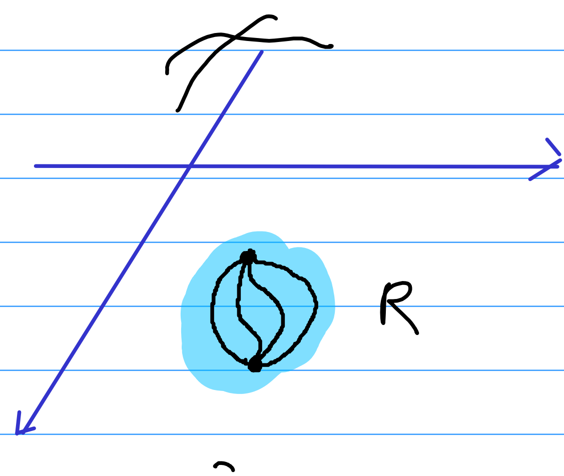




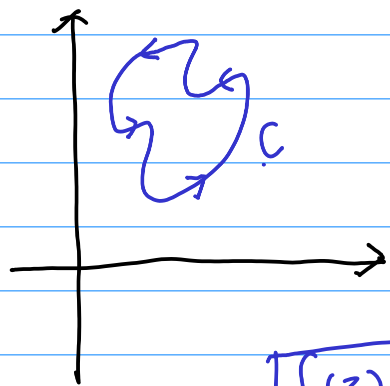
$$\int_{x_1}^{x_2} f(x)\,dx = -\int_{x_2}^{x_1} f(x)\,dx$$



$$\oint f(z)\,dz = 0$$

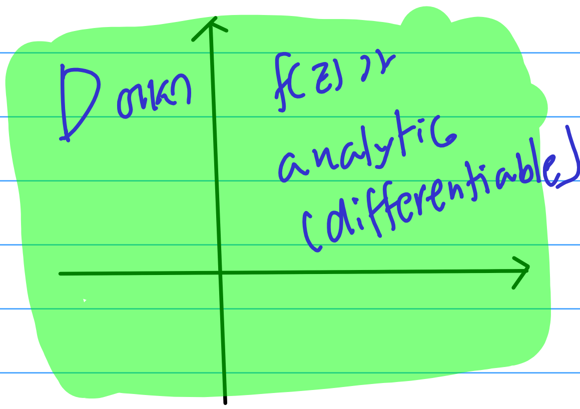


예제 1)



$$\int_C e^z \, dz = 0$$

$$\boxed{f(z) = e^z}$$

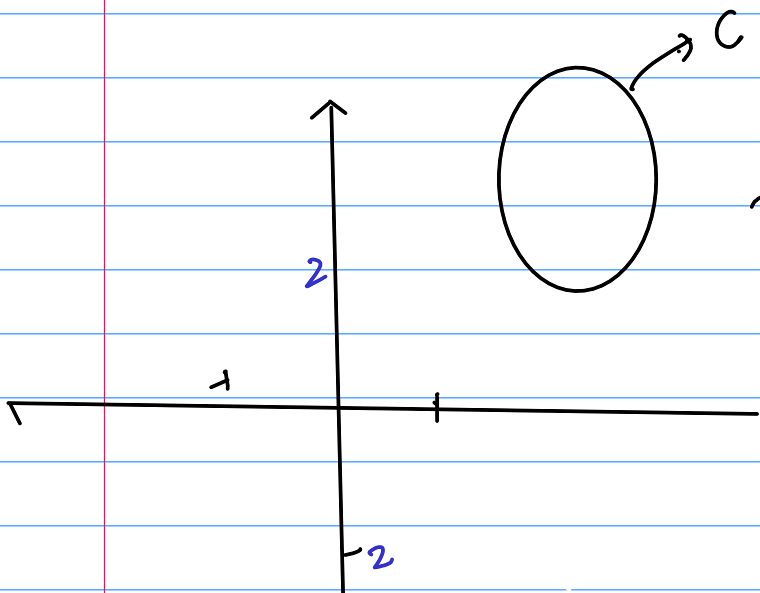


entire function e^z

entire complex domain

analytic

예제2) C가 $\overset{x}{\downarrow}$ $(x-2)^2 + \dfrac{(\overset{y}{\downarrow}\,y-5)^2}{4} = 1$ 타원

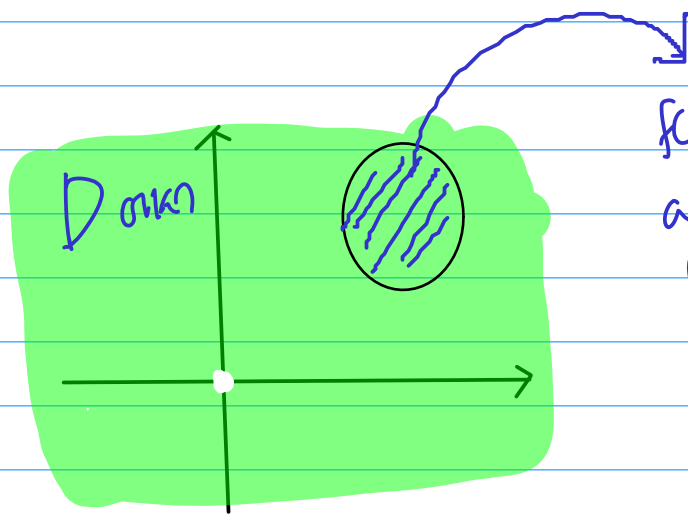


$f(z) = \dfrac{1}{z^2}$

분 $z=0$

$z=0$

$\int_C \dfrac{1}{z^2}\,dz = 0$



$f(z)$ 가 analytic (differentiable)

$z \neq 0$

복소수 $\boxed{z = x + iy}$ $\boxed{\Delta z = \Delta x + i\Delta y}$

$$\int_C \underbrace{f(z)}_{\text{복소수}} dz = \lim \sum \underbrace{(u+iv)}_{\text{함수값 복소수}} (\Delta x + i\Delta y)$$

함수값

복소수

$\boxed{f(z) = u + iv}$

$$(u+iv)(\Delta x + i\Delta y)$$

$$\lim \sum \left[u\Delta x - v\Delta y + i(v\Delta x + u\,dy) \right]$$

$$\lim \sum \left[u\Delta x - v\Delta y \right] + \sum \left[i(v\Delta x + u\,dy) \right]$$

$$\left[\int_C u\,dx - v\,dy \right] + i \left[\int_C v\,dx + u\,dy \right]$$

$$\begin{cases} z = x + iy \\ f(z) = u + iv \end{cases} = u(x,y) + i(x,y)$$

$$= u(x(t), y(t)) + i(x(t), y(t))$$

$$\boxed{x'(t) = \frac{dx}{dt}} \qquad \boxed{y'(t) = \frac{dy}{dt}}$$

$$x'(t)\,dt = \boxed{\frac{dx}{dt}\,dt} \qquad y'(t)\,dt = \boxed{\frac{dy}{dt}\,dt}$$

$$x'(t)\,dt = dx \qquad y'(t)\,dt = dy$$

$$\int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

$$= \int_a^b u\,x'(t)\,dt - \int_a^b v\,y'(t)\,dt + i\left[\int_a^b v\,x'(t)\,dt + \int_a^b u\,y'(t)\,dt\right]$$

$$= \int_a^b u(x,y)\,x'\,dt - v(x,y)\,y'\,dt$$

$$+ i\left[\int_a^b v(x,y)\,x'\,dt + u(x,y)\,y'\,dt\right]$$

$$z = x + i y$$

$$f(z) = u + i v$$

$$z(t) = x(t) + i y(t) \qquad z'(t) = x'(t) + i y'(t)$$

$$f(z) = u(x,y) + i v(x,y)$$

$$= u(x(t), y(t)) + i v(x(t), y(t))$$

$$\boxed{f(z)\, z'} = [u(x,y) + i v(x,y)][x'(t) + i y'(t)]$$

$$= [u(x,y)x'(t) - v(x,y)y'(t)]$$

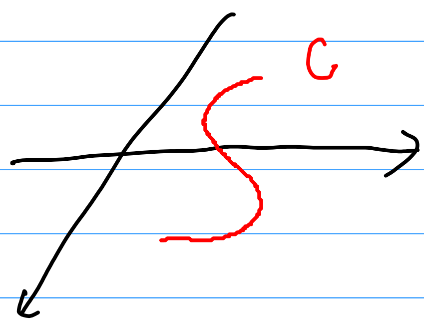
$$+ i [v(x,y)x'(t) + u(x,y)y'(t)]$$

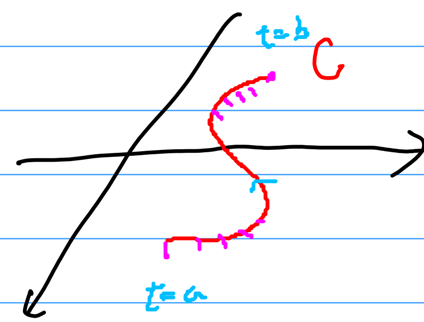
$$\int_C f(z)\,dz = \int_a^b [u(x,y)\, x'dt - v(x,y)\, y'] dt$$

$$+ i \int_a^b [v(x,y)\, x'dt + u(x,y)\, y'] dt$$

$$= \int_a^b f(z) \cdot z'(t)\, dt$$

$$\int_C f(z)\,dz = \int_a^b f(z)\cdot z'(t)\,dt$$



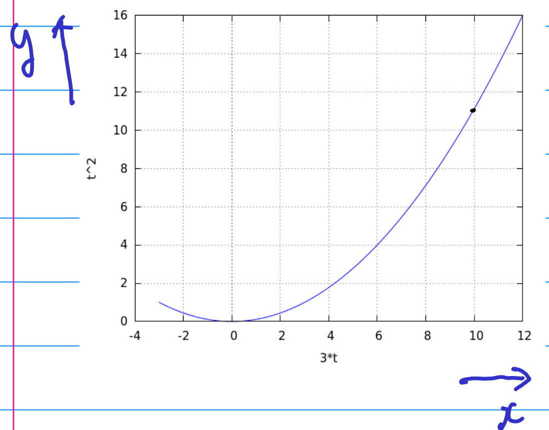


Contour C 위에 있는 복소수

$z(t) = x(t) + i\,y(t)$

$t=a$, $t=b$.

C



$$\left.\begin{aligned} x(t) &= 3t \\ y(t) &= t^2 \end{aligned}\right\} \text{ parameter } t$$

$$t \in [-1, 4]$$

$$f(z) = \bar{z}$$

$$z(t) = x(t) + i\, y(t)$$

$$f(z(t)) = x(t) - i\, y(t)$$

$$\int_C f(z)\, dz$$

$$= \int_{-1}^{4} f(z)\, z'(t)\, dt$$

$$= \int_{-1}^{4} \left(x(t) - i\, y(t)\right)\left(x'(t) + i\, y'(t)\right) dt$$

$$= \int_{-1}^{4} (3t + i t^2)\,(3 + i\, 2t)\, dt$$

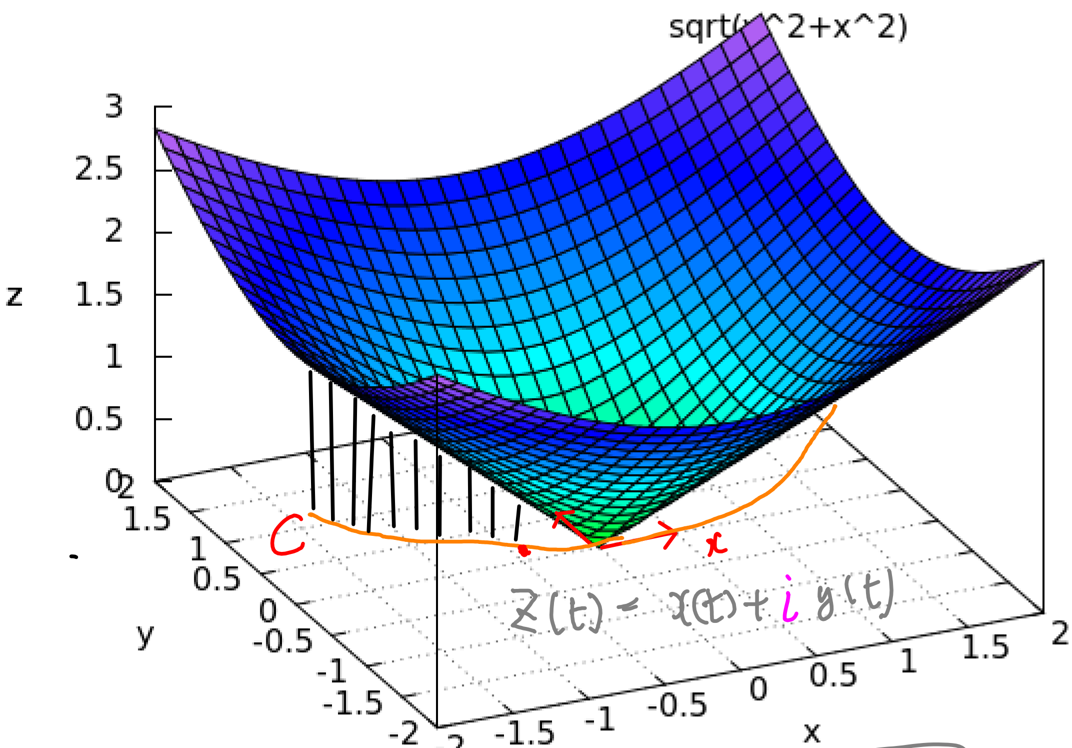
$$f(z) = \bar{z}$$

$$z(t) = x(t) + i\,y(t)$$

$$\boxed{\begin{aligned} x(t) &= 3t \\ y(t) &= t^2 \end{aligned}}$$

$$f(z(t)) = x(t) - i\,y(t)$$

$$3t - i\,t^2$$

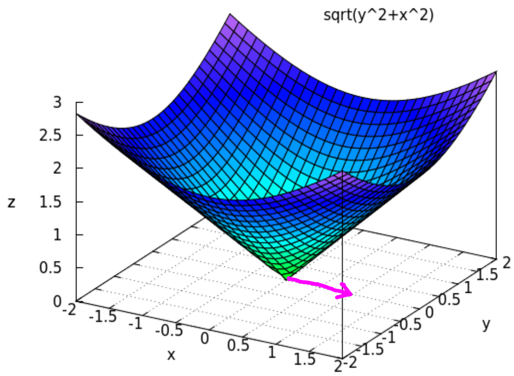


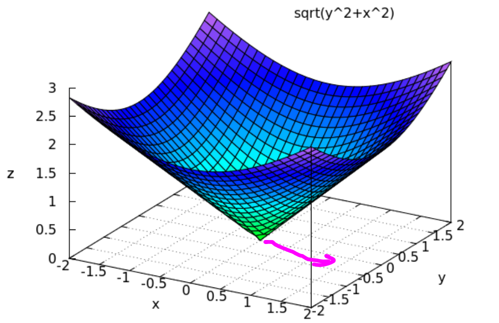
$$\boxed{z(t) = 3t - i\,t^2}$$

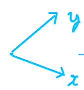
$$1 \le t \le 4$$

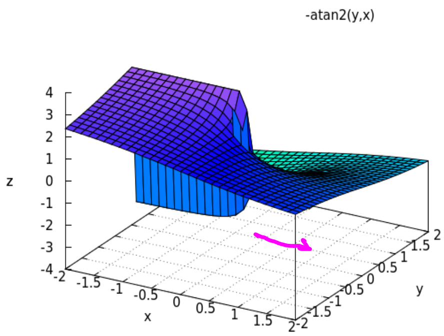
$$f(z) = \bar{z} = x - iy$$

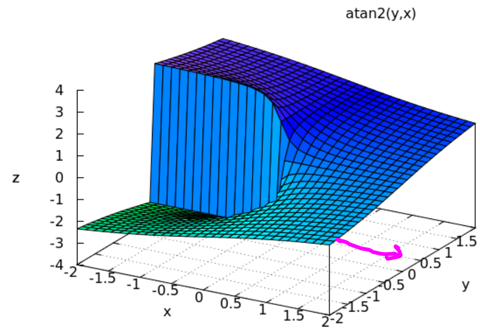
$$f(z) = z = x + iy$$



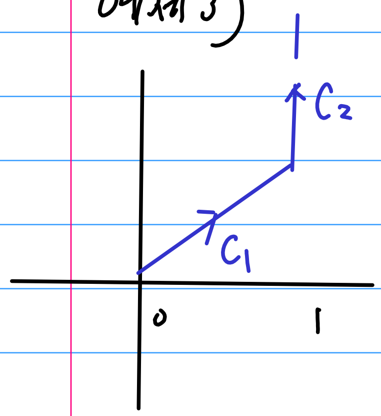








예제 3)



$$\boxed{\int_C (x^2+iy^2)\,dz}$$

$$\boxed{\int_C f(z)\,dz} = \int_a^b f(z(t))\, z'(t)\,dt$$

$$z = x+iy \rightarrow f(z) = x^2+iy^2$$

C_1

$\underline{y=x}$

$$z = x+ix \Rightarrow z(x) = x+ix$$

$$z'(x) = 1+i$$

$$f(z) = x^2+ix^2$$

$$\int_0^1 (x^2+ix^2)(1+i)\,dx$$

C_2

$x=1$

$y = 1 \sim 2$

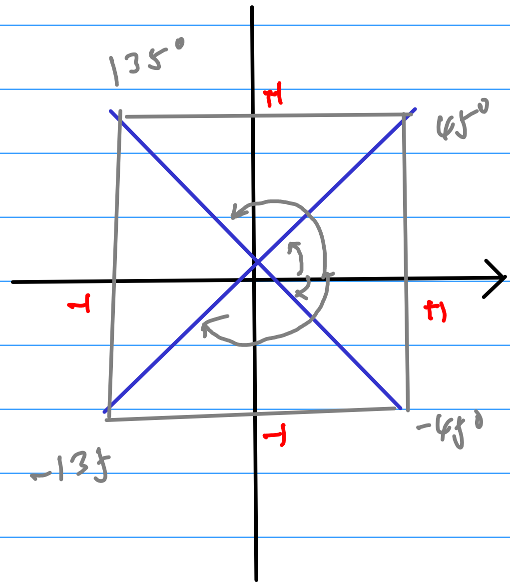
$$z = 1+iy \Rightarrow z(y) = 1+iy$$

$$z'(y) = i$$

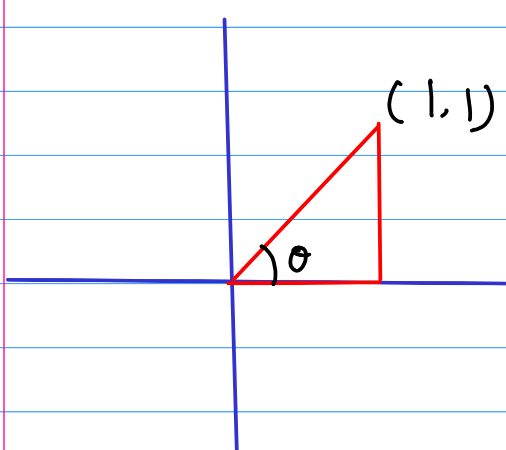
$$f(z) = 1+iy^2$$

$$\int_1^2 (1+iy^2)\cdot i \quad dy$$

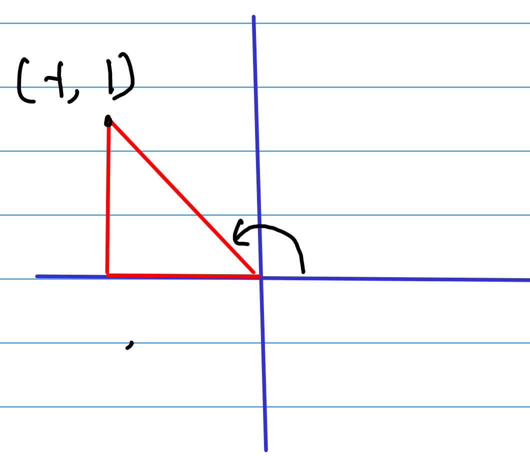
a tan(θ)



