

# Second Order ODE's (2A)

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*Homogeneous Linear Equations  
with constant coefficients*

# Types of First Order ODEs

## A General Form of First Order Differential Equations

$$\frac{d y}{d x} = g(x, y)$$

$$y' = g(x, y)$$

## Separable Equations

$$\frac{d y}{d x} = g_1(x)g_2(y)$$

$$y' = g_1(x)g_2(y)$$

$$y = f(x)$$

## Linear Equations

$$a_1(x)\frac{d y}{d x} + a_0(x)y = g(x)$$

$$a_1(x)y' + a_0(x)y = g(x)$$

$$y = f(x)$$

## Exact Equations

$$M(x, y)dx + N(x, y)dy = 0$$

$$\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = 0$$

$$z = f(x, y)$$

# Second Order ODEs

## **First Order Linear Equations**

$$a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

$$a_1(x)y' + a_0(x)y = g(x)$$

## **Second Order Linear Equations**

$$a_2(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

## **Second Order Linear Equations with Constant Coefficients**

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

$$a_2 y'' + a_1 y' + a_0 y = g(x)$$

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + c y = g(x)$$

$$a y'' + b y' + c y = g(x)$$

# Auxiliary Equation

## Homogeneous Second Order DEs with Constant Coefficients

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$a y'' + b y' + c y = 0$$

try a solution  $y = e^{mx}$

$$a \frac{d^2}{dx^2} \{e^{mx}\} + b \frac{d}{dx} \{e^{mx}\} + c \{e^{mx}\} = 0$$

$$a \{m^2 e^{mx}\} + b \{m e^{mx}\} + c \{e^{mx}\} = 0$$

$$(a m^2 + b m + c) \cdot e^{mx} = 0$$

$$a \{e^{mx}\}'' + b \{e^{mx}\}' + c \{e^{mx}\} = 0$$

$$a \{m^2 e^{mx}\} + b \{m e^{mx}\} + c \{e^{mx}\} = 0$$

$$(a m^2 + b m + c) \cdot e^{mx} = 0$$

auxiliary equation

$$(a m^2 + b m + c) = 0$$

$$(a m^2 + b m + c) = 0$$

# Roots of the Auxiliary Equation

## Homogeneous Second Order DEs with Constant Coefficients

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$a y'' + b y' + c y = 0$$

auxiliary equation

try a solution  $y = e^{mx}$



$$(am^2 + bm + c) = 0$$

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$$

$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$$

$$y_1 = e^{m_1 x}, \quad y_2 = e^{m_2 x}$$

$$y_1 = e^{m_1 x} = y_2 = e^{m_2 x}$$

$$y_1 = e^{m_1 x}, \quad y_2 = e^{m_2 x}$$



(A)  $b^2 - 4ac > 0$  Real, distinct  $m_1, m_2$



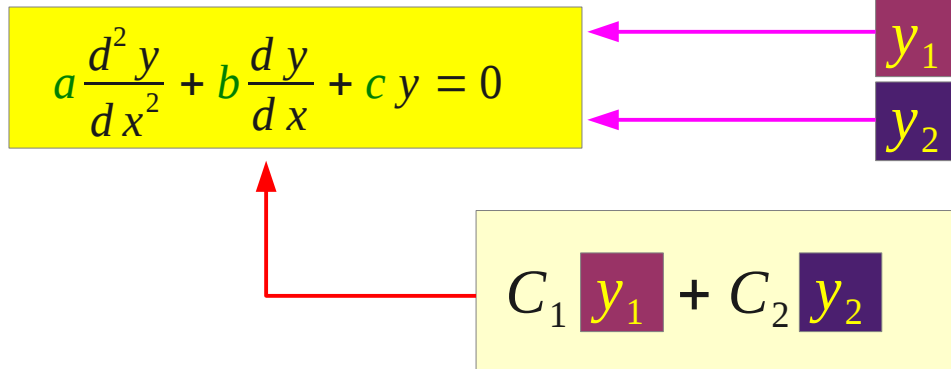
(B)  $b^2 - 4ac = 0$  Real, equal  $m_1, m_2$



