Fourier Series (2A)

- Fourier Series
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Fourier Series

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos kx\, dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin kx\, dx$$

$k = 1,\ 2,\ 3,\ \ldots$ → **one-sided spectrum**

only positive frequencies

Trigonometric Identities

$$\cos\theta\cos\varphi = \frac{1}{2}\left(\cos(\theta-\varphi) + \cos(\theta+\varphi)\right)$$

$$\sin\theta\sin\varphi = \frac{1}{2}\left(\cos(\theta-\varphi) - \cos(\theta+\varphi)\right)$$

$$\sin\theta\cos\varphi = \frac{1}{2}\left(\sin(\theta+\varphi) + \sin(\theta-\varphi)\right)$$

$$\cos\theta\sin\varphi = \frac{1}{2}\left(\sin(\theta+\varphi) - \sin(\theta-\varphi)\right)$$

$$\frac{1}{2}\left(1 + \cos(\theta+\varphi)\right) \quad \textit{when} \quad \theta=\varphi$$

$$\frac{1}{2}\left(1 - \cos(\theta+\varphi)\right) \quad \textit{when} \quad \theta=\varphi$$

$$\frac{1}{2}\left(\sin(\theta+\varphi)\right) \quad \textit{when} \quad \theta=\varphi$$

$$\frac{1}{2}\left(\sin(\theta+\varphi)\right) \quad \textit{when} \quad \theta=\varphi$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n}x \cos \boldsymbol{m}x \, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n}x \sin \boldsymbol{m}x \, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n}x \cos \boldsymbol{m}x \, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n}x \sin \boldsymbol{m}x \, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n}x \cos \boldsymbol{m}x \, dx = \pi \quad (n = m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n}x \sin \boldsymbol{m}x \, dx = \pi \quad (n = m)$$

$\boldsymbol{n}, \boldsymbol{m}$: integer

Trigonometric Orthogonality

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos kx\, dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin kx\, dx$$

$$k = 1, 2, 3, \ldots$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \cos \boldsymbol{m} x\, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \sin \boldsymbol{m} x\, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \cos \boldsymbol{m} x\, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \sin \boldsymbol{m} x\, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \cos \boldsymbol{m} x\, dx = \pi \quad (n = m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \sin \boldsymbol{m} x\, dx = \pi \quad (n = m)$$

$\boldsymbol{n}, \boldsymbol{m}$: integer

$$a_{\boldsymbol{k}} \leftarrow f(x) \cdot \cos \boldsymbol{k} x = a_0 \cdot \cos \boldsymbol{k} x + \sum_{m=1}^{\infty} (a_m \cos \boldsymbol{m} x \cdot \cos \boldsymbol{k} x + b_m \sin \boldsymbol{m} x \cdot \cos \boldsymbol{k} x)$$

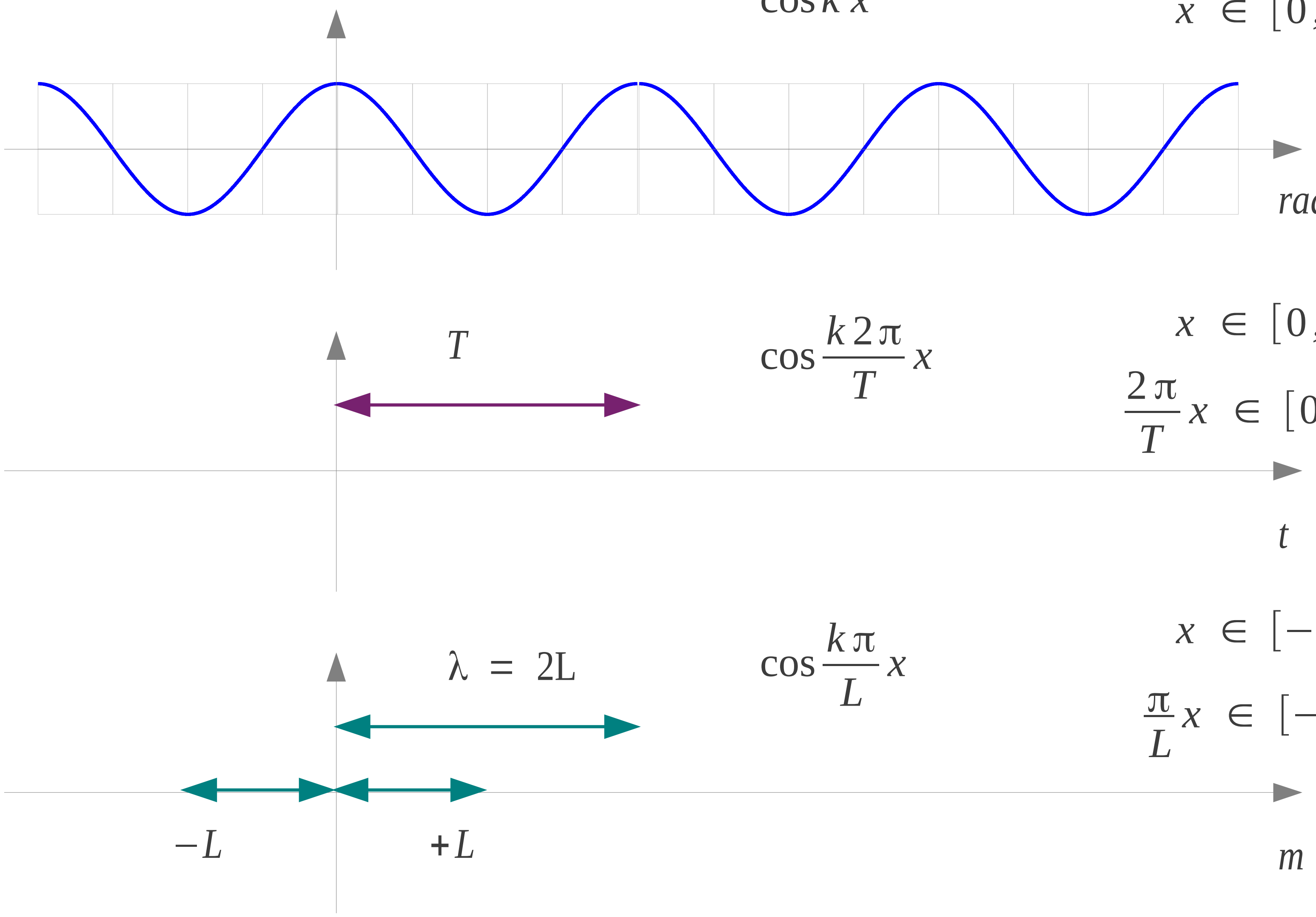
$$b_{\boldsymbol{k}} \leftarrow f(x) \cdot \sin \boldsymbol{k} x = a_0 \cdot \sin \boldsymbol{k} x + \sum_{m=1}^{\infty} (a_m \cos \boldsymbol{m} x \cdot \sin \boldsymbol{n} x + b_m \sin \boldsymbol{m} x \cdot \sin \boldsymbol{k} x)$$