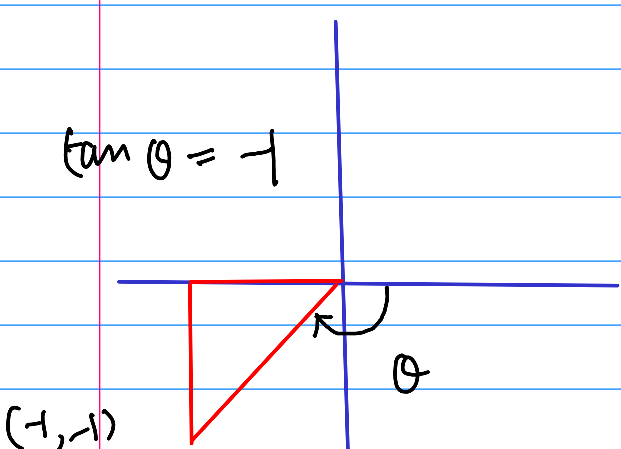
$\tan\theta = \frac{1}{1} = 1$



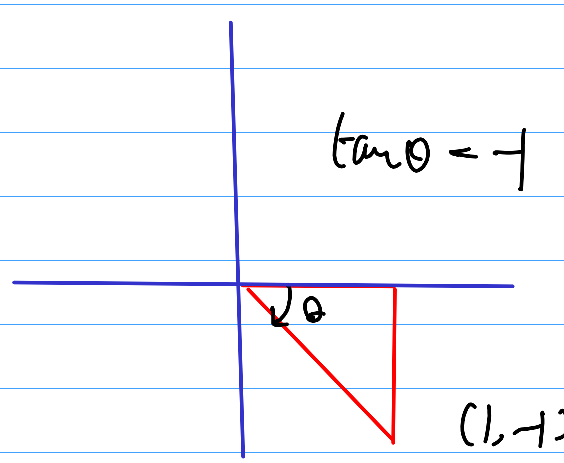
$\tan\theta = \frac{1}{1} = -1$



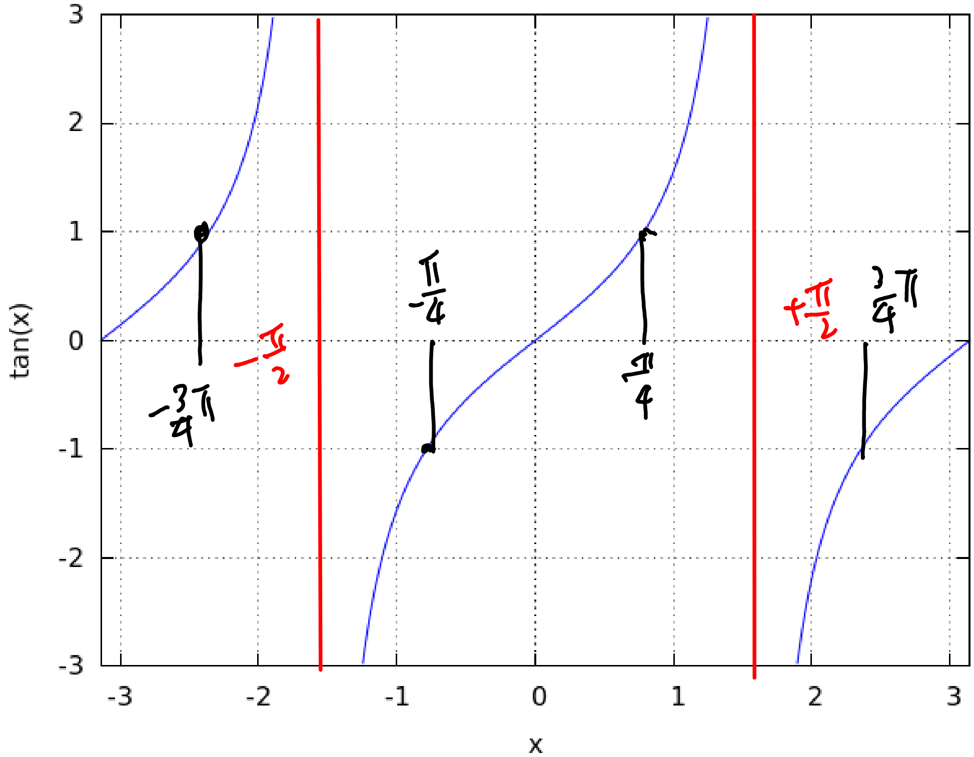
$\tan\theta = -1$



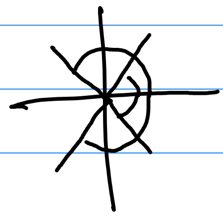
$\tan\theta = -1$



tan(x)



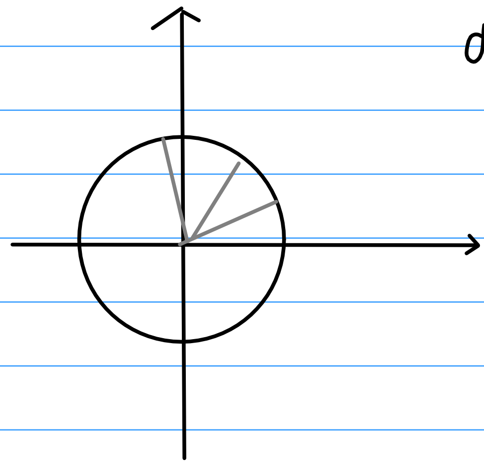
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3
x
tan(x)
$-\frac{3}{4}\pi$
$-\frac{\pi}{2}$
$-\frac{\pi}{4}$
$\frac{\pi}{4}$
$+\frac{\pi}{2}$
$\frac{3}{4}\pi$



◎ Circle Equations

$\{(x, y) \mid$ 원점에서 부터 거리가 r 인 점 $(x, y)\}$

중심 $(0, 0)$

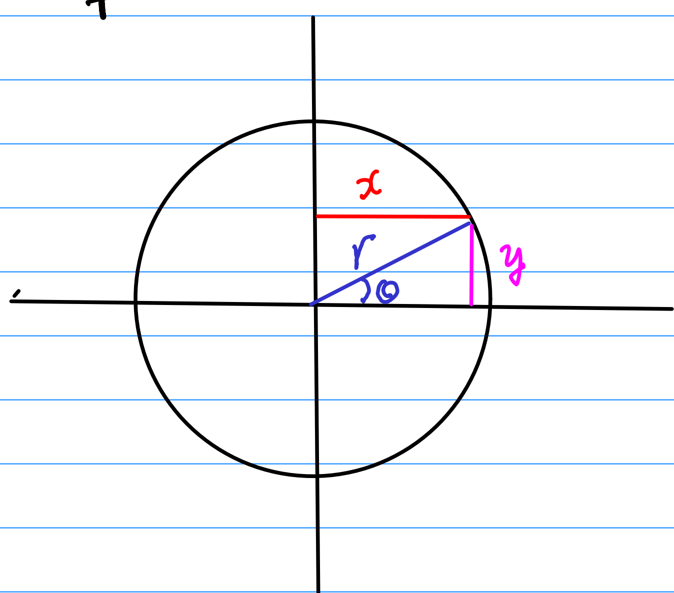


distance $(0,0) \sim (x, y)$

$$d = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2+y^2} = r$$

$$\boxed{x^2+y^2=r^2}$$

◎ Parameterized Circle Equation



$\cos\theta = \dfrac{x}{r}$ $\qquad x = r\cos\theta$

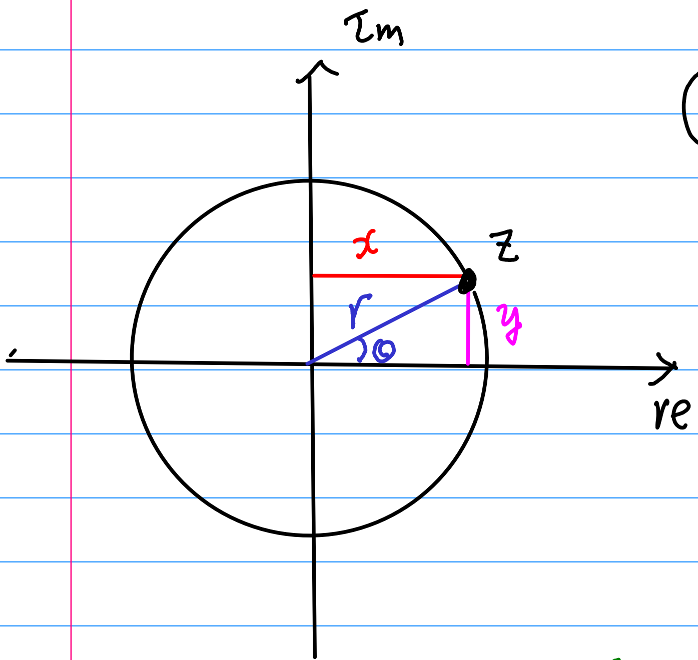
$\sin\theta = \dfrac{y}{r}$ $\qquad y = r\sin\theta$

$x = x(\theta)$

$y = y(\theta)$

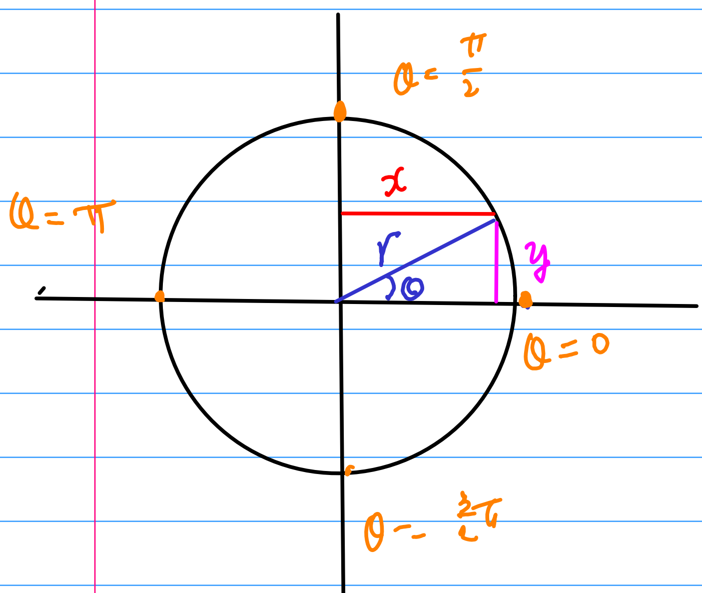
parameter θ

$x(\theta)$ $\qquad y(\theta)$



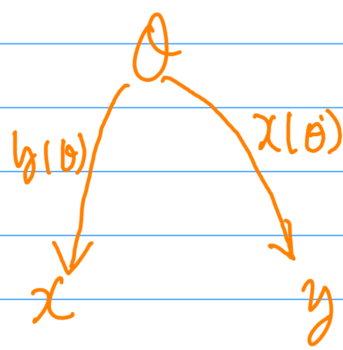
① complex number z

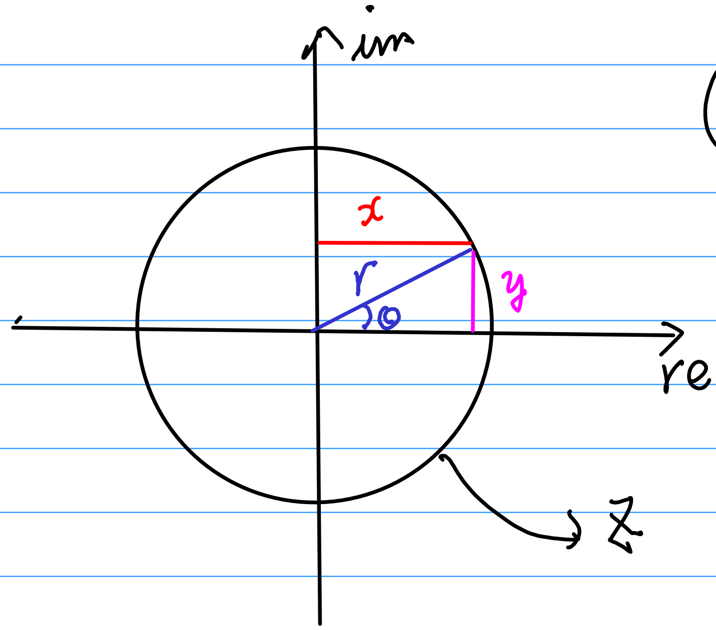
$$z = x + iy$$
$$= r\cos\theta + i\, r\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$



parameter : $0 \leq \theta \leq 2\pi$

$$\xleftarrow{\theta(t)} t$$





② complex number

$|z| = r$

absolute value
= magnitude
= distance
= radius = r

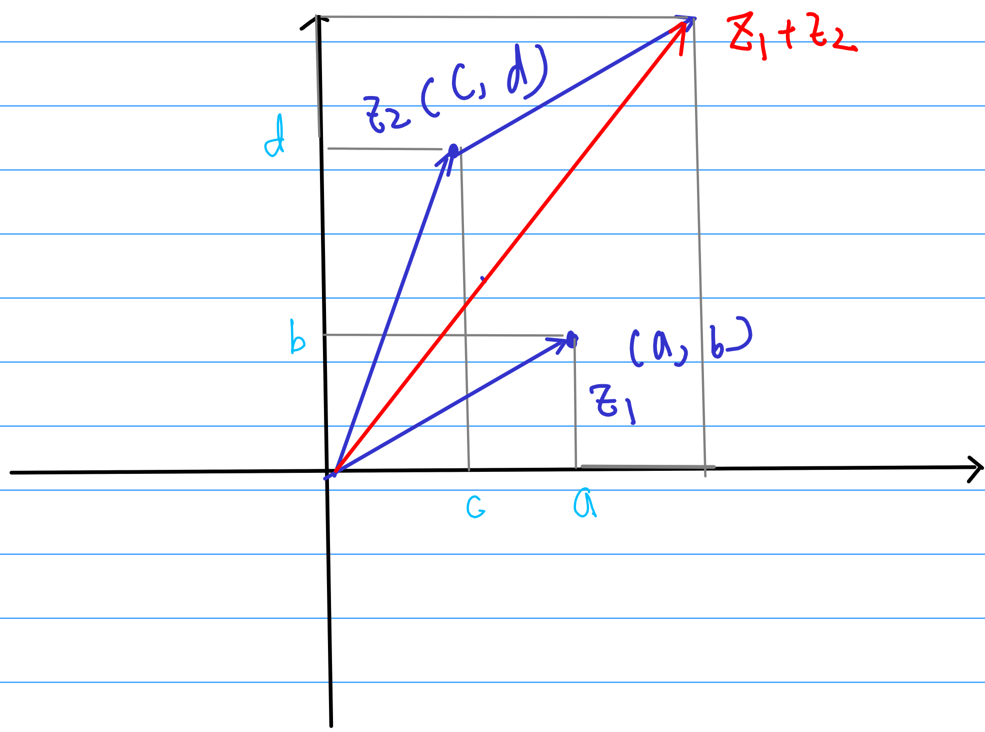
$|z-1| = 2$?