(C)  $b^2 - 4ac < 0$  Conjugate complex  $m_1, m_2$

# Linear Combination of Solutions

DEQ



$$a y_1'' + b y_1' + c y_1 = 0$$

$$a y_2'' + b y_2' + c y_2 = 0$$



$$a(y_1'' + y_2'') + b(y_1' + y_2') + c(y_1 + y_2) = 0$$

$$a(y_1 + y_2)'' + b(y_1 + y_2)' + c(y_1 + y_2) = 0$$

$$y_3 = y_1 + y_2$$

$$y_4 = y_1 - y_2$$

$$y_5 = y_3 + 2y_4$$

$$y_6 = y_3 - 2y_4$$

$$a(C_1 y_1'' + C_2 y_2'') + b(C_1 y_1' + C_2 y_2') + c(C_1 y_1 + C_2 y_2) = 0$$

$$a(C_1 y_1 + C_2 y_2)'' + b(C_1 y_1 + C_2 y_2)' + c(C_1 y_1 + C_2 y_2) = 0$$



# Solutions of 2nd Order ODEs

DEQ

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$y_1$$

$$y_2$$

$$C_1 y_1 + C_2 y_2$$

$$\begin{cases} y_1 = e^{m_1 x} \\ y_2 = e^{m_2 x} \end{cases} \quad (D > 0)$$

$$\begin{cases} y_1 = e^{m_1 x} \\ y_2 = e^{m_2 x} \end{cases} \quad (D = 0)$$

$$\begin{cases} y_1 = e^{m_1 x} \\ y_2 = e^{m_2 x} \end{cases} \quad (D < 0)$$

$$\begin{cases} y = C_1 e^{m_1 x} + C_2 e^{m_2 x} & (D > 0) \\ y = C_1 e^{m_1 x} \quad ? & (D = 0) \\ y = C_1 e^{m_1 x} + C_2 e^{m_2 x} & (D < 0) \end{cases}$$

auxiliary equation

$$(a m^2 + b m + c) = 0$$

$$m_1 = (-b + \sqrt{b^2 - 4ac}) / 2a$$

$$m_2 = (-b - \sqrt{b^2 - 4ac}) / 2a$$

# (A) Real Distinct Roots Case

## Homogeneous Second Order DEs with Constant Coefficients

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$a y'' + b y' + c y = 0$$

try a solution  $y = e^{mx}$



$$(a m^2 + b m + c) = 0$$

auxiliary equation

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$$

$$y_1 = e^{m_1 x}$$

$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$$

$$y_2 = e^{m_2 x}$$

$$b^2 - 4ac > 0 \quad \text{Real, distinct } m_1, m_2$$

$$b^2 - 4ac = 0 \quad \text{Real, equal } m_1, m_2$$

$$b^2 - 4ac < 0 \quad \text{Conjugate complex } m_1, m_2$$

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$$

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

## (B) Repeated Real Roots Case

### Homogeneous Second Order DEs with Constant Coefficients

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$a y'' + b y' + c y = 0$$

try a solution  $y = e^{mx}$



$$(am^2 + bm + c) = 0$$

auxiliary equation

$$m_1 = (-b + \sqrt{b^2 - 4ac}) / 2a$$

$$m_2 = (-b - \sqrt{b^2 - 4ac}) / 2a$$



$$b^2 - 4ac = 0$$



$$m_1 = -b/2a$$

$$m_2 = -b/2a$$

$$e^{m_1 x} = e^{m_2 x} = e^{-\frac{b}{2a} x}$$

$$b^2 - 4ac > 0 \quad \text{Real, distinct } m_1, m_2$$

$$b^2 - 4ac = 0 \quad \text{Real, equal } m_1, m_2$$

$$b^2 - 4ac < 0 \quad \text{Conjugate complex } m_1, m_2$$

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$$

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

# (C) Complex Roots of the Auxiliary Equation

## Homogeneous Second Order DEs with Constant Coefficients

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$a y'' + b y' + c y = 0$$

try a solution  $y = e^{mx}$



$$(am^2 + bm + c) = 0$$

auxiliary equation

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$$



$$m_1 = (-b + \sqrt{4ac - b^2} i)/2a$$

$$y_1 = e^{m_1 x}$$

$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$$



$$m_2 = (-b - \sqrt{4ac - b^2} i)/2a$$

$$y_2 = e^{m_2 x}$$

$$b^2 - 4ac > 0 \quad \text{Real, distinct } m_1, m_2$$

$$b^2 - 4ac = 0 \quad \text{Real, equal } m_1, m_2$$

$$b^2 - 4ac < 0 \quad \text{Conjugate complex } m_1, m_2$$

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$$

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

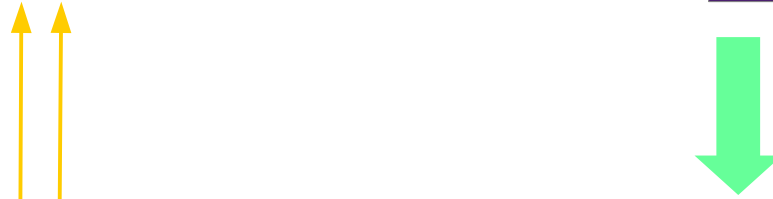
# Fundamental Set Examples (1)

Second Order EQ

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$y_1$$
$$y_2$$

$$e^{(\alpha+i\beta)x}$$
$$e^{(\alpha-i\beta)x}$$



$$y_3 = \frac{1}{2} y_1 + \frac{1}{2} y_2$$

$$y_4 = \frac{1}{2i} y_1 - \frac{1}{2i} y_2$$

$$\{e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x}\}/2 = e^{\alpha x} \cos(\beta x)$$

$$\{e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x}\}/2i = e^{\alpha x} \sin(\beta x)$$