Period and Wavelength

$$\cos k x \qquad x \in [0, +2\pi]$$

$$\cos \frac{k 2\pi}{T} x \qquad x \in [0, +T] \qquad \frac{2\pi}{T} x \in [0, 2\pi]$$

$$\cos \frac{k\pi}{L} x \qquad x \in [-L, +L] \qquad \frac{\pi}{L} x \in [-\pi, +\pi]$$



Any Period *2L*

$$g(v) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kv + b_k \sin kv)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} g(v)\, dv$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(v) \cos kv\, dv$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(v) \sin kv\, dv$$

$$k = 1, 2, \ldots$$

$$v : [-\pi, +\pi]$$

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi}{L} x + b_k \sin \frac{k\pi}{L} x \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x)\, dx$$

$$a_k = \frac{1}{L} \int_{-L}^{+L} f(x) \cos \frac{k\pi x}{L}\, dx$$

$$b_k = \frac{1}{L} \int_{-L}^{+L} f(x) \sin \frac{k\pi x}{L}\, dx$$

$$k = 1, 2, 3, \ldots$$

$$x : [-L, +L]$$

$$v = \frac{\pi}{L} x$$

$$dv = \frac{\pi}{L} dx$$

Any Period 2*p*

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi}{L} x + b_k \sin \frac{k\pi}{L} x \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x)\, dx$$

$$a_k = \frac{1}{L} \int_{-L}^{+L} f(x) \cos \frac{k\pi x}{L}\, dx$$

$$b_k = \frac{1}{L} \int_{-L}^{+L} f(x) \sin \frac{k\pi x}{L}\, dx$$

$$k = 1,\ 2,\ 3,\ \ldots$$

$$x : [-L,\ +L]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

$$a_0 = \frac{1}{p} \int_{-p}^{+p} f(x)\, dx$$

$$a_n = \frac{1}{p} \int_{-p}^{+p} f(x) \cos \frac{n\pi}{p} x\, dx$$

$$b_n = \frac{1}{p} \int_{-p}^{+p} f(x) \sin \frac{n\pi}{p} x\, dx$$

$$n = 1,\ 2,\ 3,\ \ldots$$

$$x : [-p,\ +p]$$

$$k \rightarrow n$$

$$a_0 \rightarrow \frac{a_0}{2}$$

Time and Frequency

$$f(x) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos\frac{k\pi}{L}x + b_k \sin\frac{k\pi}{L}x\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{+L} f(x)\, dx$$

$$a_k = \frac{1}{L}\int_{-L}^{+L} f(x) \cos\frac{k\pi x}{L}\, dx$$

$$b_k = \frac{1}{L}\int_{-L}^{+L} f(x) \sin\frac{k\pi x}{L}\, dx$$

$$k = 1,\ 2,\ 3,\ \ldots$$

$$x:\ [-L,\ +L]$$

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos\frac{2\pi k}{T}t + b_k \sin\frac{2\pi k}{T}t\right)$$

$$a_0 = \frac{1}{T}\int_{0}^{T} x(t)\, dt$$

$$a_k = \frac{2}{T}\int_{0}^{T} x(t) \cos\frac{2\pi k t}{T}\, dt$$

$$b_k = \frac{2}{T}\int_{0}^{T} x(t) \sin\frac{2\pi k t}{T}\, dt$$

$$k = 1,\ 2,\ \ldots$$

$$t:\ [0,\ T]$$

$$2L = T$$

Continuous Time Periodic Signal $x(t)$

Harmonic Frquency

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{2\pi k}{T} t + b_k \sin \frac{2\pi k}{T} t \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos \frac{2\pi k t}{T}\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin \frac{2\pi k t}{T}\, dt$$

$$k = 1, 2, \ldots$$

$$t : [0, T]$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(2\pi \boldsymbol{k} f_0 t) + b_k \sin(2\pi \boldsymbol{k} f_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(2\pi \boldsymbol{k} f_0 t)\, dt \qquad k = 1, 2, \ldots$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(2\pi \boldsymbol{k} f_0 t)\, dt \qquad k = 1, 2, \ldots$$

$$t : [0, T]$$

resolution frequency $\quad f_0 = \frac{1}{T}$

n-th harmonic frequency $\quad f_n = \boldsymbol{n} f_0 = \boldsymbol{n} \frac{1}{T}$