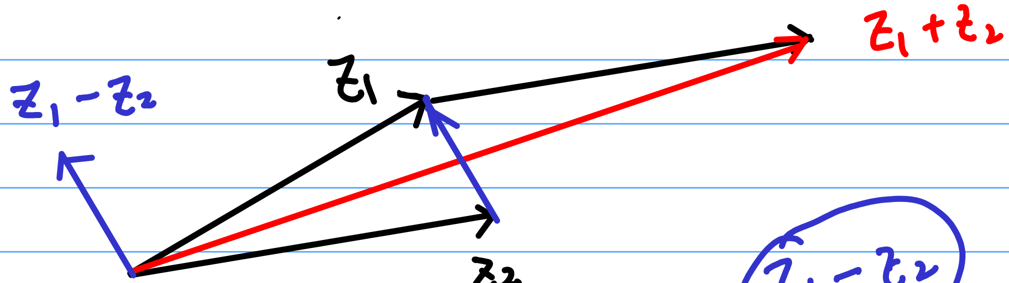
center (1, 0)
radius 2

addition / subtraction complex numbers $\Longleftrightarrow$ addition / subtraction Vectors

$$z_1 = (a + i\,b)$$
$$z_2 = (c + i\,d)$$

$$z_1 + z_2 = (a+c) + i\,(b+d)$$

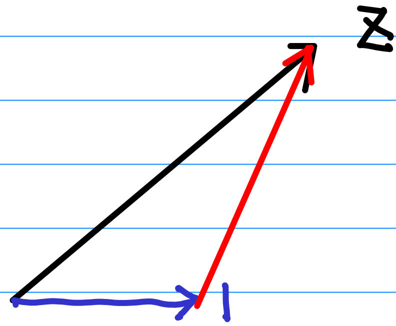


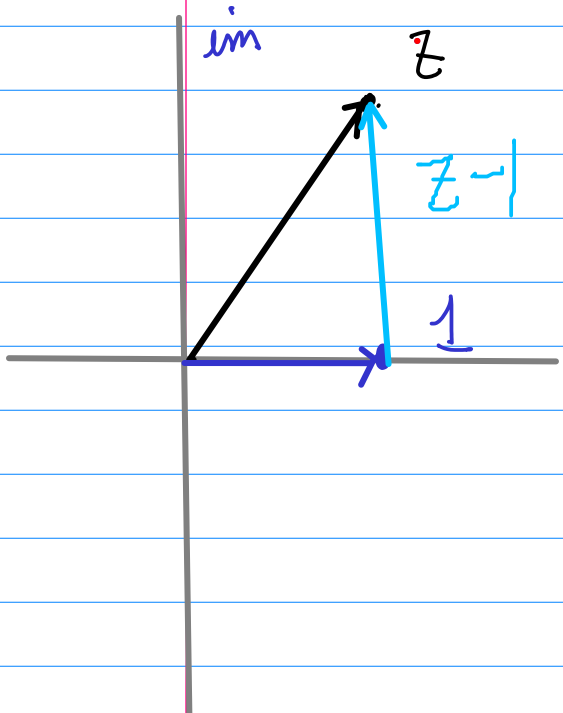


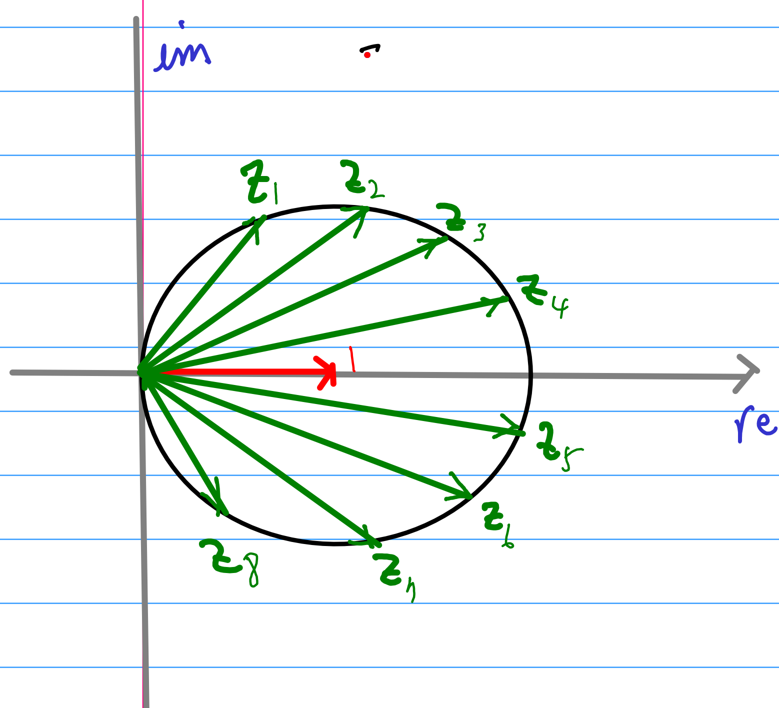
$z_1 - z_2$

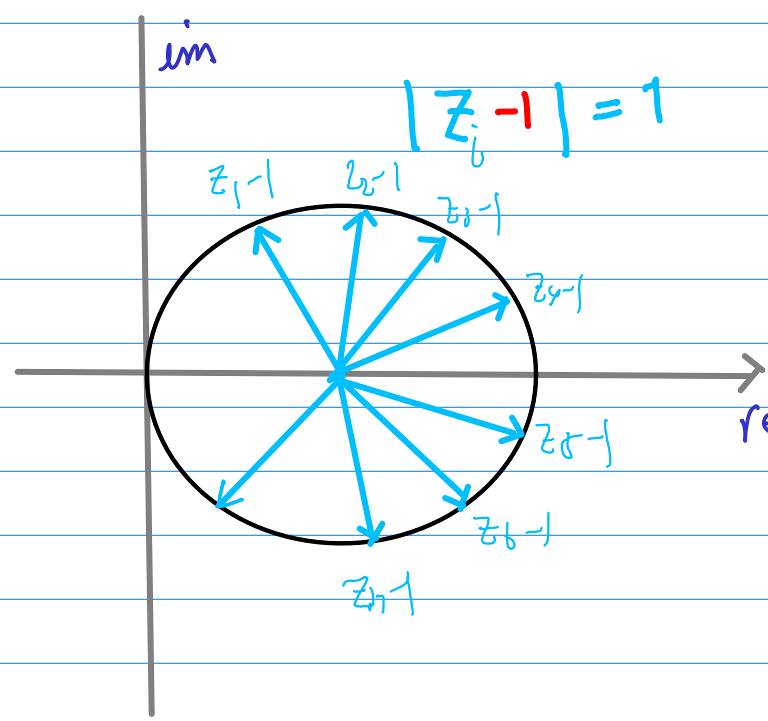
Complex number z

$z - 1$

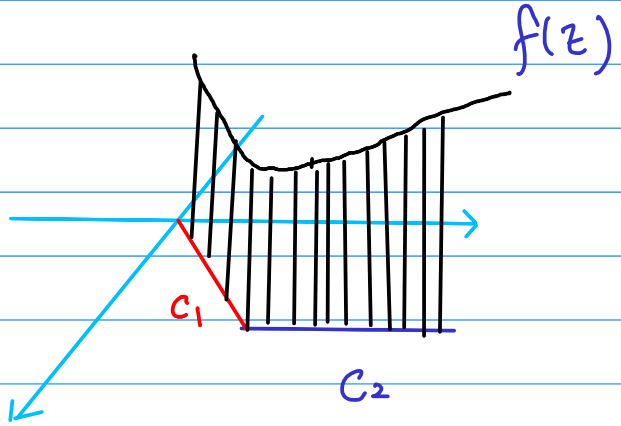








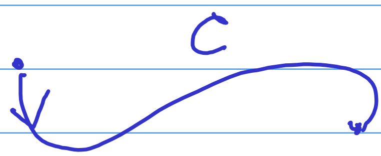
Piecewise Contours

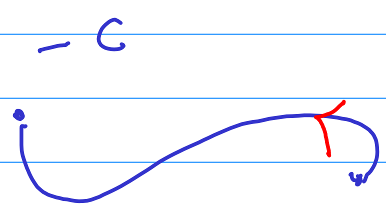


$$\int_C f(z)\,dz$$

$$= \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

Reverse direction

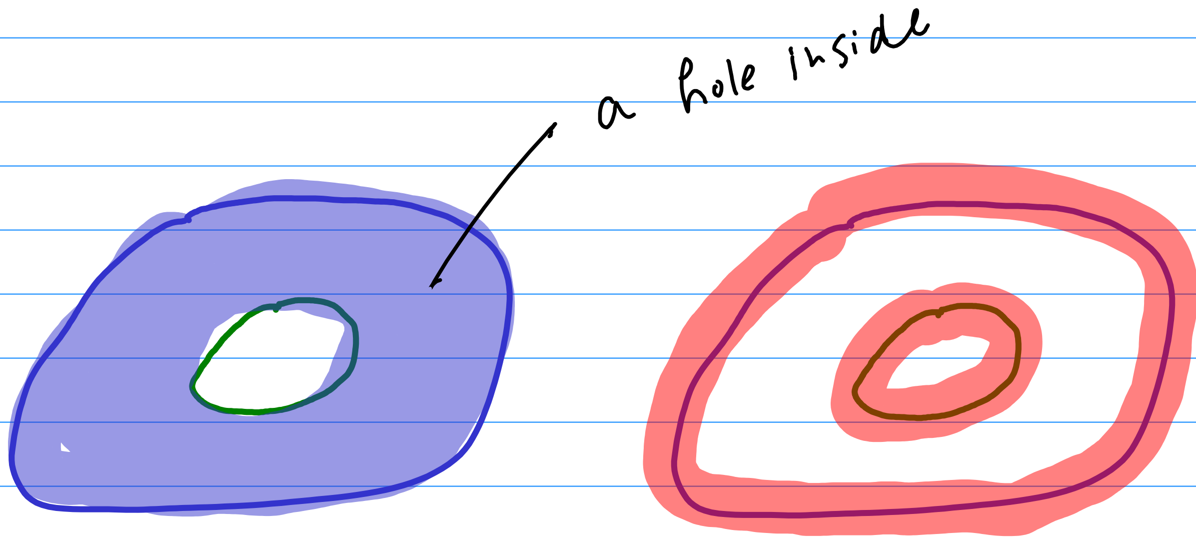


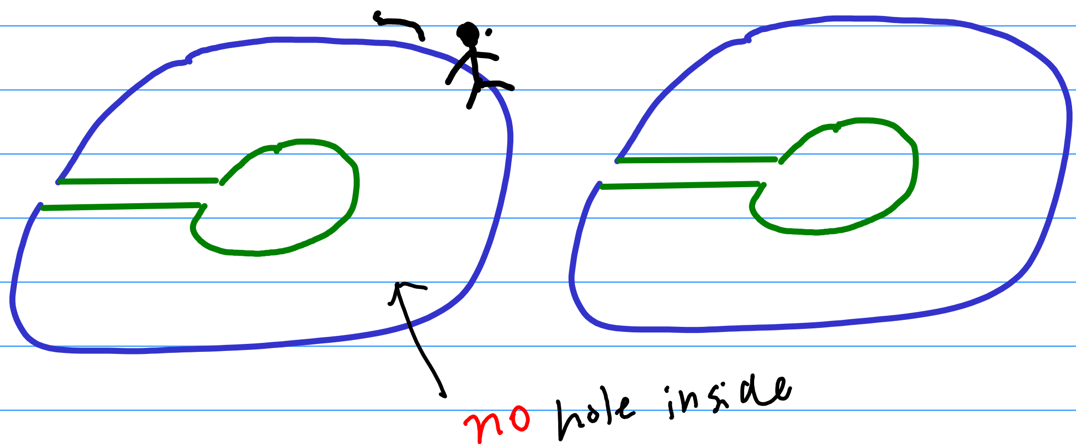


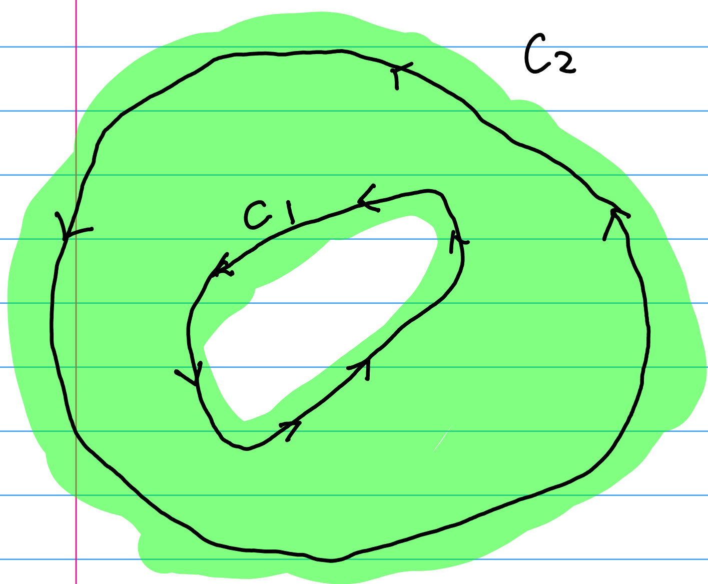
$$-\int_C f(z)\,dz = \int_{-C} f(z)\,dz$$

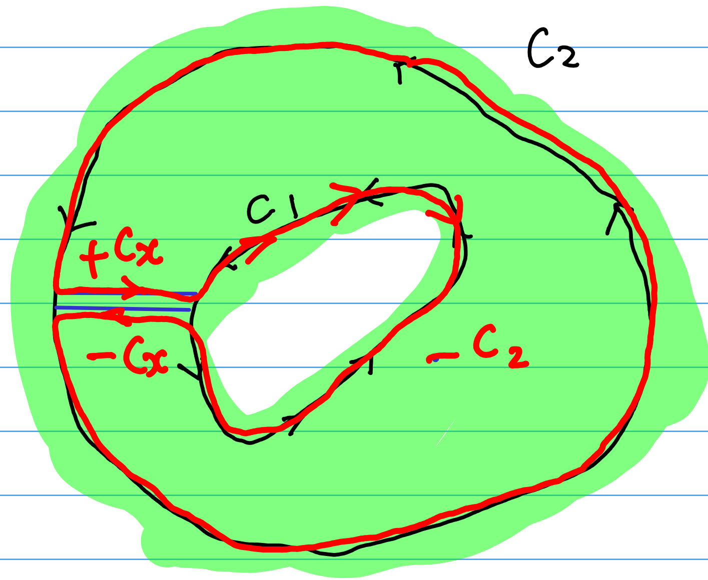
$$-\oint_C f(z)\,dz = \oint_C f(z)\,dz$$

Deformation of Contours









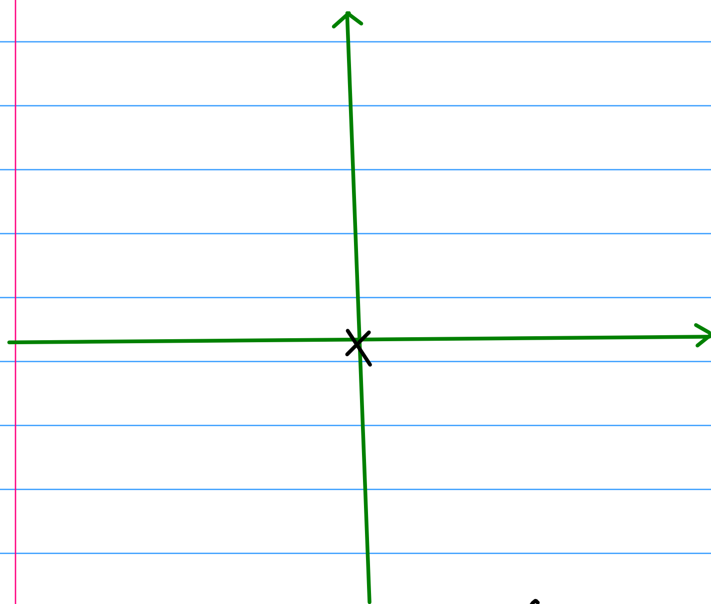
$$\int_{C_x} f(z)\,dz + \int_{-C_x} f(z)\,dz = 0$$

$$\int_{C_2} f(z)\,dz + \boxed{\int_{C_x} f(z)\,dz + \int_{-C_x} f(z)\,dz} + \int_{-C_1} f(z)\,dz$$

$$\Rightarrow \int_{C_2} f(z)\,dz + \int_{-C_1} f(z)\,dz = 0$$

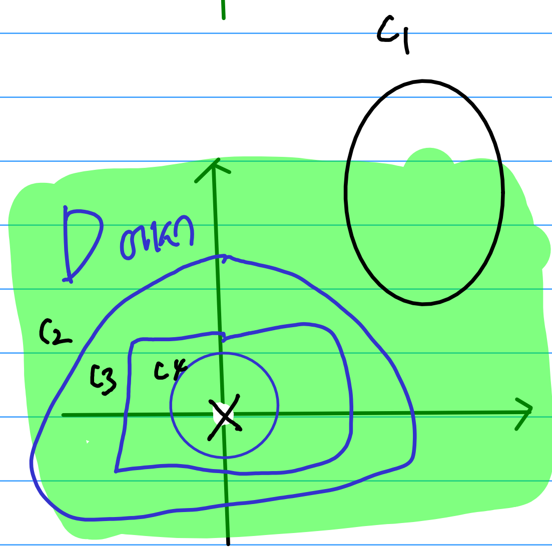
$$\int_{C_2} f(z)\,dz - \int_{C_1} f(z)\,dz = 0$$