# Fundamental Set Examples (2)

Second Order EQ

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$y_1$$

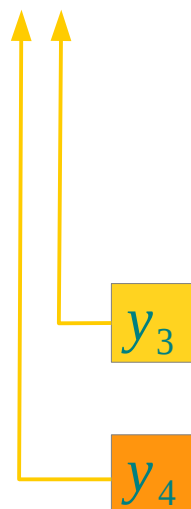
$$= y_3 + i y_4$$

$$y_2$$

$$= y_3 - i y_4$$

$$e^{(\alpha+i\beta)x}$$

$$e^{(\alpha-i\beta)x}$$



$$y_3 = e^{\alpha x} \cos(\beta x)$$

$$y_4 = e^{\alpha x} \sin(\beta x)$$

$$e^{\alpha x} [\cos(\beta x) + i \sin(\beta x)]$$

$$e^{\alpha x} [\cos(\beta x) - i \sin(\beta x)]$$

# General Solution Examples

## Second Order EQ

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

linearly independent

Fundamental Set of Solutions

$$\{y_1, y_2\} = \{e^{(\alpha+i\beta)x}, e^{(\alpha-i\beta)x}\}$$

$$C_1 y_1 + C_2 y_2$$

$$C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$$

General Solution

linearly independent

Fundamental Set of Solutions

$$\{y_3, y_4\} = \{e^{\alpha x} \cos(\beta x), e^{\alpha x} \sin(\beta x)\}$$

$$C_3 y_3 + C_4 y_4$$

$$C_3 e^{\alpha x} \cos(\beta x) + C_4 e^{\alpha x} \sin(\beta x) \\ = e^{\alpha x} (C_3 \cos(\beta x) + C_4 \sin(\beta x))$$

General Solution

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## *Reduction of Orders*



# Finding another solution $y_2$ from the known $y_1$

Second Order EQ

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$y_1 = f(x)$       *known solution*

$y_2 = u(x)f(x)$       *another solution to be found*

We know one solution

$$y_1(x) = e^{m_1 x} = e^{m_2 x} = e^{-\frac{b}{2a}x}$$

Suppose the other solution

$$y_2(x) = u(x)y_1(x) = u(x)e^{m_1 x}$$

Condition for  $y_2(t)$  to be a solution

→ Find  $u(x)$

# Conditions for $y_2$ to be another solution

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$a y'' + b y' + c y = 0$$

$$\begin{cases} y_2 = u y_1 \\ y_2' = u' y_1 + u y_1' \\ y_2'' = u'' y_1 + 2u' y_1' + u y_1'' \end{cases}$$

$$a y_2'' + b y_2' + c y_2 = 0 \quad \Rightarrow \quad a[u'' y_1 + 2u' y_1' + \underline{u y_1''}] + b[u' y_1 + \underline{u y_1'}] + \underline{c u y_1} = 0$$

$$a y_1'' + b y_1' + c y_1 = 0 \quad \Rightarrow \quad \underline{u[a y_1'' + b y_1' + c y_1]} + a[u'' y_1 + 2u' y_1'] + b[u' y_1] = 0$$

Condition for  $y_2(t)$  to be a solution

$$y_2(x) = u(x) y_1(x)$$

$$a u'' y_1 + u' [2a y_1' + b y_1] = 0$$

# Reduction of Order

We know one solution

$$y_1(x)$$

Suppose the other solution

$$y_2(x) = u(x)y_1(x)$$

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$a y'' + b y' + c y = 0$$

$$a y_2'' + b y_2' + c y_2 = 0$$

$$a u'' y_1 + u' [2a y_1' + b y_1] = 0$$

2<sup>nd</sup> Order

$$w(x) = u'(x)$$

$$a w' y_1 + w [2a y_1' + b y_1] = 0$$

1<sup>st</sup> Order

$$u = c_1 \int \frac{e^{-(b/a)x}}{y_1^2} dx + c_2$$

$$y_2 = c_1 y_1 \int \frac{e^{-(b/a)x}}{y_1^2} dx + c_2 y_1 \quad (c_1=1, c_2=0)$$

$$y_2 = y_1 \int \frac{e^{-(b/a)x}}{y_1^2} dx$$

# General Solutions for the repeated roots case

$$y_2 = y_1 \int \frac{e^{-(b/a)x}}{y_1^2} dx$$

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$$

$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$$



$$b^2 - 4ac = 0$$



$$m_1 = -b/2a$$

$$m_2 = -b/2a$$

$$e^{m_1 x} = e^{m_2 x} = e^{-\frac{b}{2a}x}$$

$$y_1(x) = e^{-\frac{b}{2a}x}$$

$$y_1^2 = e^{-\frac{b}{a}x}$$

$$y_2 = e^{-\frac{b}{2a}x} \int \frac{e^{-(b/a)x}}{e^{-(b/a)x}} dx = e^{-\frac{b}{2a}x} \int 1 dx \quad \rightarrow$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