Radial Frequency

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_k \cos(k 2\pi f_0 t) + b_k \sin(k 2\pi f_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k 2\pi f_0 t)\, dt \qquad k = 1, 2, \ldots$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k 2\pi f_0 t)\, dt \qquad k = 1, 2, \ldots$$

$$t : [0, T]$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_k \cos(\boldsymbol{k} \omega_0 t) + b_k \sin(\boldsymbol{k} \omega_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(\boldsymbol{k} \omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(\boldsymbol{k} \omega_0 t)\, dt$$

$$k = 1, 2, \ldots$$

$$t : [0, T]$$

linear frequency f

angular (radial) frequency $\omega = 2\pi f$

Complex Fourier Series Coefficients

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T}\int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T}\int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1,\ 2,\ \ldots$$

$$t : [0,\ T]$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

$$A_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

$$A_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$B_k = \frac{1}{T}\int_0^T x(t)\, e^{+jk\omega_0 t}\, dt$$

$$t : [0,\ T]$$

Real coefficients

$$a_0,\ a_k,\ b_k,\quad k = 1,\ 2,\ \ldots$$

→ **one-sided spectrum**

only positive frequencies

Complex coefficients

$$A_0,\ A_k,\ B_k,\quad k = 1,\ 2,\ \ldots$$

→ **two-sided spectrum**

Both pos and neg frequencies

Euler Equation (1)

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1,\ 2,\ \ldots$$

$$e^{+j\omega t} = \cos\omega t + j\sin\omega t$$

$$e^{-j\omega t} = \cos\omega t - j\sin\omega t$$

$$\cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$$= a_k \frac{1}{2}\left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right) + b_k \frac{1}{2j}\left(e^{jk\omega_0 t} - e^{-jk\omega_0 t}\right)$$

$$= \frac{(a_k - jb_k)}{2} e^{jk\omega_0 t} + \frac{(a_k + jb_k)}{2} e^{-jk\omega_0 t}$$

$$= A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t}$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

zero freq → $A_0 = a_0$

pos freq → $A_k = \frac{1}{2}(a_k - jb_k)$

neg freq → $B_k = \frac{1}{2}(a_k + jb_k)$

only positive frequencies

Euler Equation (2)

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)\,dt$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)\,dt$$

$$k = 1, 2, \ldots$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

zero freq → $A_0 = a_0$

pos freq → $A_k = \frac{1}{2}(a_k - j b_k)$

neg freq → $B_k = \frac{1}{2}(a_k + j b_k)$

only positive frequencies

$$A_k = \frac{1}{T}\int_0^T x(t)\left(\cos(k\omega_0 t) - j\sin(k\omega_0 t)\right)dt$$

$$B_k = \frac{1}{T}\int_0^T x(t)\left(\cos(k\omega_0 t) + j\sin(k\omega_0 t)\right)dt$$

$$A_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\,dt$$

$$B_k = \frac{1}{T}\int_0^T x(t)\, e^{+jk\omega_0 t}\,dt$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

$$x(t) = \sum_{k=0}^{\infty} \left(A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

Complex Fourier Series (1)

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1, 2, \ldots$$

$$x(t) = \sum_{k=0}^{\infty} \left(A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

$$A_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = 0, 1, 2, \ldots$$

$$B_k = \frac{1}{T} \int_0^T x(t)\, e^{+jk\omega_0 t}\, dt$$

$$k = 1, 2, \ldots$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

$$A_0 = a_0$$

$$A_k = \frac{1}{2}(a_k - jb_k)$$

$$B_k = \frac{1}{2}(a_k + jb_k)$$

$$k = 1, 2, \ldots$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = -2, -1, 0, +1, +2, \ldots$$

$$C_k = \begin{cases} A_0 & (k = 0) \\ A_k & (k > 0) \\ B_{|k|} & (k < 0) \end{cases}$$

Complex Fourier Series (2)

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t) \, e^{-jk\omega_0 t} \, dt$$

$$k = -2, \ -1, \ 0, \ +1, \ +2, \ \ldots$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\frac{2\pi}{T}t}$$

$$C_k = \frac{1}{T} \int_0^T x(t) \, e^{-jk\frac{2\pi}{T}t} \, dt$$

$$k = -2, \ -1, \ 0, \ +1, \ +2, \ \ldots$$

$$C_2 \rightarrow \quad A_2 = \frac{1}{T} \int_0^T x(t) \, e^{-j\cdot(+2)\cdot\omega_0 t} \, dt$$

$$C_1 \rightarrow \quad A_1 = \frac{1}{T} \int_0^T x(t) \, e^{-j\cdot(+1)\cdot\omega_0 t} \, dt$$

$$C_0 \rightarrow \quad A_0 = \frac{1}{T} \int_0^T x(t) \, e^{-j\cdot(0)\cdot\omega_0 t} \, dt$$

$$C_{-1} \rightarrow \quad B_1 = \frac{1}{T} \int_0^T x(t) \, e^{-j\cdot(-1)\cdot k\,\omega_0 t} \, dt$$

$$C_{-2} \rightarrow \quad B_2 = \frac{1}{T} \int_0^T x(t) \, e^{-j\cdot(-2)\cdot k\,\omega_0 t} \, dt$$

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n \, e^{+jn\pi x/p}$$

$$c_n = \frac{1}{2p} \int_{-p}^{+p} f(x) \, e^{-jn\pi x/p} \, dx$$

$$n = -2, \ -1, \ 0, \ +1, \ +2, \ \ldots$$

Square Wave CTFS (1)