$$\int_{C_2} f(z)\,dz = \int_{C_1} f(z)\,dz$$



$\frac{1}{z}, \frac{1}{z^2}, \frac{1}{z^3}, \ldots$

not analytical at $z=0$



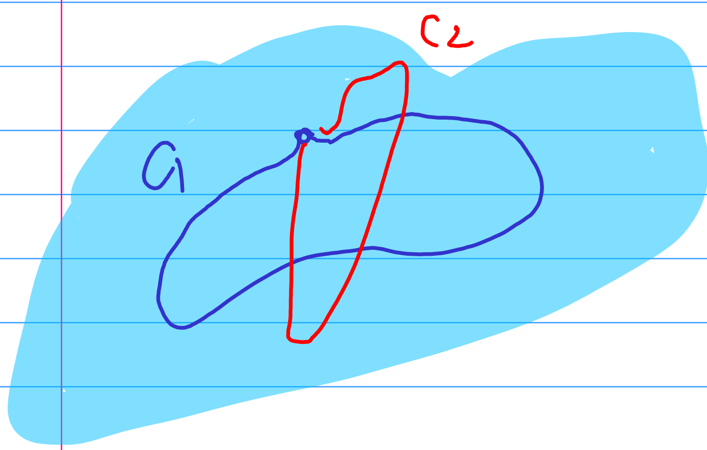
$$\oint_{C_1} f(z)\,dz = 0$$

$$\int_{C_2} f(z)\,dz =$$

$$\int_{C_3} f(z)\,dz =$$

$$\int_{C_4} f(z)\,dz$$

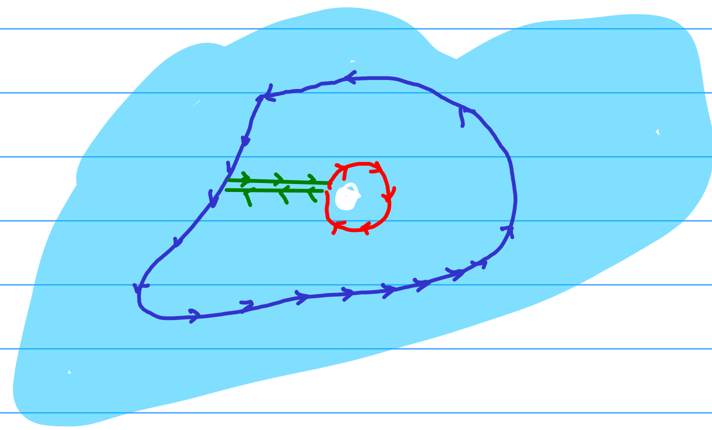
I no hole

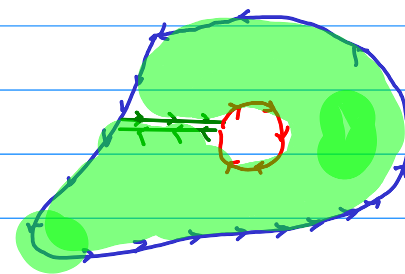


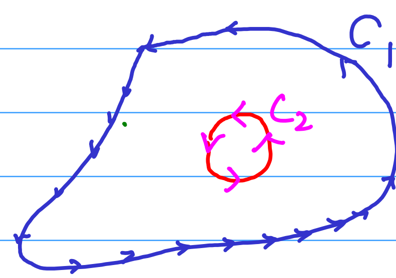
$$\int_{C_1} f(z)\,dz = 0$$

$$\int_{C_2} f(z)\,dz = 0$$

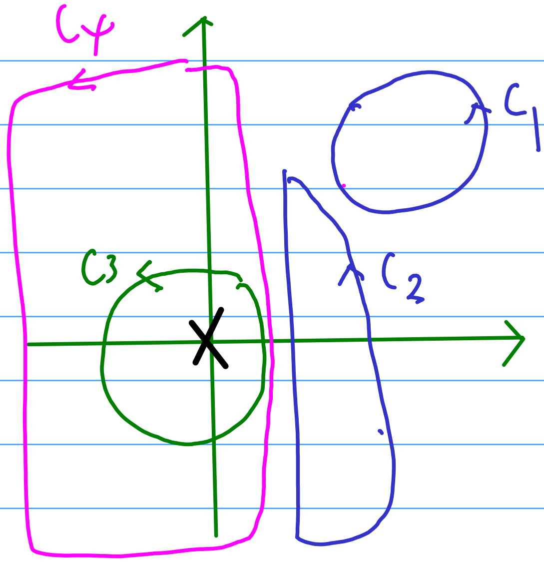
II a hole inside







$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

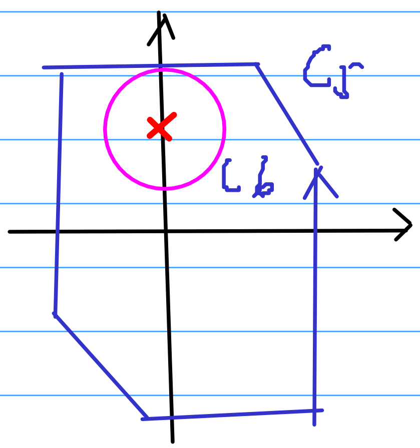


$$\int_{C_1} f(z)\,dz = 0$$

$$\int_{C_2} f(z)\,dz = 0$$

$$\int_{C_3} f(z)\,dz$$

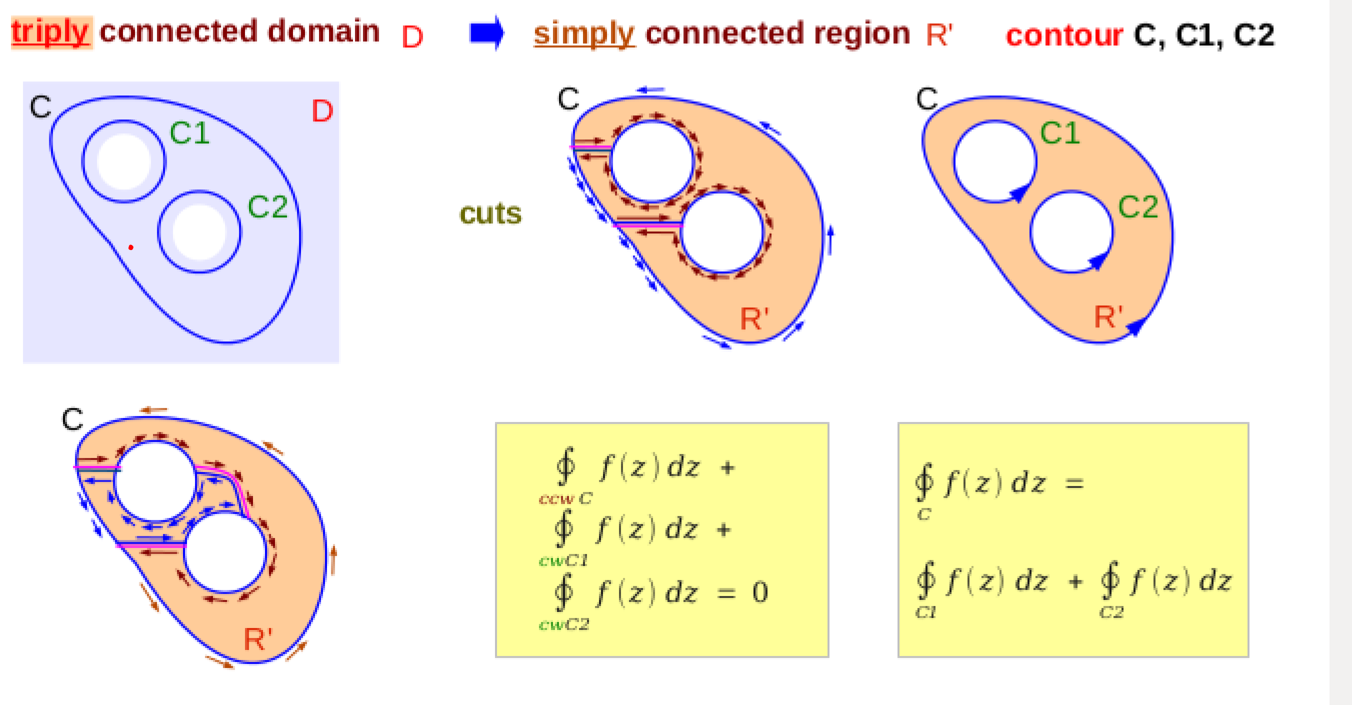
$$\int_{C_4} f(z)\,dz$$



$$\int_{C_5} f(z)\,dz$$

$$\int_{C_6} f(z)\,dz$$

multiple holes inside

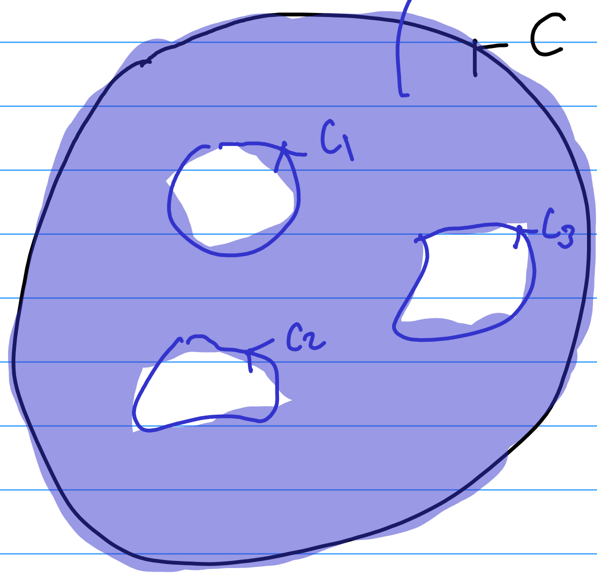


$$\oint_C f(z)\,dz + \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz = 0$$

$$\oint_C f(z)\,dz - \oint_{C_1} f(z)\,dz - \oint_{C_2} f(z)\,dz = 0$$

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz$$

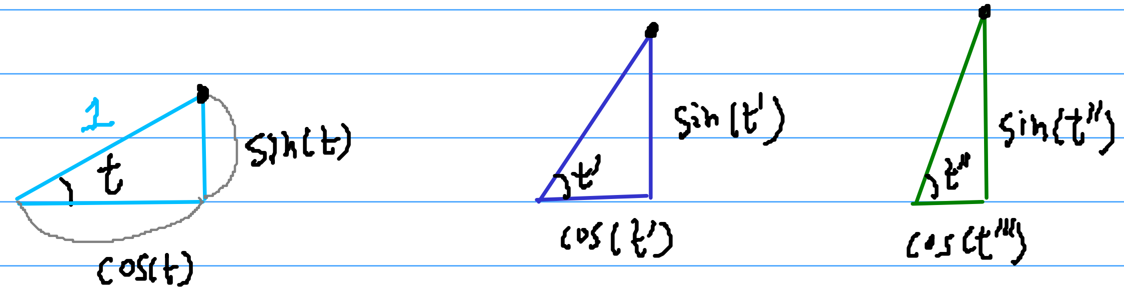
여기서 $f(z)$ analytic

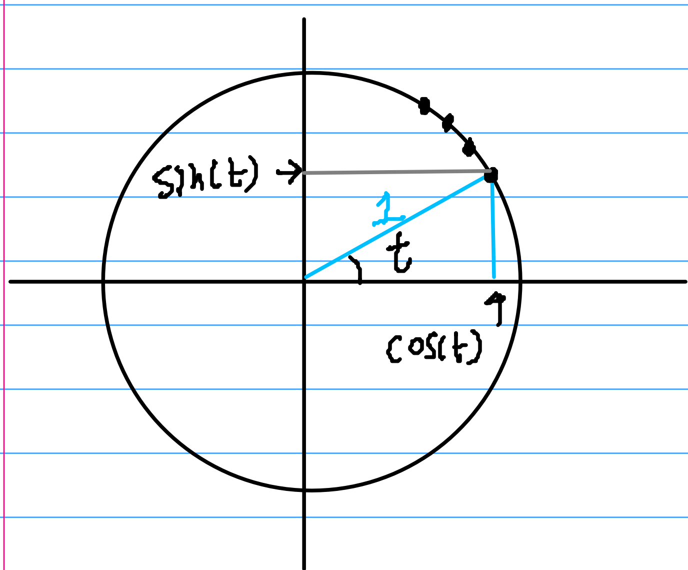


$$\boxed{\oint_C f(z)\,dz} = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz$$

Circle Contour

$$\begin{cases} x(t) = \cos(t) \\ y(t) = \sin(t) \end{cases} \qquad 0 \leq t \leq 2\pi$$

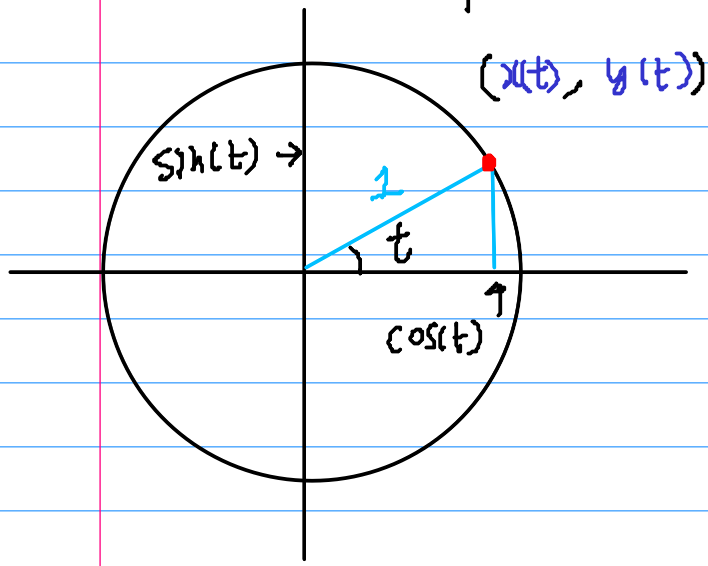




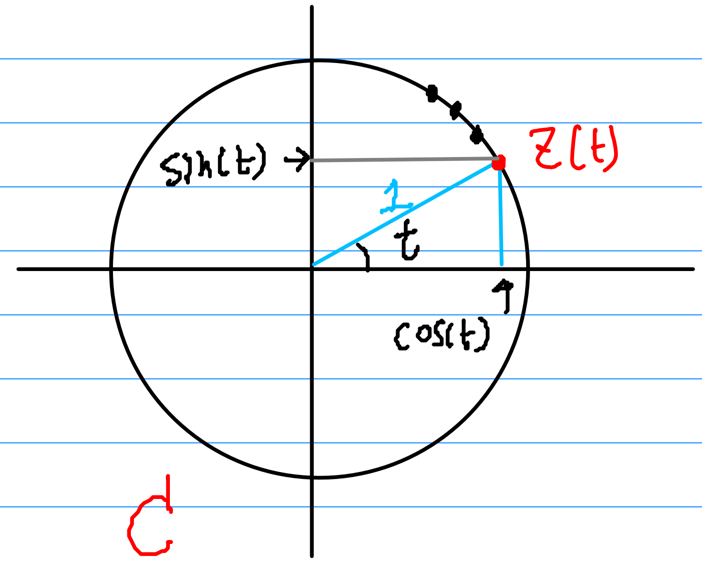
$\boxed{0 \leqslant t \leqslant 2\pi}$

all points
of the circle

ordered pair



complex number



$$\begin{cases} x(t) = \cos(t) \\ y(t) = \sin(t) \end{cases} \quad 0 \le t \le 2\pi$$

$$z(t) = x(t) + i\,y(t)$$
$$= \cos(t) + i\sin(t)$$
$$= e^{it}$$

unit circle

$$\int_C f(z)\,dz = \oint_C \frac{1}{z}\,dz$$

적분을 C를 따라서 한다

(z는 C 위에 있는 복소수만)

$$z = e^{it}$$

$$f(z) = \frac{1}{z} = \frac{1}{e^{it}} = e^{-it}$$

C 위의 z (실수부 x, 허수부 y)

매개변수 t로 연결

$$\int_C f(z)\,dz = \int_a^b f(z)\,\boxed{z'(t)\;dt}$$

$$\hookrightarrow dz = \frac{dz}{dt}\,dt$$

$$= \oint_C \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{z}\cdot z'(t)\,dt$$

$$= \int_0^{2\pi} e^{-jt}\left(e^{+jt}\right)'\,dt = \int_0^{2\pi} \cancel{e^{-jt}}\cdot j\cdot \cancel{e^{+jt}}\,dt$$

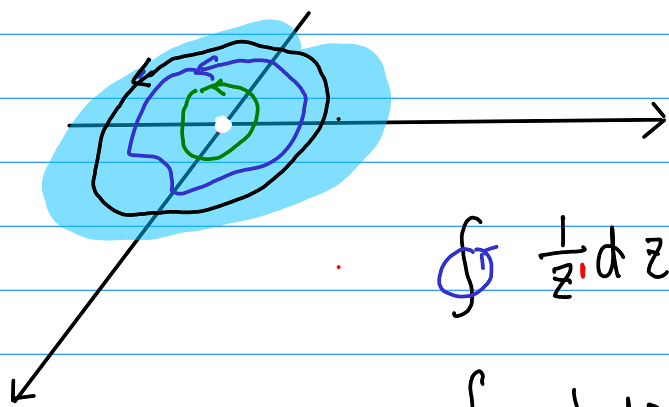
$$= j\int_0^{2\pi} 1\,dt = 2\pi j$$

$$f(z) = \frac{1}{z^n}$$ n: integer

$$f(z) = \cdots z^{-2}, \; z^{-1}, \; z^0, \; z^1, \; z^2 \cdots\cdots$$

$n=2 \quad n=1 \quad n=0 \quad n=-1 \quad n=-2$

$$\int_C f(z)\,dz = \begin{cases} 2\pi i & f(z) = \frac{1}{z} \quad n=1 \\ 0 & f(z) \neq \frac{1}{z} \quad n \neq 0 \end{cases}$$



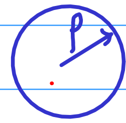
z^{-1}

$$\oint_C \frac{1}{z^1}\,dz = 2\pi i \neq 0$$

$$\oint_C \frac{1}{z^2}\,dz = \oint_C \frac{1}{z^3}\,dz = \cdots\cdots = 0$$

$$\oint_C z^0\,dz = \oint_C z^1\,dz = \cdots\cdots = 0$$

$f(z) = z^m$



$z(t) = e^{it}$

$z'(t) = i \cdot e^{it}$

$z'(t)\,dt = ie^{it}dt$

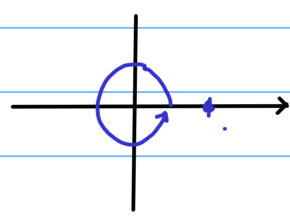
$$\int_C f(z)\,dz = \int_0^{2\pi} e^{mit}\, i e^{it}\, dt = \int_0^{2\pi} i e^{i(m+1)t}\, dt \qquad dz = \boxed{ie^{it}}dt$$

$z = \rho e^{j\theta}$ $\qquad dz = j\rho e^{j\theta}\, d\theta$

m=2	$\int_C z^2\, dz = \int_0^{2\pi} e^{i2t}\, i e^{it}\, dt = \int_0^{2\pi} i e^{i3t}\, dt = \left[\frac{1}{3} e^{i3t}\right]_0^{2\pi} = \frac{1}{3}(e^{6\pi} - e^{0}) = 0$	3
m=1	$\int_C z\, dz = \int_0^{2\pi} e^{it}\, i e^{it}\, dt = \int_0^{2\pi} i e^{i2t}\, dt = \left[\frac{1}{2} e^{i2t}\right]_0^{2\pi} = \frac{1}{2}(e^{4\pi i} - e^{0}) = 0$	2
m=0	$\int_C 1\, dz = \int_0^{2\pi} i e^{it}\, dt = \int_0^{2\pi} i e^{it}\, dt = \left[e^{it}\right]_0^{2\pi} = (\underbrace{e^{2\pi i}}_{1} - \underbrace{e^{0}}_{1}) = 0$	1
m=-1	$\int_C \frac{1}{z}\, dz = \int_0^{2\pi} e^{-it}\, i e^{it}\, dt = \int_0^{2\pi} i\, dt = \left[it\right]_0^{2\pi} = i(2\pi - 0) = \boxed{2\pi i}$	0
m=-2	$\int_C \frac{1}{z^2}\, dz = \int_0^{2\pi} e^{-i2t}\, i e^{it}\, dt = \int_0^{2\pi} i e^{-it}\, dt = \left[-e^{-it}\right]_0^{2\pi} = -(e^{-2\pi i} - e^{0}) = 0$	-1
m=-3	$\int_C \frac{1}{z^3}\, dz = \int_0^{2\pi} e^{-i3t}\, i e^{it}\, dt = \int_0^{2\pi} i e^{-i2t}\, dt = \left[-\frac{1}{2} e^{-i2t}\right]_0^{2\pi} = -\frac{1}{2}(e^{-4\pi i} - e^{0}) = 0$	-2