$$y_1(x) = e^{-\frac{b}{2a}x}$$

$$y_2(x) = x e^{-\frac{b}{2a}x}$$

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

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## *General Solutions*

- Homogeneous Equation*
- Non-homogeneous Equation*

# General Solution – Homogeneous Equations

## Homogeneous Second Order DEs with Constant Coefficients

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$a y'' + b y' + c y = 0$$

auxiliary equation

$$(a m^2 + b m + c) = 0$$

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$$

$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$$

(A)  $b^2 - 4ac > 0$  Real, distinct  $m_1, m_2$

(B)  $b^2 - 4ac = 0$  Real, equal  $m_1, m_2$

(C)  $b^2 - 4ac < 0$  Conjugate complex  $m_1, m_2$

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$$

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$$

$$= e^{\alpha x} (C_3 \cos(\beta x) + C_4 \sin(\beta x))$$

# Complementary Function

DEQ

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = g(x)$$

$$y_p$$

particular solution

$$y_p$$

+

$$y_c$$

a complementary function

that makes a general (whole) solution

the general solution of a nonhomogeneous eq

Associated DEQ

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$y_1$$

$$y_2$$

$$C_1$$

$$y_1$$

+

$$C_2$$

$$y_2$$

homogeneous solution

the general solution of a homogeneous eq

# $y_c$ and $y_p$

DEQ

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = g(x)$$

$y_p$

particular solution

$y_c + y_p$

general solution –  
nonhomogeneous eq

$$a \frac{d^2 y_c}{dx^2} + b \frac{dy_c}{dx} + c y_c \Rightarrow 0$$

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

many such complementary functions

$c_i$  many possible coefficients

$$a \frac{d^2 y_p}{dx^2} + b \frac{dy_p}{dx} + c y_p \Rightarrow g(x)$$

$$y_p \Rightarrow g(x) \text{ similar}$$

only one particular functions

coefficients can be determined

$$a \frac{d^2 (y_p + y_c)}{dx^2} + b \frac{d(y_p + y_c)}{dx} + c (y_p + y_c) \Rightarrow g(x)$$



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*Finding a Particular Solution*  
*- Undetermined Coefficients*

# Particular Solutions

DEQ

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = g(x) \leftarrow y_p$$

particular solution  
by a conjecture

(I) FORM Rule

(II) Multiplication Rule

When *coefficients* are constant

And

$$g(x) = \begin{cases} \text{A constant or } \dots\dots\dots k \\ \text{A polynomial or } \dots\dots\dots P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0 \\ \text{An exponential function or } \dots\dots\dots e^{\alpha x} \\ \text{A sine and cosine functions or } \dots\dots\dots \sin(\beta x) \quad \cos(\beta x) \\ \text{Finite sum and products of the} \quad \dots\dots\dots e^{\alpha x} \sin(\beta x) + x^2 \\ \text{above functions} \end{cases}$$

And

$$g(x) \neq \ln x \quad \frac{1}{x} \quad \tan x \quad \sin^{-1} x$$

# Form Rule

DEQ

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = g(x)$$

$y_p$

particular solution  
by a conjecture

(I) FORM Rule

(II) Multiplication Rule

When *coefficients are constant*

$$g(x) = 2$$

$$y_p = A$$

$$g(x) = 3x+4$$

$$y_p = Ax+B$$

$$g(x) = 6x^2-7$$

$$y_p = Ax^2+Bx+C$$

$$g(x) = \sin 8x$$

$$y_p = A \cos 8x + B \sin 8x$$

$$g(x) = \cos 9x$$

$$y_p = A \cos 9x + B \sin 9x$$

$$g(x) = e^{10x}$$

$$y_p = Ae^{10x}$$

$$g(x) = xe^{11x}$$

$$y_p = (Ax+B)e^{11x}$$

$$g(x) = e^{11x} \sin 12x$$

$$y_p = Ae^{11x} \sin 12x + Be^{11x} \cos 12x$$

$$g(x) = 5x \sin(3x)$$

$$y_p = (Ax+B) \cos(3x) + (Cx+D) \sin(3x)$$

# Form Rule Example

DEQ

$$1 \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 2y = x$$

$y_p$

$$y_p + y_c$$

Associated DEQ

$$1 \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 2y = 0$$

$$m^2 + 3m + 2 = 0$$

$$(m+2)(m+1) = 0$$

$$c_1 y_1 + c_2 y_2$$

assume  $y_p = Ax + B$   
 $y_p' = A$   
 $y_p'' = 0$

$$y_p'' + 3y_p' + 2y_p = x$$

$$= 3A + 2(Ax + B)$$

$$= 2Ax + 3A + 2B$$

$$= x$$

$$2A = 1 \quad A = \frac{1}{2}$$

$$3A + 2B = 0 \quad B = -\frac{3}{4}$$

$$y_p = \frac{1}{2}x - \frac{3}{4}$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{2}x - \frac{3}{4}$$

# Multiplication Rule

DEQ

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = g(x)$$

$$y_p$$

$$y_p + y_c$$

Associated DEQ

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$y_1$$

$$y_2$$

$$c_1 y_1 + c_2 y_2$$

$$\begin{aligned} \text{use } y_p &= x^n y_1 & y_p &= x^n y_2 \\ \text{if } y_p &= y_1 & y_p &= y_2 \end{aligned}$$