Continuous Time Fourier Series

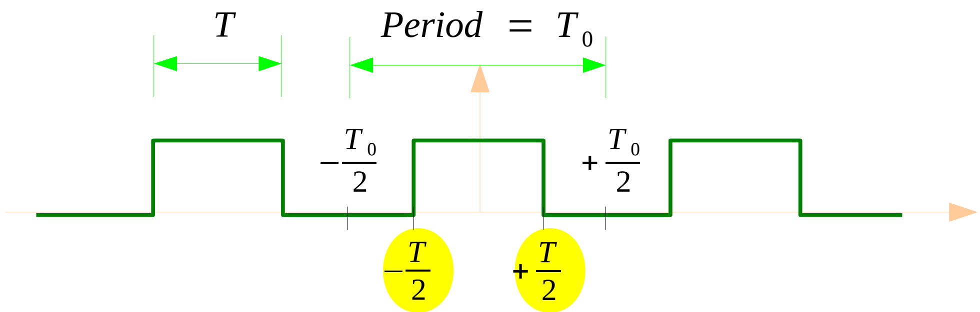
$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Leftrightarrow \quad x(t) = \sum_{n=-\infty}^{\infty} C_k\, e^{+jk\omega_0 t}$$

$$C_k T_0 = \int_{-T_0/2}^{+T_0/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \int_{-T/2}^{+T/2} e^{-jk\omega_0 t}\, dt = \left[\frac{-1}{jk\omega_0} e^{-jk\omega_0 t}\right]_{-T/2}^{+T/2}$$

$$= -\frac{e^{-jk\omega_0 T/2} - e^{+jk\omega_0 T/2}}{jk\omega_0} = \frac{e^{+jk\omega_0 T/2} - e^{-jk\omega_0 T/2}}{jk\omega_0}$$

$$= \frac{2j\sin(k\omega_0 T/2)}{jk\omega_0} = \frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$



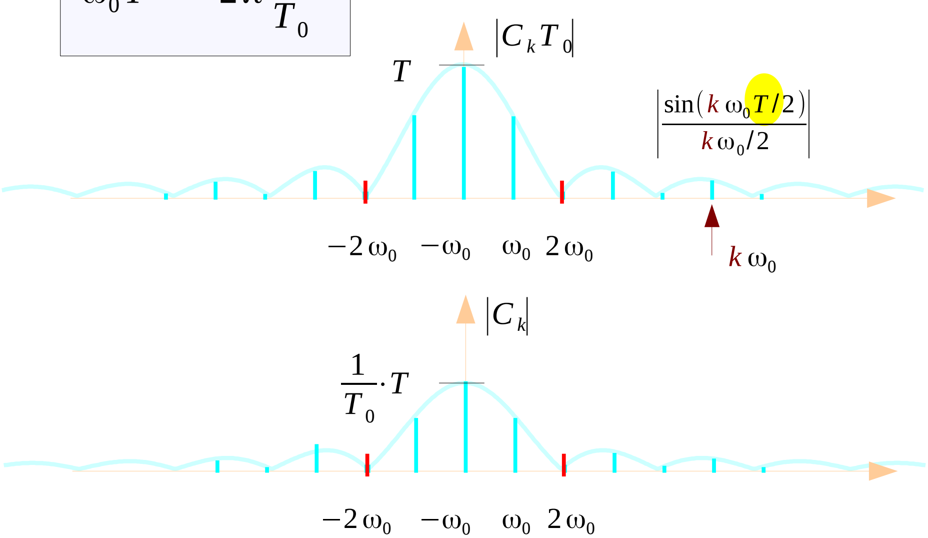
Fundamental Frequency

$$\omega_0 = \frac{2\pi}{T_0}$$

$$\omega_0 T = 2\pi \frac{T}{T_0}$$

$$\omega_0 T_0 = 2\pi$$
$$\omega_0 T = \pi$$

$$\Leftarrow \quad \frac{T_0}{T} = \frac{2}{1}$$



Square Wave CTFS (2)

$$C_k = \frac{1}{T_0}\cdot\frac{\sin(k\omega_0 T/2)}{k\omega_0/2} = \frac{1}{T_0}\cdot\frac{\sin(T k\omega_0/2)}{k\omega_0/2}$$

$$C_k = 0 \quad \Rightarrow \quad \sin(k\omega_0 T/2) = 0$$

$$\sin\left(k\frac{2\pi}{T_0}\frac{T}{2}\right) = 0 \quad \Rightarrow \quad \sin(\pm n\pi) = 0$$

$$k = \pm n\frac{T_0}{T} \quad \Rightarrow \quad \omega = k\omega_0 = \pm n\frac{T_0}{T}\omega_0$$

$$C_0 = \lim_{k\to 0}\frac{1}{T_0}\cdot\frac{\sin(T k\omega_0/2)}{k\omega_0/2}$$

$$C_0 = \lim_{k\to 0}\frac{1}{T_0}\cdot\frac{(T\omega_0/2)\cos(T k\omega_0/2)}{\omega_0/2} = \frac{T}{T_0}$$