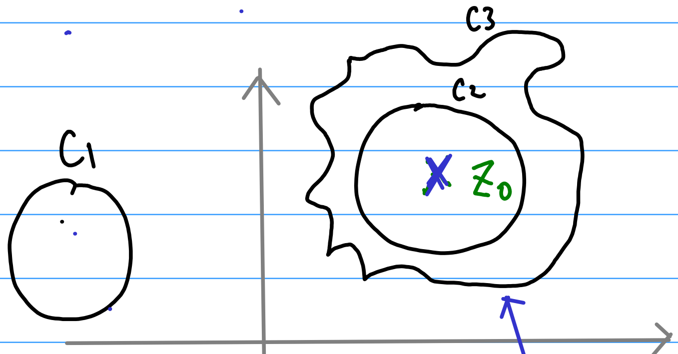
$e^{0 \cdot i} = e^0$

$e^{i2\pi}$



integer

$$\oint_C \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i & n=1 \\ 0 & n\neq 1 \end{cases}$$



$$f(z) = \frac{1}{(z-z_0)^n}$$

Fig 18.2.4

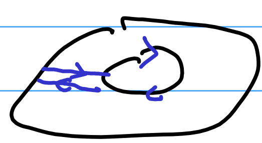
$$\boxed{\oint_{C_1} \frac{dz}{(z-z_0)^n} = 0}$$

18.2.1 Theorem

$$\boxed{\oint_{C_2} \frac{dz}{(z-z_0)^n} = \oint_{C_3} \frac{dz}{(z-z_0)^n}}$$

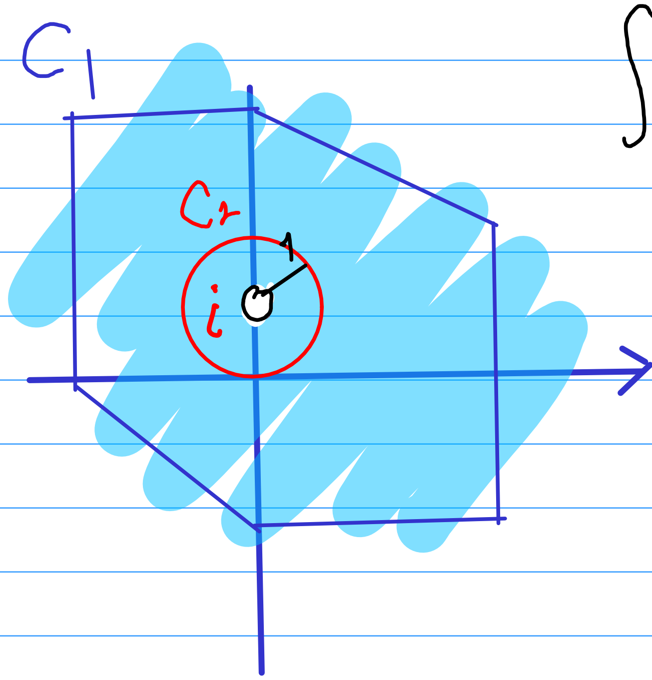
deformation of contours

식 (3)



$$\oint \underbrace{\frac{1}{z-i}}_{=f(z)} dz$$

$z = i$ on ~~analytic~~



$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

$$\oint_{C_1} \frac{dz}{z-i} = \oint_{C_2} \frac{dz}{z-i}$$

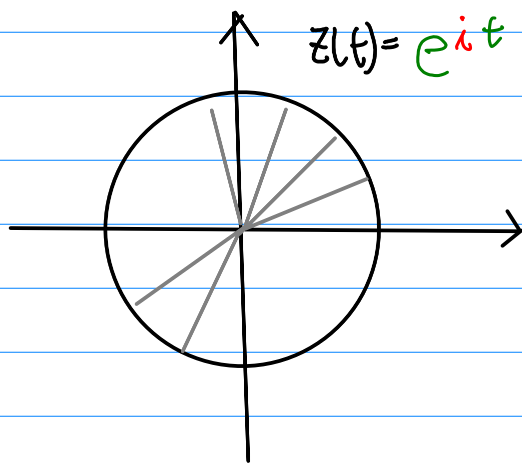
$e^{it}, \quad 0 \leq t < 2\pi$

$$z = e^{it} = \cos(t) + i\sin(t)$$

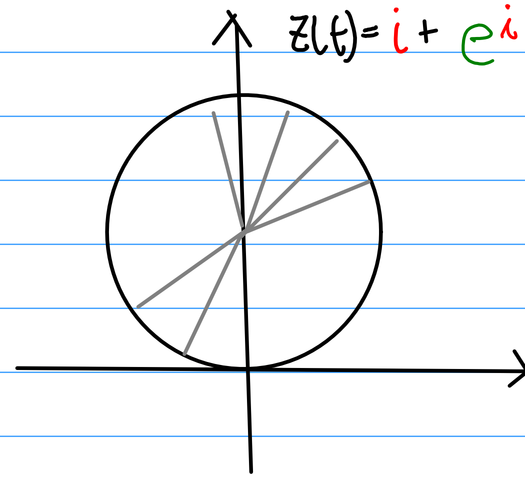
$$|z| = |e^{it}| = \sqrt{\cos^2(t) + \sin^2(t)} = 1$$

$0 \leq t \leq 2\pi$

$z(t) = e^{it}$

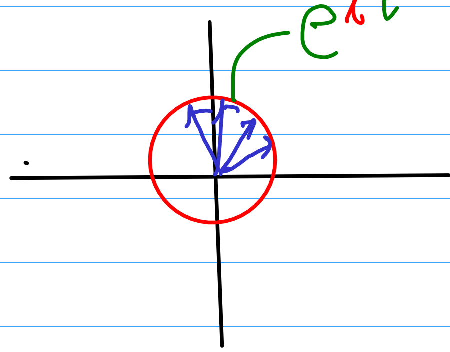


$z(t) = i + e^{it}$



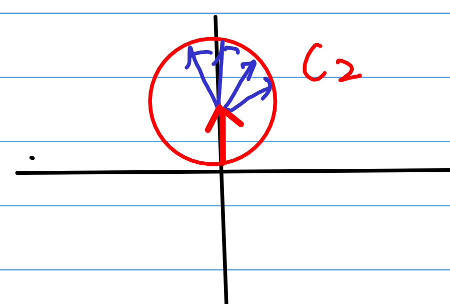
$$z(t) = e^{it}$$

e^{it}



$z(t) = i + e^{jt}$

$= i + \cos t + i \sin t$



C_2

복소수 z에 대한

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

실수 t에 대한 적분

general case

C에 있는 z를 t의 함수로 표시 가능 할 때

$$z'(t) = \frac{dz}{dt}$$

$$f(z) = \frac{1}{z-i}$$

C_2에 있는 z는 $z(t) = i + e^{it}$

$$z'(t) = ie^{it}$$

$$z(t) - i = e^{it}$$

$$\oint_{C_1} \frac{dz}{z-i} = \oint_{C_2} \frac{dz}{z-i} = \int_0^{2\pi} \frac{1}{\cancel{e^{it}}}\, i\cancel{e^{it}}\,dt$$

$$= i\int_0^{2\pi} 1\,dt = 2\pi i$$

$$\oint_C \frac{dz}{(z-i)^n} = \oint_{C_1} \frac{dz}{(z-i)^n} = \int_0^{2\pi} \frac{1}{(e^{jt})^n} je^{jt}\, dt$$

$$= j\int_0^{2\pi} \frac{1}{(e^{jt})^{n-1}}\, dt$$

$$= \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases}$$

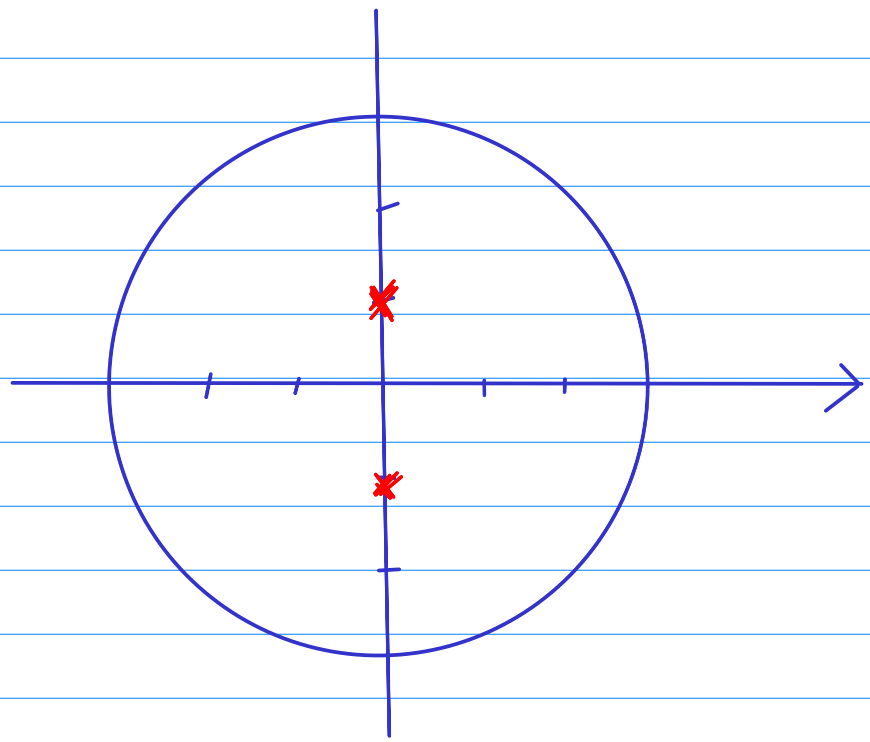
Zill & Wright

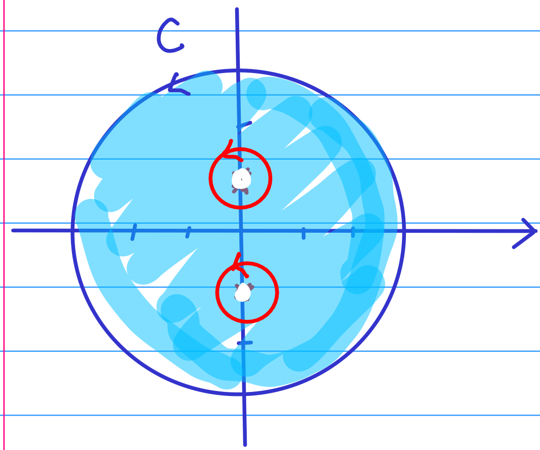
$$\oint \frac{1}{z^2+1}\,dz \qquad f(z)$$

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \left(\frac{1}{z-i} - \frac{1}{z+i}\right)\frac{1}{2i}$$

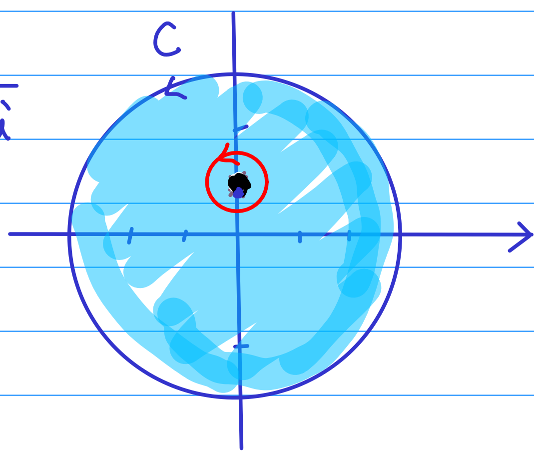
$z = i$, $z = -i$ ~~analytic~~ $\pm i$

C contour 반지름이 3인 원

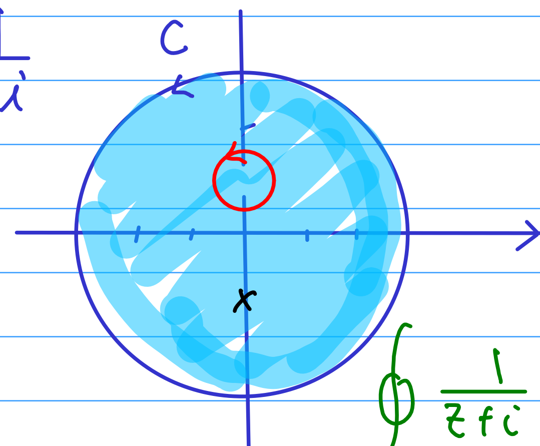




$\frac{1}{z-i}$



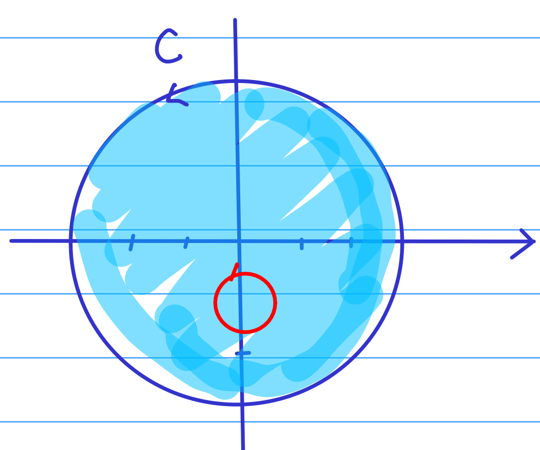
$\frac{1}{z+i}$



C_1

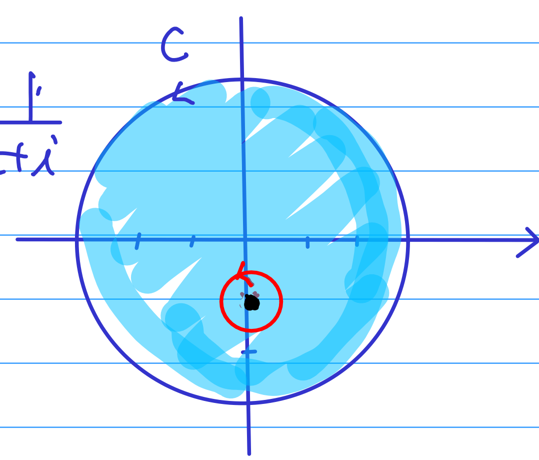
$$\oint_{C_1} \frac{1}{z+i}\, dz = 0$$

$\frac{1}{z-i}$



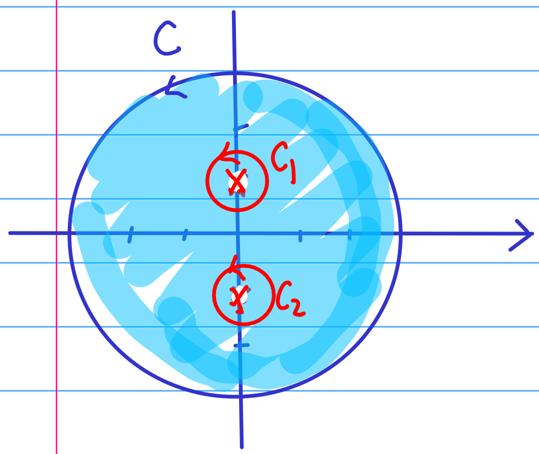
$$\oint_{C_2} \frac{1}{z-i}\, dz = 0$$

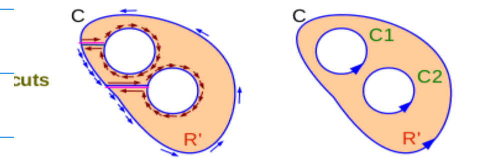
$\frac{1}{z+i}$



C_2

$$\oint_C \frac{1}{z^2+1}\,dz = \oint_{C_1} \frac{1}{z^2+1}\,dz + \oint_{C_2} \frac{1}{z^2+1}\,dz$$





$$\oint_{ccw\,C} f(z)\,dz + \oint_{cw\,C1} f(z)\,dz + \oint_{cw\,C2} f(z)\,dz = 0$$

$$\oint_C f(z)\,dz = \oint_{C1} f(z)\,dz + \oint_{C2} f(z)\,dz$$

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \frac{\frac{1}{2i}}{(z-i)} - \frac{\frac{1}{2i}}{(z+i)}$$

$$\frac{1}{2i}\oint_C \left[\frac{1}{(z-i)} - \frac{1}{(z+i)}\right]dz = \frac{1}{2i}\oint_{C_1} \left[\frac{1}{(z-i)} - \frac{1}{(z+i)}\right]dz$$

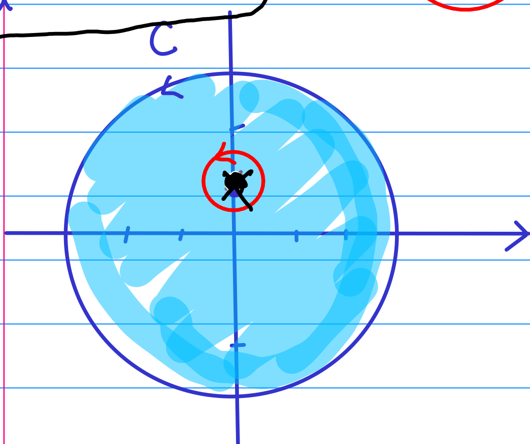
$$+ \frac{1}{2i}\oint_{C_2} \left[\frac{1}{(z-i)} - \frac{1}{(z+i)}\right]dz$$

$$\frac{1}{2i}\oint_{C}\left[\frac{1}{(z-i)}-\frac{1}{(z+i)}\right]dz = \frac{1}{2i}\oint_{C_1}\left[\frac{1}{(z-i)}-\frac{1}{(z+i)}\right]dz$$

$$+\ \frac{1}{2i}\oint_{C_2}\left[\frac{1}{(z-i)}-\frac{1}{(z+i)}\right]dz$$

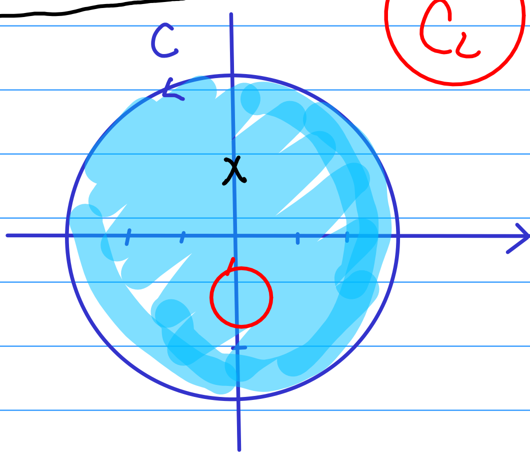
$$\boxed{\oint_{C_1}\frac{1}{z-i}\,dz = 2\pi i}$$