When  $y_p$  contains a term which is the same term in  $y_c$

Use  $y_p$  multiplied by  $x^n$

$n$  is the **smallest** positive integer that eliminates the duplication

# Multiplication Rule Example (1)

$$y'' - 2y' + y = 2e^x$$

$$y_p = \cancel{Ae^x} \rightarrow \cancel{Ax e^x} \rightarrow Ax^2 e^x$$

$$y_1 = e^x \quad y_2 = x e^x$$

$$y'' - 2y' + y = 0$$

$$y'' - 2y' + y = 6x e^x$$

$$y_p = \cancel{Ae^x} \rightarrow \cancel{Ax^2 e^x} \rightarrow Ax^3 e^x$$

$$2Ae^x \neq 6x e^x$$

$$y_1 = e^x \quad y_2 = x e^x$$

$$y'' - 2y' + y = 0$$

$$y_p = x(Ax + B)e^x \rightarrow Bx e^x$$

$$y_p = x^2(Ax + B)e^x$$

## Multiplication Rule Example (2)

$$y' + 4y = e^x \sin(2t) + 2t \cos(2t)$$

$$y_p(t) = e^x (A \cos(2t) + B \sin(2t)) + (Ct + D) \cos(2t) + (Et + F) \sin(2t)$$

$$y_p(t) = e^x (A \cos(2t) + B \sin(2t)) + t(Ct + D) \cos(2t) + t(Et + F) \sin(2t)$$

$$y_h(t) = c_1 e^{+i2t} + c_2 e^{-i2t}$$

$$= (c_3 \cos(2t) + c_4 \sin(2t))$$



$$y'' + 5y' + 6y = t^2 e^{-3t}$$

$$y_p(t) = (At^2 + Bt + C) e^{-3t}$$

$$y_p(t) = t(At^2 + Bt + C) e^{-3t}$$

$$y_h = c_1 e^{-2t} + c_2 e^{-3t}$$



# Superposition (1)

$$\frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 2x^2 + 3 + \cos 8x$$

$$(2x^2 + 3) + (\cos 8x)$$

$$\frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$y_c$$

$$\frac{d^2 y_c}{dx^2} + b \frac{dy_c}{dx} + c y_c = 0$$

$$\frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 2x^2 + 3$$

$$y_{p1}$$

$$\frac{d^2 y_{p1}}{dx^2} + b \frac{dy_{p1}}{dx} + c y_{p1} = (2x^2 + 3)$$

$$\frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = \cos 8x$$

$$y_{p2}$$

$$\frac{d^2 y_{p2}}{dx^2} + b \frac{dy_{p2}}{dx} + c y_{p2} = \cos 8x$$

$$y_c + y_{p1} + y_{p2}$$

$$\frac{d^2}{dx^2} [y_c + y_{p1} + y_{p2}] + b \frac{d}{dx} [y_c + y_{p1} + y_{p2}] + c [y_c + y_{p1} + y_{p2}] = 2x^2 + 3 + \cos 8x$$



# Superposition (2)

DEQ

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = (2x^2 + 3) \cdot \cos 8x$$

$$y_p = (Ax^2 + Bx + C) \cdot (\cos 8x + \sin 8x)$$

$\frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$	← <span style="background-color: blue; color: yellow; padding: 2px 5px;"><math>y_c</math></span>
$\frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = (2x^2 + 3)$	← <span style="background-color: blue; color: yellow; padding: 2px 5px;"><del><math>y_{p1}</math></del></span>
$\frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = \cos 8x$	← <span style="background-color: blue; color: yellow; padding: 2px 5px;"><del><math>y_{p2}</math></del></span>

$$\frac{d^2}{dx^2} [y_c + \cancel{y_{p1}} + \cancel{y_{p2}}] + b \frac{d}{dx} [y_c + \cancel{y_{p1}} + \cancel{y_{p2}}] + c [y_c + \cancel{y_{p1}} + \cancel{y_{p2}}] = (2x^2 + 3) \cdot \cos 8x$$

# Finite Number of Derivative Functions

$$y = x e^{mx}$$

$$\dot{y} = e^{mx} + m x e^{mx}$$

$$\begin{aligned}\ddot{y} &= m e^{mx} + m(e^{mx} + m x e^{mx}) \\ &= 2m e^{mx} + m^2 x e^{mx}\end{aligned}$$

$$\begin{aligned}\ddot{\ddot{y}} &= 2m e^{mx} + m^2(e^{mx} + m x e^{mx}) \\ &= (m^2 + 2m)e^{mx} + m^3 x e^{mx}\end{aligned}$$

- 
- 
- $\{e^{mx}, x e^{mx}\}$

$$y = 2x^2 + 3x + 4$$

$$\dot{y} = 4x + 3$$

$$\ddot{y} = 4$$

$$\ddot{\ddot{y}} = 0$$

---

$$\{2x^2 + 3x + 4, 4x + 3, 4\}$$

# Infinite Number of Derivative Functions

$$y = +x^{-1}$$

$$\dot{y} = -x^{-2}$$

$$\ddot{y} = +2x^{-3}$$

$$\ddot{y} = -6x^{-4}$$



$$y = \ln x$$

$$y = +x^{-1}$$

$$\dot{y} = -x^{-2}$$

$$\ddot{y} = +2x^{-3}$$

$$\ddot{y} = -6x^{-4}$$



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*Finding a Particular Solution*  
*- Variation of Parameters*