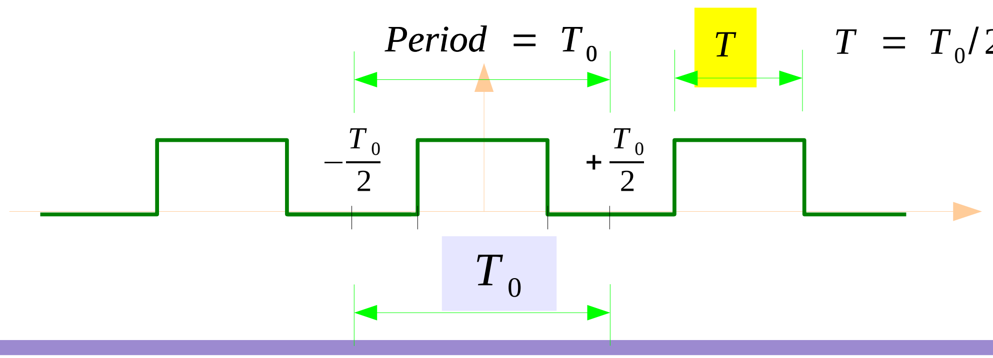
$$\omega_0 = \frac{2\pi}{T_0}$$

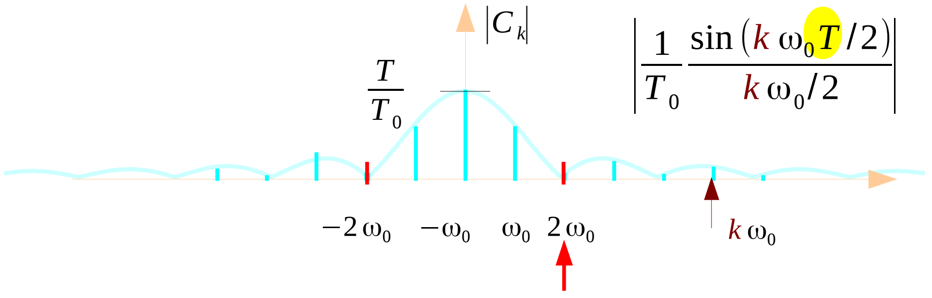
$$\omega_0\frac{T_0}{T} = \frac{2\pi}{T}$$

$$\omega_0 T = 2\pi\frac{T}{T_0}$$

$$C_k = \frac{1}{T_0}\cdot\frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

$$C_0 = \frac{T}{T_0}$$





Square Wave CTFS (3)

$$C_k = \frac{1}{T_0}\cdot\frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

$$C_0 = \frac{T}{T_0}$$

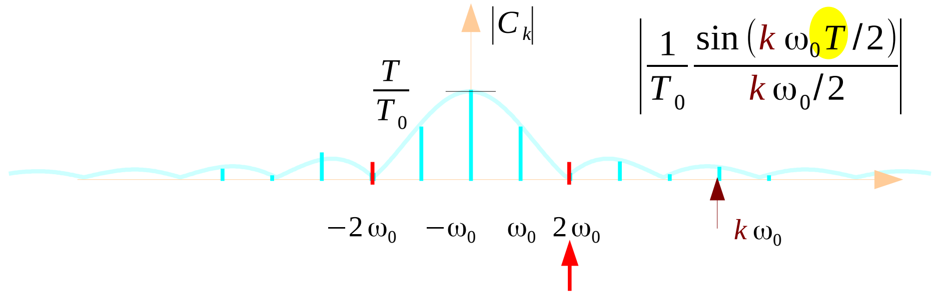
$$\omega_0 = \frac{2\pi}{T_0}$$

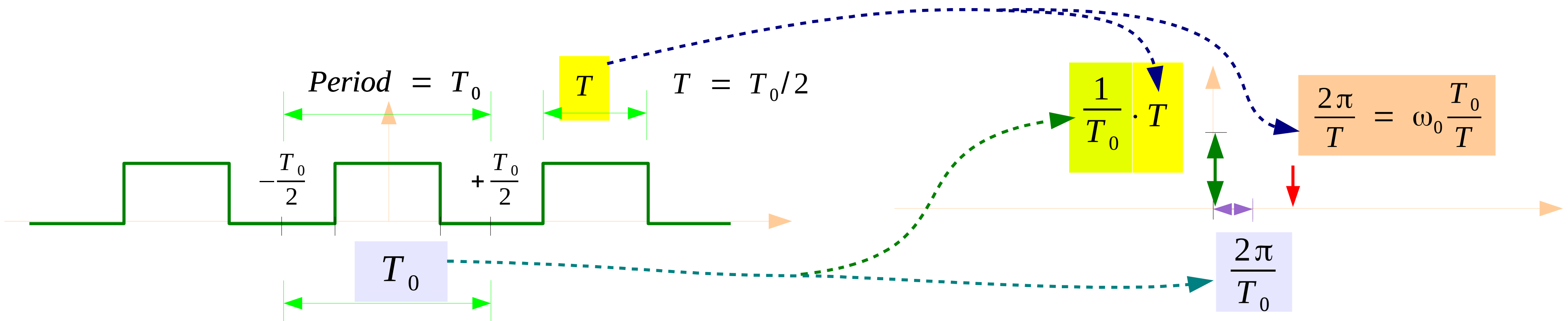
$$\omega_0\frac{T_0}{T} = \frac{2\pi}{T}$$

$$\omega_0 T = 2\pi\frac{T}{T_0}$$

$$\omega = k\omega_0 = \pm n\frac{T_0}{T}\omega_0$$

$$C_k = 0 \quad \left(k = \pm n\frac{T_0}{T}\right)$$





Continuous Time Fourier Series

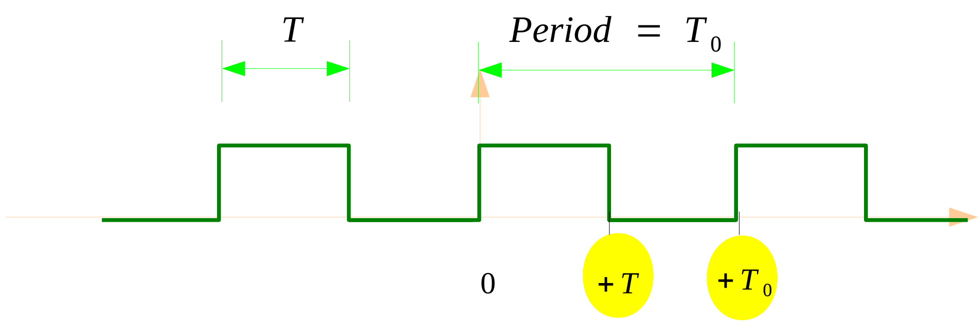
$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Leftrightarrow \quad x(t) = \sum_{k=-\infty}^{\infty} C_k\, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T_0}\int_0^{+T_0} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