C_1

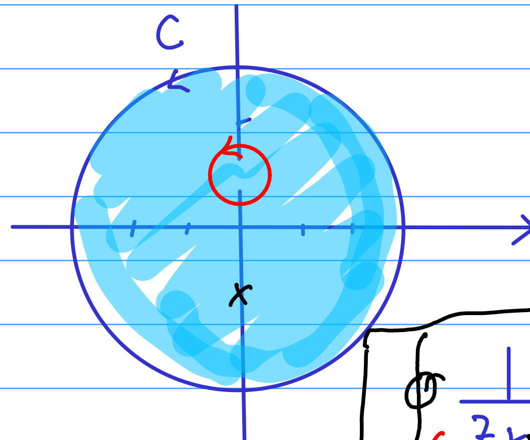


$$\boxed{\oint_{C_2}\frac{1}{z-i}\,dz = 0}$$

C_2

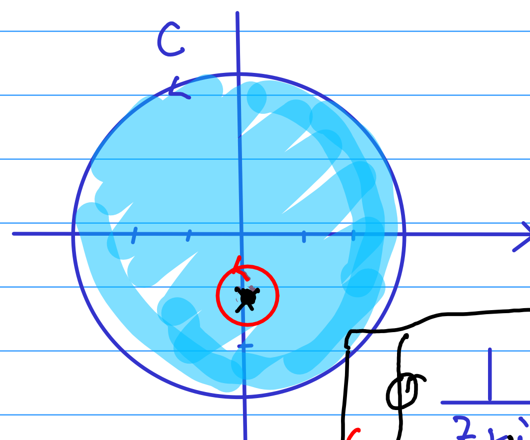


C_1



$$\boxed{\oint_{C_1}\frac{1}{z+i}\,dz = 0}$$

C_2



$$\boxed{\oint_{C_2}\frac{1}{z+i}\,dz = 2\pi i}$$

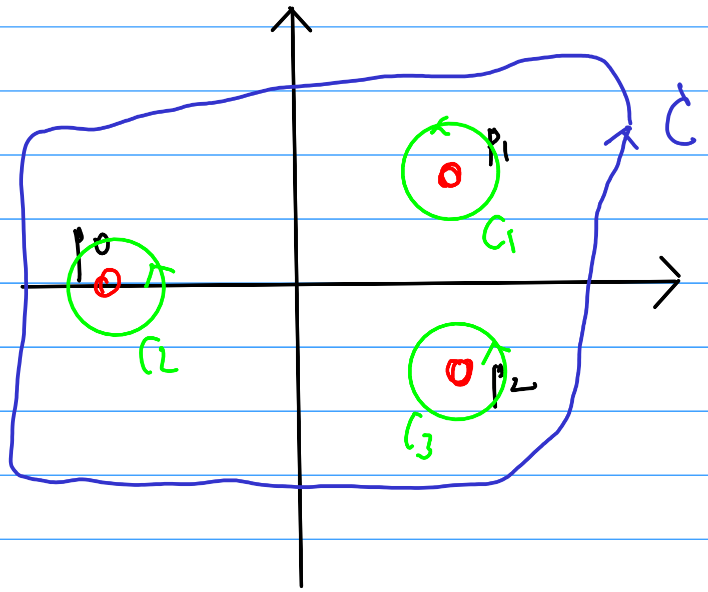
$$\frac{1}{2i}\oint_C \left[\frac{1}{(z-i)} - \frac{1}{(z+i)}\right] dz$$

$$= \frac{1}{2i}\overbrace{\boxed{\oint_{C_1} \frac{1}{(z-i)}\, dz}}^{2\pi i} - \frac{1}{2i}\overbrace{\boxed{\oint_{C_1} \frac{1}{(z+i)}\, dz}}^{0}$$

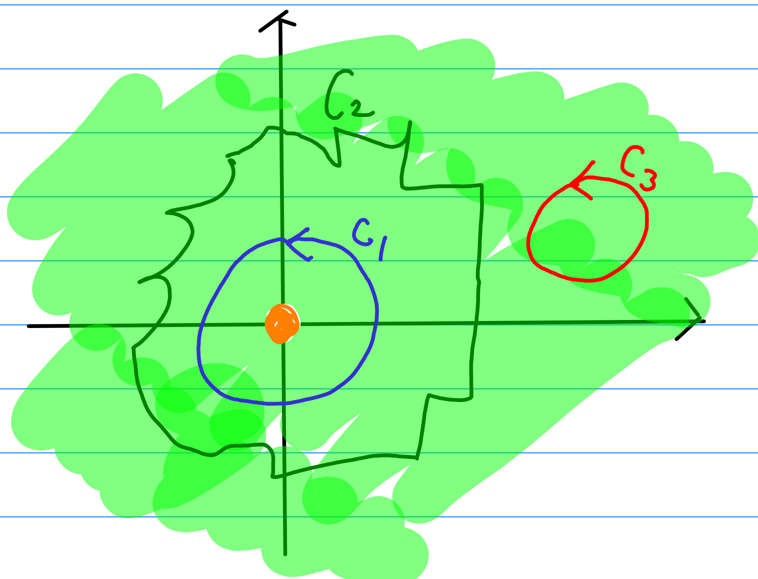
$$+ \frac{1}{2i}\overbrace{\boxed{\oint_{C_2} \frac{1}{(z-i)}\, dz}}^{0} - \frac{1}{2i}\overbrace{\boxed{\oint_{C_2} \frac{1}{(z+i)}\, dz}}^{2\pi i}$$

$$= \pi - \pi = 0$$

$$\oint_C \frac{az^2 + bz + c}{(z-p_0)(z-p_1)(z-p_2)} \, dz$$



$$\oint_C \frac{A}{(z-p_0)} + \frac{B}{(z-p_1)} + \frac{C}{(z-p_2)} \, dz$$

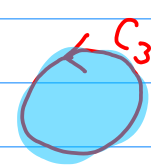


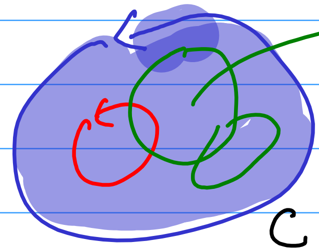
$$f(z) = \frac{1}{z}$$

$$\oint_{C_1} f(z)\,dz = 2\pi i$$

$$\oint_{C_2} f(z)\,dz = 2\pi i$$

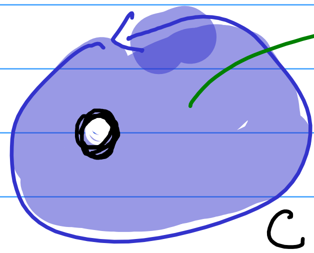
$$\oint_{C_3} f(z)\,dz = 0$$





여기서 $f(z)$는 해석적

$$\oint f(z) = 0$$



여기서 $f(z)$가 해석적이지 않은 곳이 있으면

$$\oint f(z) \neq 0$$

not necessarily zero $\begin{cases} \text{sometimes zero} \\ \text{sometimes non-zero} \end{cases}$

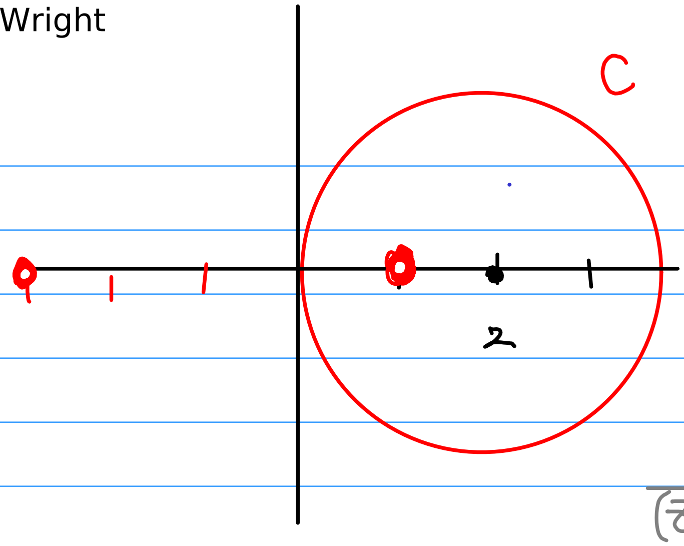
$f(z) = \frac{1}{z}$

$f(z) = \frac{1}{z^2}$

$$\oint \frac{1}{z} dz = 2\pi i$$

$$\oint \frac{1}{z^2} dz = 0$$

Zill & Wright

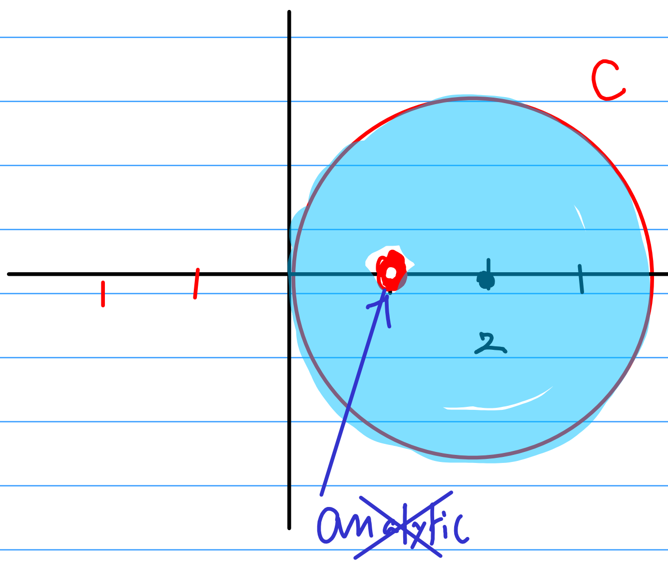


$$\oint \frac{5z+7}{(z-1)(z+3)}\,dz$$

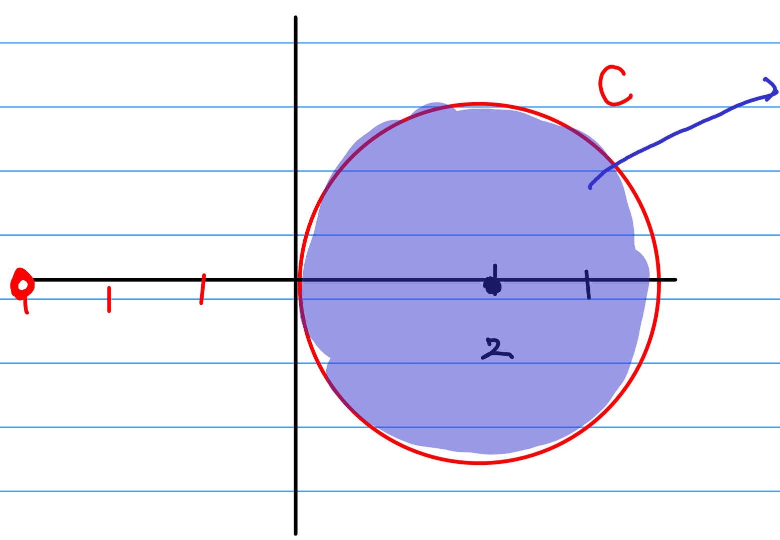
partial fraction

$$\frac{5z+7}{(z-1)(z+3)} = \frac{3}{z-1} + \frac{2}{z+3}$$

$$\boxed{\oint_C \frac{3}{z-1}\,dz} + \boxed{\oint_C \frac{2}{z+3}\,dz}$$



$$\oint \frac{3}{z-1}\,dz = 3\cdot 2\pi i$$



analytic

$$\oint \frac{2}{z+3}\,dz = 0$$

$f(z)$: **analytic** on and inside simple close curve C

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(w)}{w-z}dw$$

the value of $f(z)$ at a point $z=a$ inside C

$$\frac{d}{dz}f(z) = \frac{d}{dz}\left[\frac{1}{2\pi i}\oint \frac{f(w)}{w-z}dw\right] \qquad f'(z) = \frac{1}{2\pi i}\oint \frac{f(w)}{(w-z)^2}dw$$

$$\frac{d}{dz}f(z) = \frac{d}{dz}\left[\frac{1}{2\pi i}\oint \frac{f(w)}{w-z}dw\right] \qquad f''(z) = \frac{2}{2\pi i}\oint \frac{f(w)}{(w-z)^3}dw$$

$$\frac{d}{dz}f(z) = \frac{d}{dz}\left[\frac{1}{2\pi i}\oint \frac{f(w)}{w-z}dw\right] \qquad f^{(3)}(z) = \frac{3!}{2\pi i}\oint \frac{f(w)}{(w-z)^4}dw$$

• • •

적분은 w에대해서
미분은 z에대해서

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(w)}{(w-z)^{n+1}}dw$$

$f(z)$ is *infinitely differentiable* in that neighborhood

$$\frac{d}{dz}\left(\frac{1}{w-z}\right) = \frac{d}{dz}\left((w-z)^{-1}\right) = -1(w-z)^{-2}(-1)$$
$$= (w-z)^{-2}$$

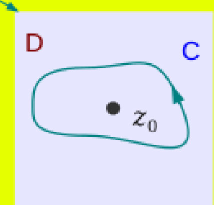
$$\frac{d}{dz}\left(\frac{1}{(w-z)^2}\right) = \frac{d}{dz}\left((w-z)^{-2}\right) = -2(w-z)^{-3}(-1)$$
$$= 2(w-z)^{-3}$$

Contour Integration for $f(z)/(z-z_0)$

$\boxed{f(z)}$ **analytic** in D

no singularity in D

D: **simply** connected domain (no hole)

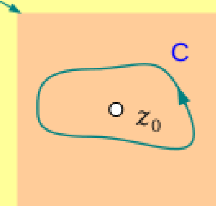


$\Rightarrow \oint_C f(z)\, dz = 0$

$\boxed{f(z_0)}$

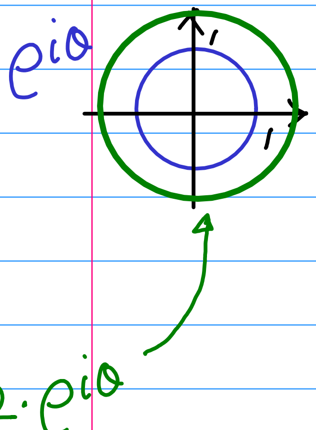
$\boxed{\dfrac{f(z)}{(z-z_0)}}$ not **analytic** at z_0 in D

D': **multiply** connected domain

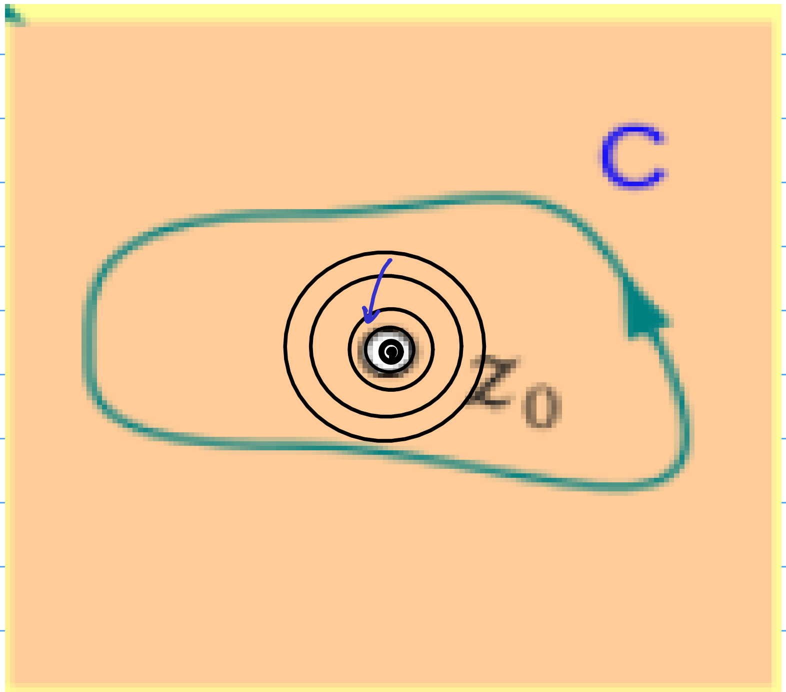


$\Rightarrow \oint_C \dfrac{f(z)}{(z-z_0)}\, dz \times 0$ not necessarily zero

$$\oint_C \frac{f(z)}{(z-z_0)}\, dz$$

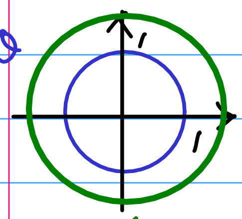


반지름=2인 원



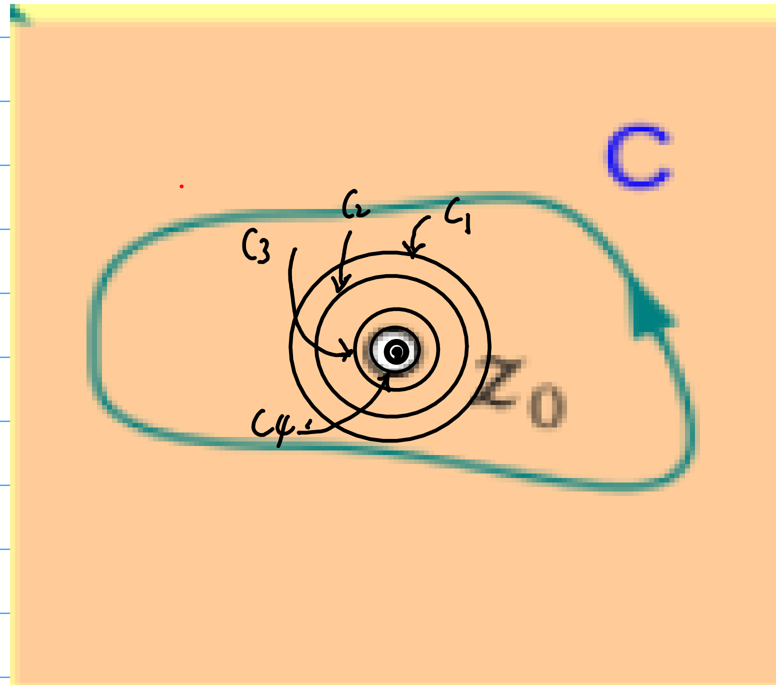
$$\oint \frac{f(z)}{(z-z_0)}\, dz$$

$e^{i\theta}$



$2 \cdot e^{i\theta}$

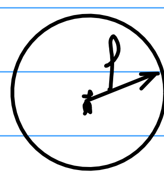
반지름=2인 원



$$\oint_{C_1} \frac{f(z)}{(z-z_0)}\, dz = \oint_{C_2} \frac{f(z)}{(z-z_0)}\, dz = \oint_{C_3} \frac{f(z)}{(z-z_0)}\, dz$$

z는 C_1위의 복소수 z는 C_2위의 복소수 z는 C_3위의 복소수

$\rho \to 0$



$\boxed{\rho \to 0}$

$z \to z_0$

$f(z) \to f(z_0)$

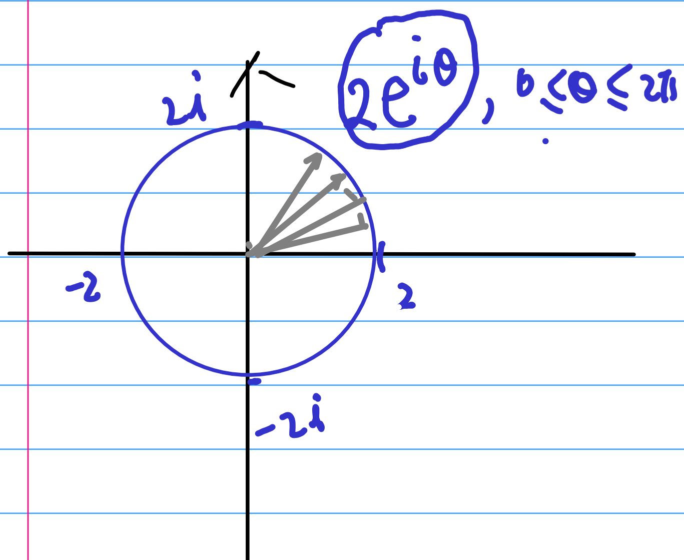
$$= \oint_{C_\infty} \frac{f(z_0)}{(z-z_0)}\, dz$$