# Variation of Parameter [ $c \rightarrow u(x)$ ]

$$y' + P(x)y = 0$$

$$y = c e^{-\int P(x)dx}$$

$$y_h = c y_1$$

$$\frac{dy_p}{dx} + P(x)y_p = Q(x)$$

$$y_p = u(x)y_1$$

**Integrating factor**

$$\frac{1}{y_1} = e^{+\int P(x)dx}$$

$$y_1 = e^{-\int P(x)dx}$$

$$y' + P(x)y = 0$$

$$y_h = c_1 y_1$$

$$+ c_2 y_2$$

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y_p = u(x)y_1$$

$$+ u_2(x)y_2$$

# Variation of Parameter : Conditions

$$y'' + P(x)y' + Q(x)y = 0$$

$$y_h = c_1 y_1 + c_2 y_2$$

*If the associated homogeneous solution can be solved*

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y_p = u_1(x)y_1 + u_2(x)y_2$$

*then, always a particular solution can be found*



*No restriction*

*constant **coefficients***

*A constant **or***

*A polynomial **or***

*An exponential function **or***

*A sine and cosine functions **or***

# Variation of Parameter : Wronskians

$$y'' + P(x)y' + Q(x)y = 0$$

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y_h = c_1 y_1 + c_2 y_2$$

$$y_p = u_1(x)y_1 + u_2(x)y_2$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{W_1}{W}$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{W_2}{W}$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W$$

$$u_1'(x) = -\frac{y_2(x)f(x)}{W}$$

$$u_2'(x) = \frac{y_1(x)f(x)}{W}$$

# Variation of Parameter : Particular Solutions

$$y'' + P(x)y' + Q(x)y = 0$$

$$y_h = c_1 y_1 + c_2 y_2$$

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y_p = u_1(x)y_1 + u_2(x)y_2$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W$$

$$y_p(x) = y_1 \int -\frac{y_2 f}{W} dx + y_2 \int \frac{y_1 f}{W} dx$$

$$y_p(x) = y_1 \int -\left(\frac{y_2 f}{y_1 y_2' - y_2 y_1'}\right) dx + y_2 \int \left(\frac{y_1 f}{y_1 y_2' - y_2 y_1'}\right) dx$$



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*Homogeneous Linear Equations  
with variable coefficients*

# Cauchy-Euler Equation

## Second Order Linear Equations with Constant Coefficients

$$a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

$$a_2 y'' + a_1 y' + a_0 y = g(x)$$

## Second Order Linear Equations with Variable Coefficients

$$a_2(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$$

$$a_2(x) y'' + a_1(x) y' + a_0(x) y = g(x)$$

## Cauchy-Euler Equation

$$a_2 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

$$a_2 x^2 y'' + a_1 x y' + a_0 y = g(x)$$

# Auxiliary Equation of Cauchy-Euler Equation

## Homogeneous Second Order Cauchy-Euler Equation

$$a x^2 \frac{d^2 y}{d x^2} + b x \frac{d y}{d x} + c y = 0$$

$$a x^2 y'' + b x y' + c y = 0$$

try a solution  $y = x^m$

$$a x^2 \frac{d^2}{d x^2} \{x^m\} + b x \frac{d}{d x} \{x^m\} + c \{x^m\} = 0$$

$$a x^2 \{x^m\}'' + b x \{x^m\}' + c \{x^m\} = 0$$

$$a \{m(m-1)x^m\} + b \{m x^m\} + c \{x^m\} = 0$$

$$a \{m(m-1)x^m\} + b \{m x^m\} + c \{x^m\} = 0$$

$$(a m^2 + (b-a)m + c) \cdot x^m = 0$$

$$(a m^2 + (b-a)m + c) \cdot x^m = 0$$

auxiliary equation

$$(a m^2 + (b-a)m + c) = 0$$

$$(a m^2 + (b-a)m + c) = 0$$

# General Solution – $y_h$ of Cauchy-Euler Equations

## Homogeneous Second Order Cauchy-Euler Equation

$$a x^2 \frac{d^2 y}{d x^2} + b x \frac{d y}{d x} + c y = 0$$

$$a x^2 y'' + b x y' + c y = 0$$

try a solution  $y = x^m$

auxiliary equation

$$(a m^2 + (b-a)m + c) = 0$$

$$m_1 = \{-(b-a) + \sqrt{(b-a)^2 - 4ac}\} / 2a$$

$$m_2 = \{-(b-a) - \sqrt{(b-a)^2 - 4ac}\} / 2a$$

**(A)**  $(b-a)^2 - 4ac > 0$  Real, distinct  $m_1, m_2$

**(B)**  $(b-a)^2 - 4ac = 0$  Real, equal  $m_1, m_2$

**(C)**  $(b-a)^2 - 4ac < 0$  Conjugate complex  $m_1, m_2$

$$y = C_1 x^{m_1} + C_2 x^{m_2}$$

$$y = C_1 x^{m_1} + C_2 x^{m_1} \ln x$$

$$y = C_1 x^{m_1} + C_2 x^{m_2} = C_1 x^{(\alpha+i\beta)} + C_2 x^{(\alpha-i\beta)}$$