$$C_k T_0 = \int_0^{+T_0} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \int_0^{+T} e^{-jk\omega_0 t}\, dt = \left[\frac{-1}{jk\omega_0} e^{-jk\omega_0 t}\right]_0^{+T}$$

$$= -\frac{e^{-jk\omega_0 T} - e^0}{jk\omega_0} = \frac{1 - e^{-jk\omega_0 T}}{jk\omega_0} = \frac{1 - e^{-jk2\pi T/T_0}}{jk2\pi/T_0}$$



Fundamental Frequency

$$\omega_0 = \frac{2\pi}{T_0}$$

$$\omega_0 T_0 = 2\pi \qquad \omega_0 T = \pi \quad \Leftarrow \quad \frac{T_0}{T} = \frac{2}{1}$$

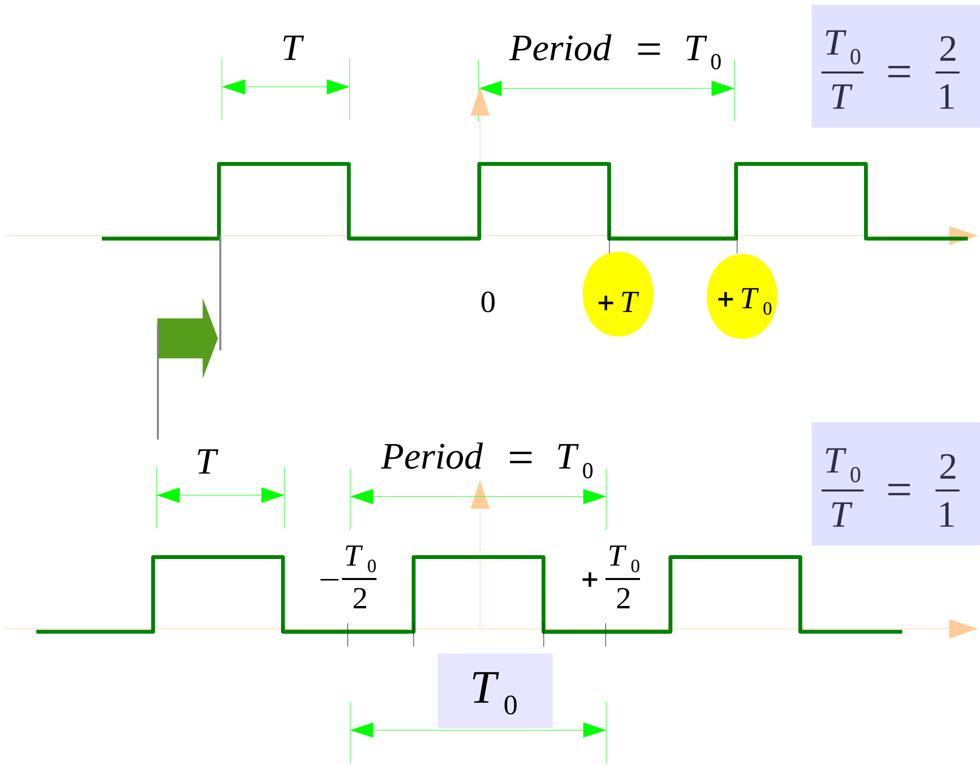
$$\omega_0 T = 2\pi \frac{T}{T_0}$$

$$C_k T_0 = \frac{1 - e^{-jk2\pi T/T_0}}{jk2\pi/T_0} \quad \Rightarrow \quad C_k = \frac{1 - (-1)^k}{j2\pi k}$$

$$C_0 T_0 = \int_0^{+T} e^{-j0\omega_0 t}\, dt = T \quad \Rightarrow \quad C_0 = \frac{1}{2}$$

C_{-4}	C_{-3}	C_{-2}	C_{-1}	C_0	C_1	C_2	C_3	C_4
0	$\frac{-1}{j3\pi}$	0	$\frac{-1}{j\pi}$	$\frac{1}{2}$	$\frac{1}{j\pi}$	0	$\frac{1}{j3\pi}$	0

Square Wave CTFS (5)



$$C_k = \frac{1 - e^{-jk2\pi T/T_0}}{jk2\pi}$$

$$C_0 = \frac{T}{T_0}$$

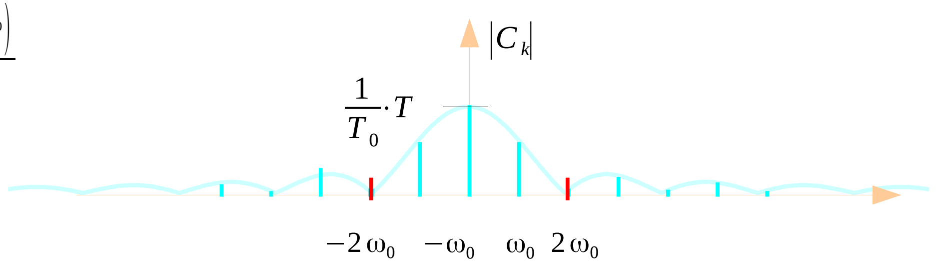
$e^{+jk\omega_0 T/2}$

$$C_k = \frac{1}{T_0} \cdot \frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

$$C_0 = \frac{T}{T_0}$$

$$\frac{1}{T_0} \cdot \frac{\sin(k\omega_0 T/2)}{k\omega_0/2} = \frac{\sin(k\pi T/T_0)}{k\pi} = \frac{\left(e^{+jk\pi T/T_0} - e^{-jk\pi T/T_0}\right)}{jk2\pi}$$

$$= e^{+jk\pi T/T_0}\left(\frac{1 - e^{-jk2\pi T/T_0}}{jk2\pi}\right) = e^{+jk\omega_0 T/2}\left(\frac{1 - e^{-jk2\pi T/T_0}}{jk2\pi}\right)$$