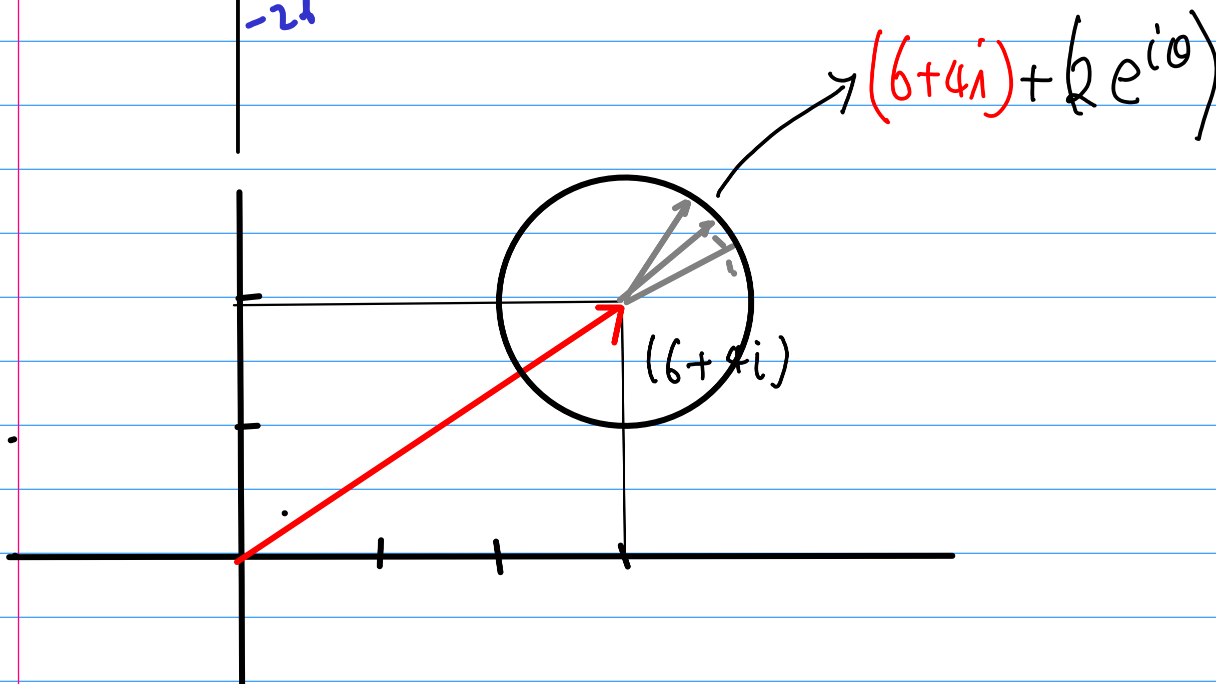
$$= f(z_0) \oint_{C_\infty} \frac{1}{(z-z_0)}\, dz$$

$$= f(z_0) \cdot 2\pi i$$

$(1+i) + e^{i\theta}$

$a + \rho e^{i\theta}$

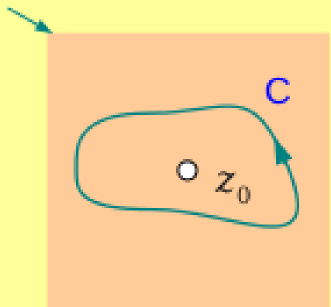




Simply Connected Region R

$\boxed{\dfrac{f(z)}{(z-z_0)}}$ not **analytic** at z_0 in D

D': multiply connected domain

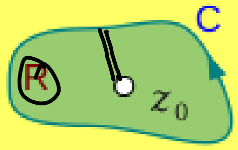


$$\oint_C \frac{f(z)}{(z-z_0)}\,dz \cancel{=} 0$$ not necessarily zero

$\boxed{\dfrac{f(z)}{(z-z_0)}}$ **analytic** at z_0 in a region R

R: all points within and on C

C: any simple closed path



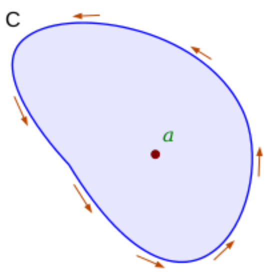
$$\oint_C \frac{f(z)}{(z-z_0)}\,dz \cancel{=} 0$$ not necessarily zero

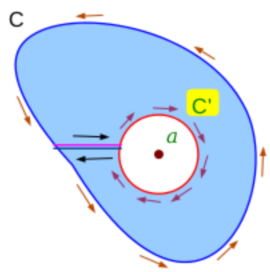
$$\boxed{\oint_C \frac{f(z)}{(z-z_0)}\,dz = 2\pi i \boxed{f(z_0)}}$$

Contour Integration in R

$$\oint_C f(z)\,dz = 0$$

$$\oint_{ccw\ C} \frac{f(z)}{z-a}\,dz + \oint_{cw\ C'} \frac{f(z)}{z-a}\,dz = 0$$





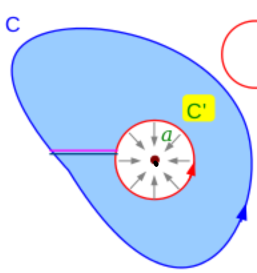
$$\oint_{ccw\ C} \frac{f(z)}{z-a}\,dz = \oint_{ccw\ C'} \frac{f(z)}{z-a}\,dz$$

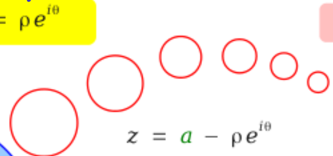
As z approaches to a

$z = a + \rho e^{i\theta}$

along C' $\quad z - a = \rho e^{i\theta}$

$z \to a \Rightarrow \rho \to 0,\ f(z) \to f(a)$





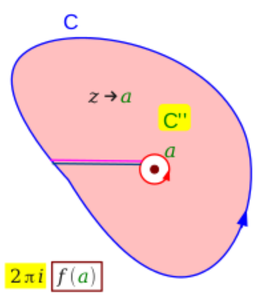
$$z = a - \rho e^{i\theta}$$

$$dz = i\rho e^{i\theta}\,d\theta$$

$$\frac{dz}{z-a} = \frac{i\cancel{\rho e^{i\theta}}\,d\theta}{\cancel{\rho e^{i\theta}}}$$

$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a} = \int_0^{2\pi} f(z)\,i\,d\theta = 2\pi i\,\boxed{f(a)}$$

$f(a)$



$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a} = \oint_{ccw\ C'} \frac{f(z)\,dz}{z-a}$$

$$= 2\pi i\ f(a)$$

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$$

$$f'(z_0) = \frac{1!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2}\, dz$$

differentiate w.r.to z_0

$$\oint \frac{z^3+3}{z(z-i)^2}\, dz \Rightarrow \oint \frac{\dfrac{z^3+3}{(z-i)^2}}{z-0}\, dz$$

$\frac{z^3+3}{(z-i)^2} \Rightarrow f(z)$ analytic at 0

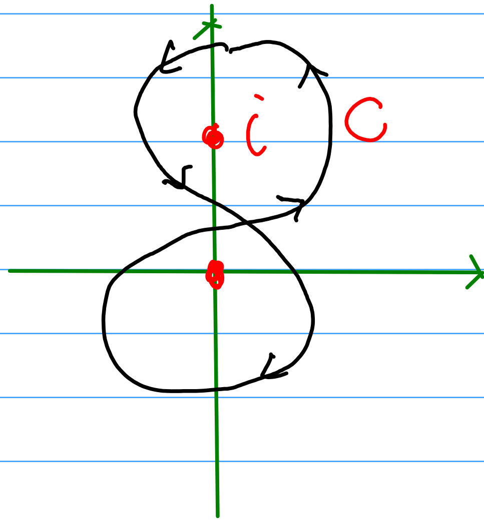
a hole at 0

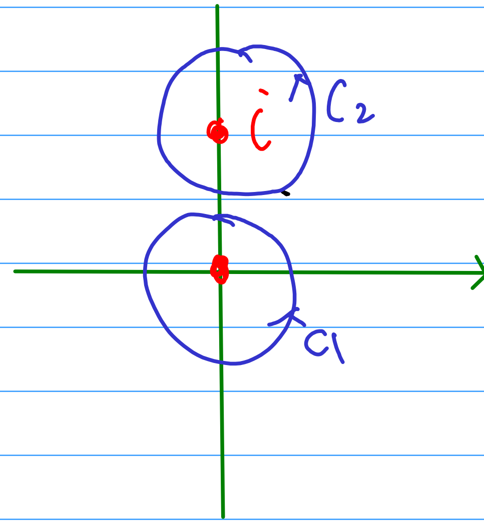
$$\oint \frac{z^3+3}{z(z-i)^2}\, dz \Rightarrow \oint \frac{\dfrac{z^3+3}{z}}{(z-i)^2}\, dz$$

$\frac{z^3+3}{z} \Rightarrow g(z)$ analytic at i

a hole at i

Zill & Wright





$$\oint_C \frac{z^3+3}{z(z-i)^2}\,dz = -\oint_{C_1} \frac{\frac{z^3+3}{(z-i)^2}}{z-0}\,dz + \oint_{C_2} \frac{\frac{z^3+3}{z}}{(z-i)^2}\,dz$$

$$= \left(-f(0) + g'(i)\right) 2\pi i$$

$$f(z) = \frac{z^3+3}{(z-i)^2}$$

$$f(0) = \frac{0^3+3}{(0-i)^2} = -3$$

$$g(z) = \frac{z^3+3}{z}$$

$$g'(z) = \frac{(z^3+3)'z - (z^3+3)(z)'}{z^2}$$

$$= \frac{3z^3 - z^3 - 3}{z^2}$$

$$g'(z) = \frac{2z^3-3}{z^2}$$

$$g'(i) = \frac{2i^3-3}{i^2}$$

$$= \frac{-2i-3}{-1} = (2i+3)$$

$$2\pi i(2i+3)$$

$$2\pi(-2+3i)$$

Cauchy-Rieman Equation

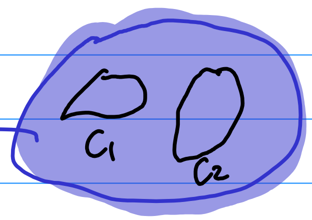
$z = x + iy$ $\qquad$ $w = u + iv$

$w = f(z)$ analytic in D

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = - \dfrac{\partial v}{\partial x} \end{cases}$$

Cauchy - Goursat Theorem

여기서 $f(z)$가 analytic

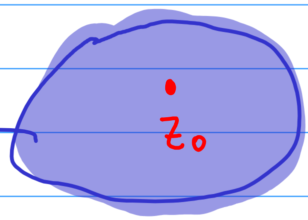


$$\oint_{C_1} f(z)\,dz = 0$$

$$\oint_{C_2} f(z)\,dz = 0$$

Cauch Integration Formula

여기서 $f(z)$가 analytic

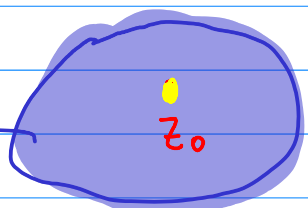


$$f(z_0) = \frac{1}{2\pi i} \oint_C \boxed{\frac{f(z)}{z - z_0}}\, dz \qquad (\mathrm{I})$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \boxed{\frac{f(z)}{(z - z_0)^{n+1}}}\, dz \qquad (\mathrm{II})$$

여기서

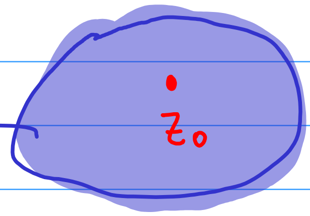
$\dfrac{f(z)}{z - z_0}$



~~continuous~~ ~~diff~~ ~~analytic~~

Cauch Integration Formula

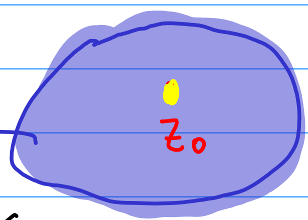
여기서
$f(z)$가
analytic



$$f(z_0) = \frac{1}{2\pi i} \oint_C \boxed{\frac{f(z)}{z - z_0}} dz \quad (\mathrm{I})$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \boxed{\frac{f(z)}{(z - z_0)^{n+1}}} dz \quad (\mathrm{II})$$

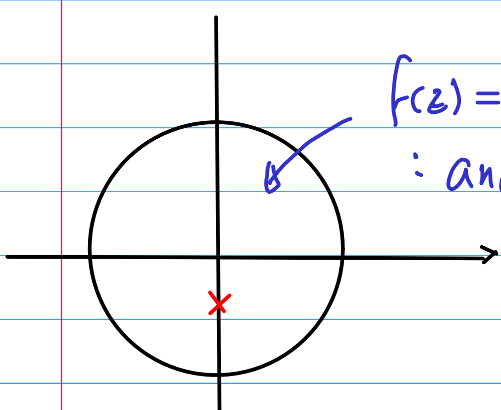
여기서
$\frac{f(z)}{z - z_0}$



~~continuous~~ ~~diff~~ ~~analytic~~

예제 1) $C : |z| = 2$ $\oint_C \frac{z^2 - 4z + 4}{z + i} dz$

$$\oint_C \frac{\overbrace{z^2 - 4z + 4}^{f(z)}}{z + i} dz = \oint_C \frac{f(z)}{z + i} dz$$



$f(z) = z^2 - 4z + 4$
: analytic

$\boxed{\frac{f(z)}{z + i}}$: analytic except $z = -i$

$$f(-i) = \frac{1}{2\pi i} \oint_C \boxed{\frac{f(z)}{z - (-i)}} dz$$

$$\oint_C \boxed{\frac{f(z)}{z - (-i)}} dz = 2\pi i \cdot f(-i)$$

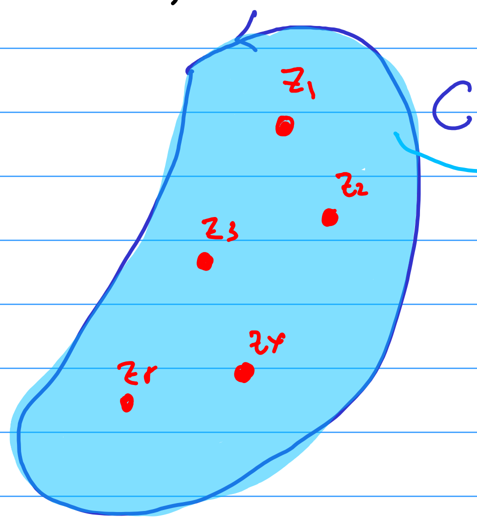
$$= 2\pi i \left((-i)^2 - 4(-i) + 4 \right)$$