# Constant v.s. Non-constant Coefficients

## Homogeneous Second Order DEs with Constant Coefficients

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$a y'' + b y' + c y = 0$$

- (A)  $b^2 - 4ac > 0$   $\longrightarrow$   $y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$   
(B)  $b^2 - 4ac = 0$   $\longrightarrow$   $y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$   
(C)  $b^2 - 4ac < 0$   $\longrightarrow$   $y = C_1 e^{\alpha x} e^{+i\beta x} + C_2 e^{\alpha x} e^{-i\beta x} = e^{\alpha x} (C_3 \cos(\beta x) + C_4 \sin(\beta x))$

## Homogeneous Second Order Cauchy-Euler Equation

$$a x^2 \frac{d^2 y}{dx^2} + b x \frac{dy}{dx} + c y = 0$$

$$a x^2 y'' + b x y' + c y = 0$$

$$\begin{aligned} x &= e^{\ln x} \\ x^{i\beta} &= e^{+i\beta \ln x} \end{aligned}$$

- (A)  $(b-a)^2 - 4ac > 0$   $\longrightarrow$   $y = C_1 x^{m_1} + C_2 x^{m_2}$   
(B)  $(b-a)^2 - 4ac = 0$   $\longrightarrow$   $y = C_1 x^{m_1} + C_2 x^{m_1} \ln x$   
(C)  $(b-a)^2 - 4ac < 0$   $\longrightarrow$   $y = C_1 x^\alpha \cdot e^{+i\beta \ln x} + C_2 x^\alpha \cdot e^{-i\beta \ln x} = x^\alpha (C_3 \cos(\beta \ln x) + C_4 \sin(\beta \ln x))$

# A Unifying View

## Constant Coefficients

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

## Non-constant Coefficients

$$a x^2 \frac{d^2 y}{dx^2} + b x \frac{dy}{dx} + c y = 0$$

$$x = e^{\ln x}$$
$$x^{i\beta} = e^{+i\beta \ln x}$$

(A)  $y = C_1 x^{m_1} + C_2 x^{m_2}$

(B)  $y = C_1 x^{m_1} + C_2 x^{m_1} \ln x$

(C)  $y = C_1 x^\alpha \cdot e^{+i\beta \ln x} + C_2 x^\alpha \cdot e^{+i\beta \ln x}$   
 $= x^\alpha (C_3 \cos(\beta \ln x) + C_4 \sin(\beta \ln x))$

(A)  $y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

(B)  $y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$

(C)  $y = C_1 e^{\alpha x} e^{+i\beta x} + C_2 e^{\alpha x} e^{-i\beta x}$   
 $= e^{\alpha x} (C_3 \cos(\beta x) + C_4 \sin(\beta x))$

$x$

(A)  $y = C_1 e^{m_1 \ln x} + C_2 e^{m_2 \ln x}$

(B)  $y = C_1 e^{m_1 \ln x} + C_2 e^{m_1 \ln x} \ln x$

(C)  $y = C_1 e^{\alpha \ln x} \cdot e^{+i\beta \ln x} + C_2 e^{\alpha \ln x} \cdot e^{+i\beta \ln x}$   
 $= e^{\alpha \ln x} (C_3 \cos(\beta \ln x) + C_4 \sin(\beta \ln x))$

$\ln x$

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## *Green's Function*

# Initial Value Problems

$$y'' + P(x)y' + Q(x)y = 0$$

$$y'(x_0) = y_1$$

$$y(x_0) = y_0$$

$$y_h = c_1 y_1 + c_2 y_2$$

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y'(x_0) = 0$$

$$y(x_0) = 0$$

$$y_p = u_1(x)y_1 + u_2(x)y_2$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W$$

$$y_p(x_0) = 0$$

$$u_1'(x) = \frac{W_1}{W} = -\frac{y_2(x)f(x)}{W}$$

$$u_2'(x) = \frac{W_2}{W} = \frac{y_1(x)f(x)}{W}$$

$$u_1(x) = \int u_1'(x) dx$$

$$u_2(x) = \int u_2'(x) dx$$

$$\text{anti-derivative} = \int -\frac{y_2(t)f(t)}{W(t)} dt + c_1$$

$$\text{anti-derivative} = \int \frac{y_1(t)f(t)}{W(t)} dt + c_2$$

$$= \int_{x_0}^x -\frac{y_2(t)f(t)}{W(t)} dt$$

$$= \int_{x_0}^x \frac{y_1(t)f(t)}{W(t)} dt$$

$$u_1(x_0) = 0 \quad \rightarrow \quad u_1(x_0)y_1(x_0) = 0$$

$$u_2(x_0) = 0 \quad \rightarrow \quad u_2(x_0)y_2(x_0) = 0$$



# Green's Function and IVP's (1)

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$[x_0, x] \subset I$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W(x)$$