Complex Fourier Series (2)

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\frac{2\pi}{T}t}$$

$$C_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\frac{2\pi}{T}t}\, dt$$

$$k = -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

$$C_2 \rightarrow \quad A_2 = \frac{1}{T}\int_0^T x(t)\, e^{-j\cdot(+2)\cdot\omega_0 t}\, dt$$

$$C_1 \rightarrow \quad A_1 = \frac{1}{T}\int_0^T x(t)\, e^{-j\cdot(+1)\cdot\omega_0 t}\, dt$$

$$C_0 \rightarrow \quad A_0 = \frac{1}{T}\int_0^T x(t)\, e^{-j\cdot(0)\cdot\omega_0 t}\, dt$$

$$C_{-1} \rightarrow \quad B_1 = \frac{1}{T}\int_0^T x(t)\, e^{-j\cdot(-1)\cdot k\,\omega_0 t}\, dt$$

$$C_{-2} \rightarrow \quad B_2 = \frac{1}{T}\int_0^T x(t)\, e^{-j\cdot(-2)\cdot k\,\omega_0 t}\, dt$$

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n \, e^{+jn\pi x/p}$$

$$c_n = \frac{1}{2p}\int_{-p}^{+p} f(x)\, e^{-jn\pi x/p}\, dx$$

$$n = -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

Cosine and Sine Series (1)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi}{p} x + b_n \sin\frac{n\pi}{p} x \right)$$

$$a_0 = \frac{1}{p} \int_{-p}^{+p} f(x)\, dx$$

$$a_n = \frac{1}{p} \int_{-p}^{+p} f(x) \cos\frac{n\pi}{p} x\, dx$$

$$b_n = \frac{1}{p} \int_{-p}^{+p} f(x) \sin\frac{n\pi}{p} x\, dx$$

$$n = 1,\ 2,\ 3,\ \ldots$$

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{+j n\pi x/p}$$

$$c_n = \frac{1}{2p} \int_{-p}^{+p} f(x)\, e^{-j n\pi x/p}\, dx$$

$$n = -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p} x$$

$$a_0 = \frac{2}{p} \int_{0}^{p} f(x)\, dx$$

$$a_n = \frac{2}{p} \int_{0}^{p} f(x) \cos\frac{n\pi}{L} x\, dx$$

$$n = 1,\ 2,\ 3,\ \ldots$$

Sine Series

$$f(x) = b_n \sin\frac{n\pi}{p} x$$

$$\frac{2}{p} \int_{0}^{p} f(x) \sin\frac{n\pi}{p} x\, dx$$

$$n = 1,\ 2,\ 3,\ \ldots$$

Cosine and Sine Series (2)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p} x$$

$$a_0 = \frac{2}{p}\int_0^p f(x)\, dx$$

$$a_n = \frac{2}{p}\int_0^p f(x) \cos\frac{n\pi}{L} x\, dx$$

$$n = 1,\ 2,\ 3,\ \ldots$$

$$f(x) = b_n \sin\frac{n\pi}{p} x$$

$$\frac{2}{p}\int_0^p f(x) \sin\frac{n\pi}{p} x\, dx$$

$$n = 1,\ 2,\ 3,\ \ldots$$

even function f(x) on –p, +p]

even even

$$a_n = \frac{1}{p}\int_{-p}^{+p} f(x) \cos\frac{n\pi}{p} x\, dx$$

$$= \frac{2}{p}\int_{-p}^{0} f(x) \cos\frac{n\pi}{p} x\, dx$$

even

odd function f(x) on –p, +p]

odd odd

$$b_n = \frac{1}{p}\int_{-p}^{+p} f(x) \sin\frac{n\pi}{p} x\, dx$$

$$= \frac{2}{p}\int_{0}^{+p} f(x) \sin\frac{n\pi}{p} x\, dx$$

even

Conditions for Convergence

$f(x)$ and $f'(x)$ are piecewise continuous on the interval $(-p, +p)$

continuous except at a finite number of points in the interval

have only finite discontinuities at these points

The Fourier series of $f(x)$ on the interval $(-p, +p)$ converges

$$\begin{cases} f(x) & \text{at a point of continuity} \\ \dfrac{f(x+)+f(x-)}{2} & \text{at a point of discontinuity} \end{cases}$$

Orthogonal Functions

Inner Product of Functions

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x)\, dx$$

Orthogonal Functions

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x)\, dx = 0$$

Orthogonal Sets

$$\{\Phi_0(x),\ \Phi_1(x),\ \Phi_2(x),\ \cdots\}$$ Orthogonal

$$(\Phi_m, \Phi_n) = \int_a^b \Phi_m(x) \Phi_n(x)\, dx = 0, \quad m \neq n$$

Orthogonal Series Expansion

$$f(x) = c_0\Phi_0(x) + c_0\Phi_0(x) + \cdots + c_n\Phi_n(x) + \cdots$$

$$\int_a^b f(x)\Phi_m(x)\,dx = c_0\int_a^b \Phi_0(x)\Phi_m(x)\,dx + c_1\int_a^b \Phi_1(x)\Phi_m(x)\,dx + \cdots + c_n\int_a^b \Phi_n(x)\Phi_m(x)\,dx + \cdots$$