$f(z) = z^2 - 4z + 4$

$$\oint_C \boxed{\frac{f(z)}{z-(-i)}}\,dz = 2\pi i \cdot f(-i)$$

$$\oint_C \boxed{\frac{1}{z-(-i)}}\,dz = 2\pi i$$

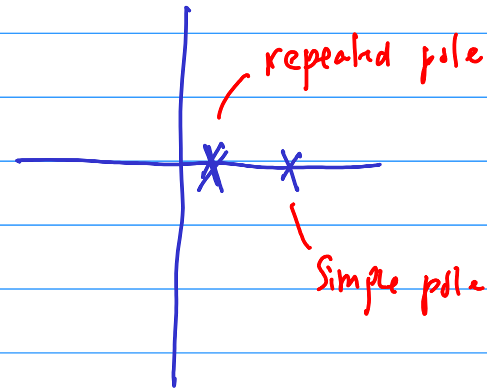
Cauchy Residue Theorem



Singular pt z_i는
제외한 모든 곳에서
$f(z)$가 analytic

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}\left(f(z),\, z_k\right)$$

$$f(z) = \frac{1}{(z-1)^2 (z-3)^1}$$



Singular pt 3

1

$$\mathrm{Res}(f(z), 3) = \lim_{z \to 3} \frac{1}{(z-1)^2 (z-3)^1} (z-3)$$

$$= \lim_{z \to 3} \frac{1}{(z-1)^2} = \frac{1}{4}$$

$$\mathrm{Res}(f(z), 1) = \frac{1}{1!} \lim_{z \to 1} \frac{d}{dz} \left[\frac{1}{(z-1)^2 (z-3)^1} (z-1)^2 \right]$$

$$\frac{d}{dz}\left[\frac{1}{(z-3)}\right] = \frac{-1}{(z-3)^2}$$

$$= \lim_{z \to 1} \frac{-1}{(z-3)^2} = -\frac{1}{4}$$

Finding Residue

$$\boxed{\mathrm{Res}\left(f(z),\ z_0\right) = \lim_{z \to z_0} (z - z_0) f(z)}$$

singular pt

$f(z)$ 분모에 $(z - z_0)$가 있을때

$$f(z) = \frac{\cdots}{\cdots (z - z_0) \cdots}$$

$$\boxed{\mathrm{Res}\left(f(z),\ z_0\right) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z) \right]}$$

singular pt

$f(z)$ 분모에 $(z - z_0)^n$가 있을때

$$f(z) = \frac{\cdots}{\cdots (z - z_0)^n \cdots}$$

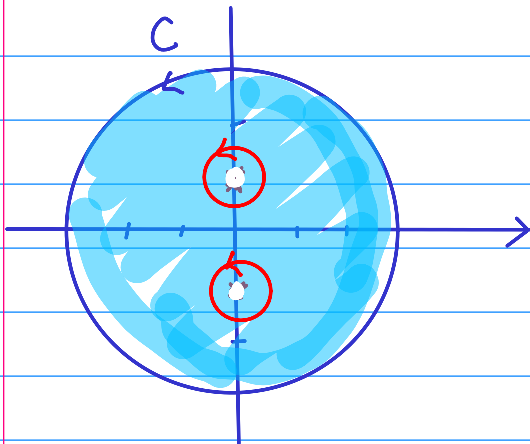
(%i10) `(s^2+6*s+9) /((s-1)*(s-2)*(s+4));`

(%o10) $$\frac{s^2+6s+9}{(s-2)(s-1)(s+4)} = \frac{A}{(s-2)} + \frac{B}{(s-1)} + \frac{C}{(s+4)}$$

(%i11) `partfrac(%, s);`

(%o11) $$\frac{1}{30(s+4)} - \frac{16}{5(s-1)} + \frac{25}{6(s-2)}$$

$A = \frac{25}{6}$ $B = -\frac{16}{5}$ $C = \frac{1}{30}$



$$\oint_C \frac{1}{z^2+1}\,dz$$

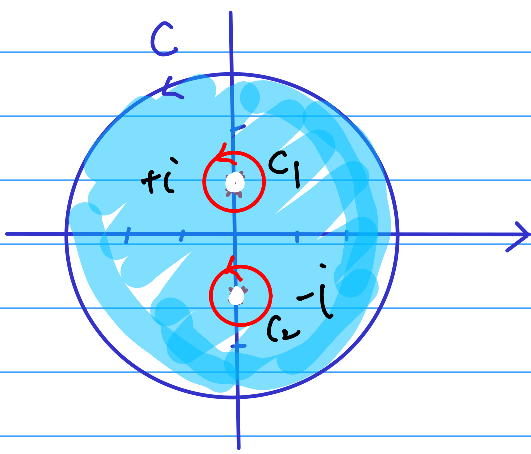
$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$$

$$= \frac{A}{(z+i)} + \frac{B}{(z-i)}$$

$$\boxed{A = \frac{1}{\cancel{(z+i)}(z-i)}\cancel{(z+i)}\Bigg|_{z=-i} = \frac{1}{(z-i)}\Bigg|_{z=-i} = -\frac{1}{2i}}$$

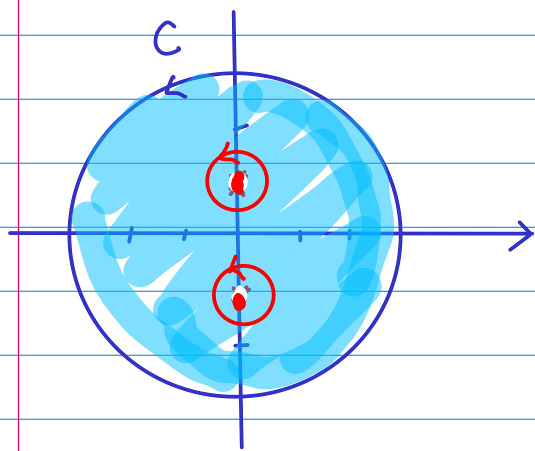
$$\boxed{B = \frac{1}{(z+i)\cancel{(z-i)}}\cancel{(z-i)}\Bigg|_{z=i} = \frac{1}{(z+i)}\Bigg|_{z=i} = \frac{1}{2i}}$$

$$\frac{1}{z^2+1} = \frac{1}{2i}\left(\frac{-1}{z+i} + \frac{1}{z-i}\right)$$



$$\oint_{c_1} \frac{1}{z-i}\,dz = 2\pi i$$

$$-\oint_{c_2} \frac{1}{z+i}\,dz = -2\pi i$$



$$f(z) = \frac{1}{z^2+1}$$

$$= \frac{1}{(z+i)(z-i)}$$

$$= \frac{1}{2i}\left(\frac{-1}{z+i} + \frac{1}{z-i}\right)$$

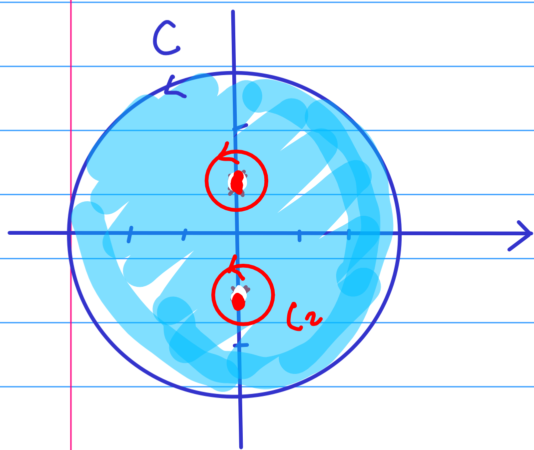
Singular points $+i$, $-i$

$$\oint_C f(z)\,dz = ?$$

$$\mathrm{Res}\left(f(z), +i\right) = \lim_{z \to i} \frac{1}{(z+i)\cancel{(z-i)}}\cancel{(z-i)} = \frac{1}{2i}$$

$$\mathrm{Res}\left(f(z), -i\right) = \lim_{z \to -i} \frac{1}{\cancel{(z+i)}(z-i)}\cancel{(z+i)} = \frac{-1}{2i}$$

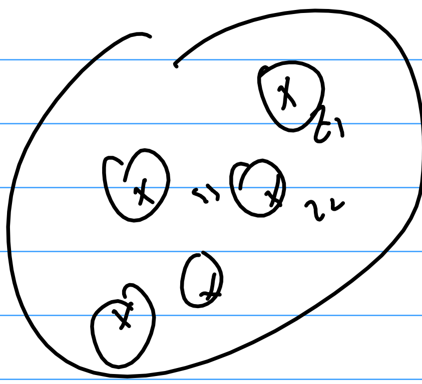
$$\int_C \frac{1}{(z+i)} + \frac{1}{(z+i)^2} + \frac{1}{(z+i)^3} \quad dz$$



$$\oint_{C_2} \frac{1}{(z+i)} dz$$

$$\oint_{C_2} \frac{1}{(z+i)^2} dz$$

$$\oint_{C_2} \frac{1}{(z+i)^3} dz$$



$$\int \frac{\quad}{(z-z_0)^2 (z-z_1)(z-z_2)} \quad dz$$

$$\frac{a}{z-z_0} + \frac{1}{z-z_1}$$

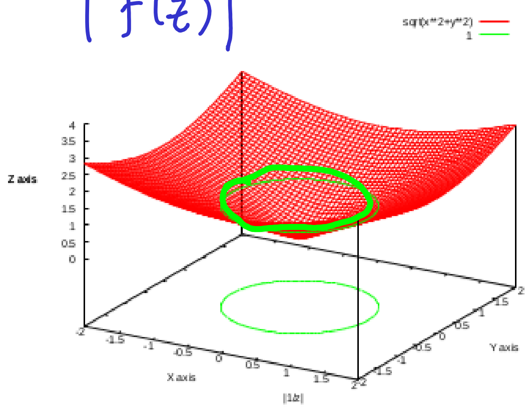
$$\frac{A}{(z-z_0)^2} + \frac{B}{(z-z_1)}$$

$$\frac{\cdots}{\cdots (z-z_0)^m \cdots}$$

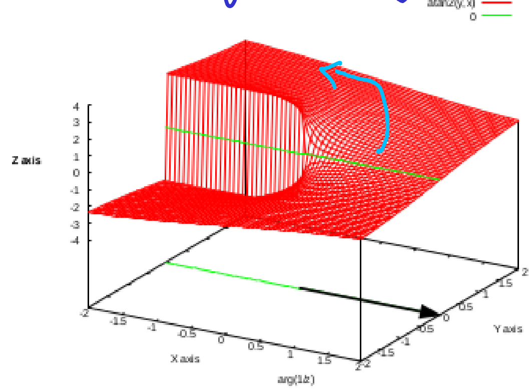
$$= \cdots \frac{A}{(z-z_0)^m} + \frac{B}{(z-z_0)^{m-1}} + \frac{\cdots}{(z-z_0)} \cdots ?$$

$f(z) = z$

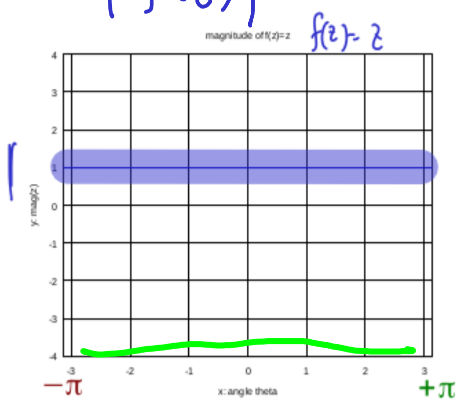
$|f(z)|$



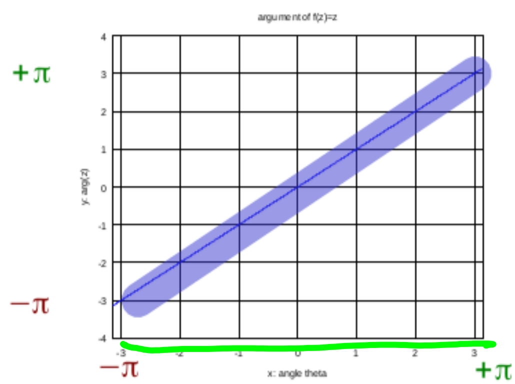
$\arg\{f(z)\}$



$|f(z)|$



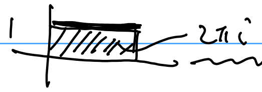
$\arg\{f(z)\}$



$$\oint z \, dz$$

$$f(z) = \frac{1}{z}$$

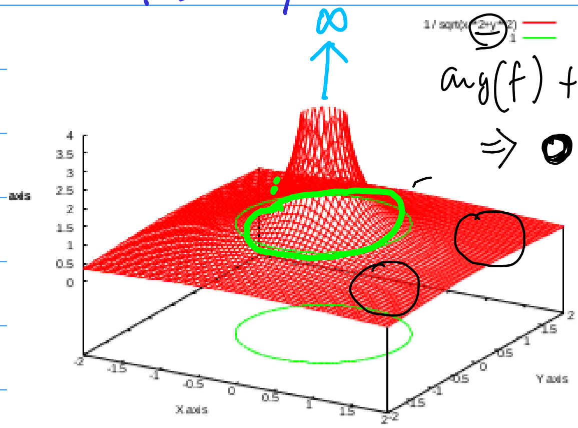
$|f(z)|$



$2\pi i$

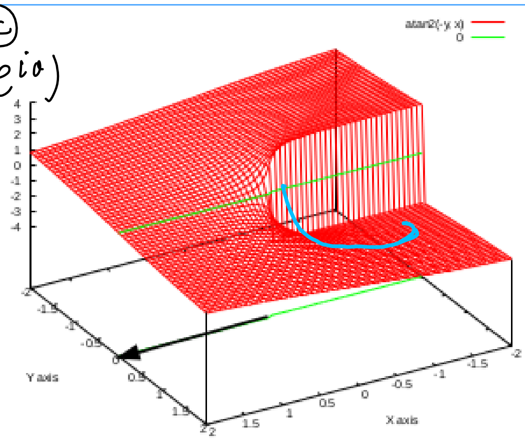
$\arg\{f(z)\}$

∞

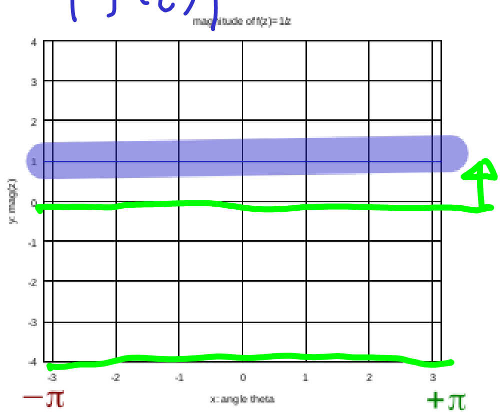


$\arg(f) + \arg(e^{i\theta})$

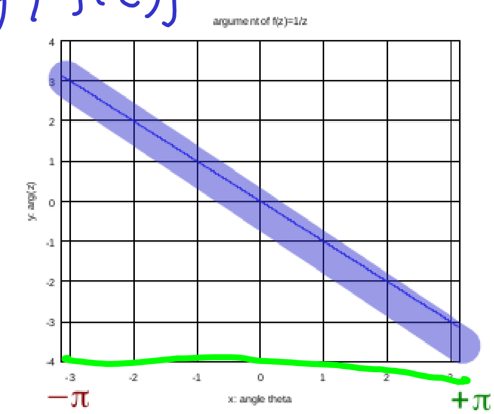
$\Rightarrow 0$



$|f(z)|$



$\arg\{f(z)\}$

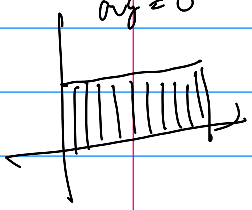


$$\oint \frac{1}{z} \, dz$$

$r = 1$

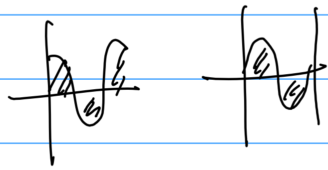
$$\frac{dz}{d\theta} = i r e^{i\theta} = i e^{i\theta}$$

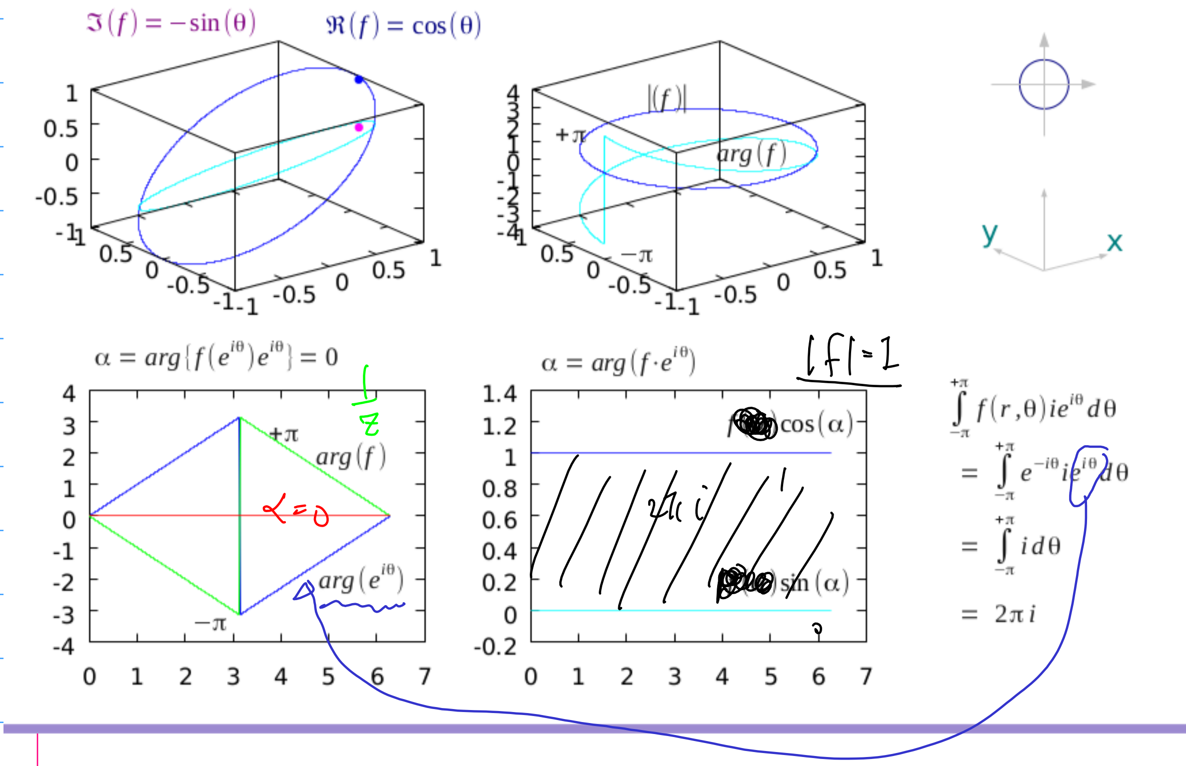
arg = 0



$$\arg(f) + \arg(e^{i\theta}) = 0$$

$\neq 0$





$\Im(f) = -\sin(\theta)$
$\Re(f) = \cos(\theta)$
$|(f)|$
$+\pi$
$arg(f)$
$-\pi$
y
x
$\alpha = arg\{f(e^{i\theta})e^{i\theta}\} = 0$
$+\pi$
$arg(f)$
$\alpha = 0$
$arg(e^{i\theta})$
$-\pi$
$\alpha = arg(f \cdot e^{i\theta})$
$|f| = 1$
$\cos(\alpha)$
$2\pi i$
$\sin(\alpha)$
$\int_{-\pi}^{+\pi} f(r,\theta) i e^{i\theta} d\theta$
$= \int_{-\pi}^{+\pi} e^{-i\theta} i e^{i\theta} d\theta$
$= \int_{-\pi}^{+\pi} i \, d\theta$
$= 2\pi i$