$$u_1'(x) = -\frac{y_2(x)f(x)}{W(x)}$$

$$u_1(x) = \int_{x_0}^x -\frac{y_2(t)f(t)}{W(t)} dt$$

$$u_2'(x) = \frac{y_1(x)f(x)}{W(x)}$$

$$u_2(x) = \int_{x_0}^x \frac{y_1(t)f(t)}{W(t)} dt$$

$$\begin{aligned} y_p &= u_1(x)y_1 + u_2(x)y_2 \\ &= \left[ \int_{x_0}^x -\frac{y_2(t)f(t)}{W(t)} dt \right] y_1(x) + \left[ \int_{x_0}^x \frac{y_1(t)f(t)}{W(t)} dt \right] y_2(x) \\ &= \left[ \int_{x_0}^x -\frac{y_1(x)y_2(t)}{W(t)} f(t) dt \right] + \left[ \int_{x_0}^x \frac{y_1(t)y_2(x)}{W(t)} f(t) dt \right] \\ &= \int_{x_0}^x \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} f(t) dt \\ &= \int_{x_0}^x G(x, t) f(t) dt \end{aligned}$$

# Green's Function and IVP's (2)

$$y'' + P(x)y' + Q(x)y = 0$$

$$y'(x_0) = y_1$$

$$y(x_0) = y_0$$

$$y_h = c_1 y_1 + c_2 y_2$$

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y'(x_0) = 0$$

$$y(x_0) = 0$$

$$y_p = u_1(x)y_1 + u_2(x)y_2$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W$$

$$y_p = u_1(x)y_1 + u_2(x)y_2 = \int_{x_0}^x \left[ \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} \right] f(t) dt = \int_{x_0}^x G(x, t) f(t) dt$$

at the end, this  $x$  will replace the literal  $t$

$$= \int_{x_0}^x G(x, t) f(t) dt$$

this  $x$  and  $t$  appear in the indefinite integral

# Green's Function

$$G(x, t) = \left[ \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} \right]$$

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

$$y'' + P(x)y' + Q(x)y = 0$$

$y_1, y_2$

$$y'' + P(x)y' + Q(x)y = f(x)$$

the same Green's function

$$y'' + P(x)y' + Q(x)y = g(x)$$

$$G(x, t) = \left[ \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} \right]$$

$$y'' + P(x)y' + Q(x)y = h(x)$$

$$y_p = \int_{x_0}^x G(x, t)f(t)dt$$

$$y_p = \int_{x_0}^x G(x, t)g(t)dt$$

$$y_p = \int_{x_0}^x G(x, t)h(t)dt$$

# Three Initial Value Problem

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y(x_0) = y_0$$

$$y'(x_0) = y_1$$

$$y'' + P(x)y' + Q(x)y = 0$$

$$y(x_0) = y_0$$

$$y'(x_0) = y_1$$

*Homogeneous DEQ*

*Nonhomogeneous Initial Conditions*

*Nonzero Initial Conditions*

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y(x_0) = 0$$

$$y'(x_0) = 0$$

*Nonhomogeneous DEQ*

*Zero Initial Conditions*

*Initially at rest*

*Rest Solution*

# General Solutions of the Initial Value Problem

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y(x_0) = y_0$$

$$y'(x_0) = y_1$$

$$y = y_h + y_p$$

$$y(x_0) = y_h(x_0) + y_p(x_0) = y_0 + 0 = y_0$$

$$y'(x_0) = y_h'(x_0) + y_p'(x_0) = y_1 + 0 = y_1$$

$$y'' + P(x)y' + Q(x)y = 0$$

$$y(x_0) = y_0$$

$$y'(x_0) = y_1$$

$$y_h$$

*Nonhomogeneous Initial Conditions*  
*Nonzero Initial Conditions*

*Response due to the initial conditions*

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y(x_0) = 0$$

$$y'(x_0) = 0$$

$$y_p = \int_{x_0}^x G(x, t) f(t) dt$$

*Zero Initial Conditions*  
*Initially at rest*

*Response due to the forcing function  $f$*

*Rest Solution*

# Rest Solution

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y(x_0) = 0$$

$$y'(x_0) = 0$$

*Nonhomogeneous DEQ*

*Zero Initial Conditions*

*Initially at rest*

*Rest Solution*

$$y_p = u_1(x)y_1 + u_2(x)y_2 = \int_{x_0}^x \left[ \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} \right] f(t) dt = \int_{x_0}^x G(x, t) f(t) dt$$

$$\begin{cases} y_p(x) = \int_{x_0}^x G(x, t) f(t) dt \\ y_p'(x) = G(x, x) f(x) + \int_{x_0}^x \frac{\partial}{\partial x} [G(x, t) f(t)] dt = \int_{x_0}^x \left[ \frac{y_1(t)y_2'(x) - y_1'(x)y_2(t)}{W(t)} \right] f(t) dt \end{cases}$$

$$\begin{cases} y_p(x_0) = \int_{x_0}^{x_0} G(x, t) f(t) dt = 0 \\ y_p'(x_0) = \int_{x_0}^{x_0} \left[ \frac{y_1(t)y_2'(x_0) - y_1'(x_0)y_2(t)}{W(t)} \right] f(t) dt = 0 \end{cases}$$

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