$$= c_0(\Phi_0, \Phi_m) + c_1(\Phi_1, \Phi_m) + \cdots + c_n(\Phi_n, \Phi_m) + \cdots$$

$$\int_a^b f(x)\Phi_n(x)\,dx = c_n\int_a^b \Phi_n^2(x)\,dx$$

$$c_n = \frac{\int_a^b f(x)\Phi_n(x)\,dx}{\int_a^b \Phi_n^2(x)\,dx} \qquad n = 0,\ 1,\ 2,\ \cdots$$

$$f(x) = \sum_{n=0}^{\infty} c_n\Phi_n(x)$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(f, \Phi_n)}{\|\Phi(x)\|^2}\Phi_n(x)$$

$$c_n = \frac{\int_a^b f(x)\Phi_n(x)\,dx}{\|\Phi_n(x)\|^2}$$

Generalized Fourier Series Expansion

Orthogonal with respect to a weight Functions

$$\int_a^b w(x)\Phi_m(x)\Phi_n(x)\,dx = 0 \quad m \neq n$$

$$c_n = \frac{\int_a^b w(x) f(x)\Phi_n(x)\,dx}{\|\Phi_n(x)\|^2}$$

$$\|\Phi_n(x)\|^2 = \int_a^b w(x)\Phi_N^2(x)\,dx$$

Inner Product Space

Hilbert Space real / complex inner product space

$$\langle f , g \rangle = \int_a^b f(t)\overline{g(t)}\,dt$$

complex conjugate

$$\langle y , x \rangle = \overline{\langle x , y \rangle}$$

linear

$$\langle a x_1 + b x_{2,} \ y \rangle = a \langle x_{1,} \ y \rangle + b \langle x_{2,} \ y \rangle$$

positive semidefinite

$$\langle x , x \rangle \geq 0$$

Norm

$$\|x\| = \sqrt{\langle x , x \rangle}$$

Cauchy-Schwartz Inequality

$$|\langle x , y \rangle| \leq \|x\| \, \|y\|$$

Orthogonality

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$k = \ldots, -2, -1, 0, +1, +2, \ldots$

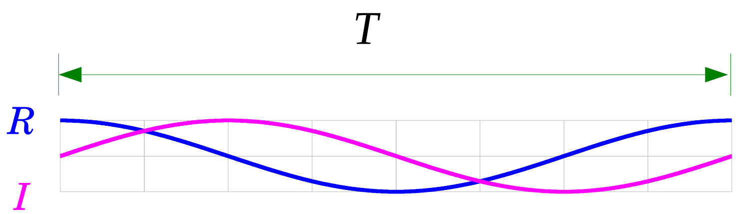
fundamental frequency $\quad f_0 = \dfrac{1}{T} \qquad \omega_0 = 2\pi f_0 = \dfrac{2\pi}{T}$

n-th harmonic frequency $\quad f_n = n\, f_0 \qquad \omega_n = 2\pi f_n = \dfrac{2\pi n}{T}$

$$\langle e^{jm\omega_0 t}, \; e^{jn\omega_0 t} \rangle = \int_0^T e^{+j(m-n)\omega_0 t}\, dt = \begin{cases} 0 & (m \neq n) \\ T & (m = n) \end{cases}$$

m, n : integer

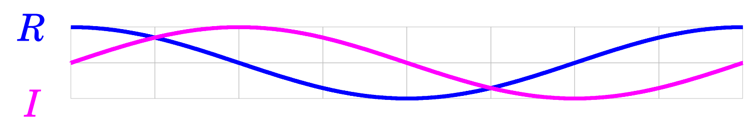
Inner Product Examples



$$f_0 = 1/T \qquad \omega_0 = 2\pi/T$$

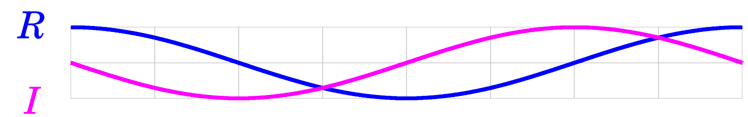
$e^{j1\omega_0 t}$

$$\langle e^{j1\omega_0 t}, e^{j1\omega_0 t}\rangle = \int_0^T e^{+j(1-1)\omega_0 t}\, dt = T$$



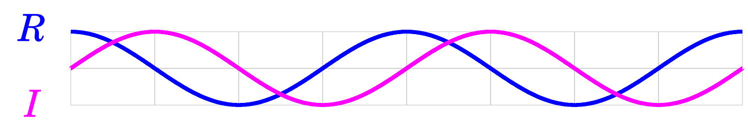
$e^{j1\omega_0 t}$

$$\langle e^{j1\omega_0 t}, e^{j-1\omega_0 t}\rangle = \int_0^T e^{+j(1+1)\omega_0 t}\, dt = 0$$



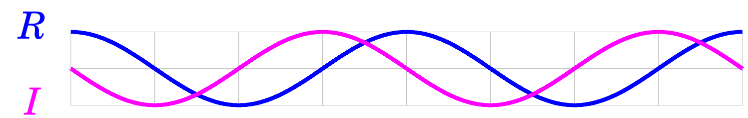
$e^{j-1\omega_0 t}$

$$\langle e^{j1\omega_0 t}, e^{j2\omega_0 t}\rangle = \int_0^T e^{+j(1-2)\omega_0 t}\, dt = 0$$



$e^{j2\omega_0 t}$

$$\langle e^{j1\omega_0 t}, e^{j-2\omega_0 t}\rangle = \int_0^T e^{+j(1+2)\omega_0 t}\, dt = 0$$



$e^{j-2\omega_0 t}$

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