Gauss Elimination (H1)

20160105

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Solving a System of Linear Equations

3x3 A A^{-1} Inverse Matrix.

$$\begin{aligned} p_1 x + p_2 y + p_3 z &= b_1 \\ q_1 x + q_2 y + q_3 z &= b_2 \\ r_1 x + r_2 y + r_3 z &= b_3 \end{aligned}$$

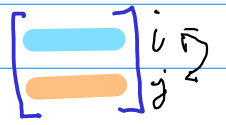
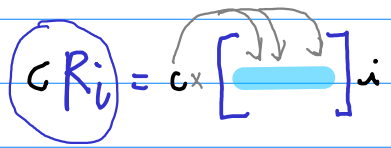
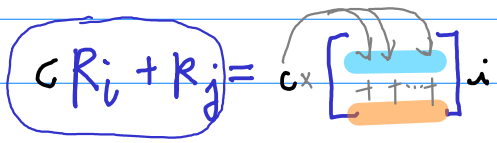
$$\begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$A \quad x = b$$

① A^{-1} $\qquad x = A^{-1} \cdot m$

② Cramer's Rule $\qquad x = \dfrac{\begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}}{\begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}} \quad y = \dfrac{\begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}}{\begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}} \quad z = \dfrac{\begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}}{\begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}}$

③ Gauss-Jordan Elimination RREF

R_{ij} =  cR_i =  $cR_i + R_j$ = 

Gauss-Jordan Elimination – Step 1

$$
\begin{aligned}
+2x_1 + x_2 - x_3 &= 8 && (L_1)\\
-3x_1 - x_2 + 2x_3 &= -11 && (L_2)\\
-2x_1 + x_2 + 2x_3 &= -3 && (L_3)
\end{aligned}
\qquad
\left(\begin{array}{cccc}
\textcircled{+2} & +1 & -1 & +8\\
-3 & -1 & +2 & -11\\
-2 & +1 & +2 & -3
\end{array}\right)
$$

$$
+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = 4 \qquad (\tfrac{1}{2}\times L_1) \qquad +2/2\ \ +1/2\ \ -1/2\ \ +8/2
$$

$$
\begin{aligned}
+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 &= 4 && (\tfrac{1}{2}\times L_1)\\
-3x_1 - x_2 + 2x_3 &= -11 && (L_2)\\
-2x_1 + x_2 + 2x_3 &= -3 && (L_3)
\end{aligned}
\qquad
\left(\begin{array}{cccc}
\textcircled{+1} & +1/2 & -1/2 & +4\\
-3 & -1 & +2 & -11\\
-2 & +1 & +2 & -3
\end{array}\right)
$$

Gauss-Jordan Elimination – Step 2

$$
\begin{aligned}
+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 &= +4 && (L_1)\\
-3x_1 - x_2 + 2x_3 &= -11 && (L_2)\\
-2x_1 + x_2 + 2x_3 &= -3 && (L_3)
\end{aligned}
\qquad
\left(\begin{array}{cccc}
+1 & +1/2 & -1/2 & +4\\
\boxed{-3} & -1 & +2 & -11\\
\boxed{-2} & +1 & +2 & -3
\end{array}\right)
$$

$$
\begin{aligned}
+3x_1 + \tfrac{3}{2}x_2 - \tfrac{3}{2}x_3 &= +12 && \boxed{3\times L_1}\\
-3x_1 - x_2 + 2x_3 &= -11 && (L_2)
\end{aligned}
\qquad
\begin{array}{cccc}
+3 & +3/2 & -3/2 & +12\\
-3 & -1 & +2 & -11
\end{array}
$$

$$
\begin{aligned}
+2x_1 + \tfrac{2}{2}x_2 - \tfrac{2}{2}x_3 &= +8 && \boxed{2\times L_1}\\
-2x_1 + x_2 + 2x_3 &= -3 && (L_3)
\end{aligned}
\qquad
\begin{array}{cccc}
+2 & +2/2 & -2/2 & +8\\
-2 & +1 & +2 & -3
\end{array}
$$

$$
\begin{aligned}
+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 &= +4 && (L_1)\\
0x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 &= +1 && \boxed{3\times L_1} + L_2)\\
0x_1 + 2x_2 + 1x_3 &= +5 && \boxed{2\times L_1} + L_3)
\end{aligned}
\qquad
\left(\begin{array}{cccc}
+1 & +1/2 & -1/2 & +4\\
\boxed{0} & +1/2 & +1/2 & +1\\
\boxed{0} & +2 & +1 & +5
\end{array}\right)
$$

Gauss-Jordan Elimination – Step 3

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \quad (L_1)$$
$$0x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 = +1 \quad (L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \quad (L_3)$$

+1	+1/2	–1/2	+4
0	+1/2	+1/2	+1
0	+2	+1	+5

$$0x_1 + 1x_2 + 1x_3 = +2 \quad (2 \times L_2)$$

0	+1	+1	+2

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \quad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \quad (2 \times L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \quad (L_3)$$

+1	+1/2	–1/2	+4
0	+1	+1	+2
0	+2	+1	+5

Gauss-Jordan Elimination – Step 4

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \quad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \quad (L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \quad (L_3)$$

+1	+1/2	–1/2	+4
0	+1	+1	+2
0	+2	+1	+5

$$0x_1 - 2x_2 - 2x_3 = -4 \quad (-2 \times L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \quad (L_3)$$

0	–2	–2	–4
0	+2	+1	+5

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \quad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \quad (L_2)$$
$$0x_1 + 0x_2 - 1x_3 = +1 \quad (-2 \times L_2 + L_3)$$

+1	+1/2	–1/2	+4
0	+1	+1	+2
0	0	–1	+1

Gauss-Jordan Elimination – Step 5

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \quad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \quad (L_2)$$
$$0x_1 + 0x_2 - 1x_3 = +1 \quad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & \textcircled{-1} & +1 \end{array}\right)$$

$$0x_1 - 0x_2 + 1x_3 = -1 \quad (-1 \times L_3)$$

$$\left(\begin{array}{ccc|c} 0 & 0 & +1 & -1 \end{array}\right)$$

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \quad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \quad (L_2)$$
$$0x_1 + 0x_2 + 1x_3 = -1 \quad (-1 \times L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & \textcircled{+1} & -1 \end{array}\right)$$

Forward Phase

$$\left(\begin{array}{ccc|c} \textcircled{+2} & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} \textcircled{+1} & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ \boxed{0} & +1/2 & +1/2 & +1 \\ \boxed{0} & +2 & +1 & +5 \end{array}\right) \Rightarrow$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & \textcircled{+1} & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & \boxed{0} & -1 & +1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & \textcircled{+1} & -1 \end{array}\right) \Rightarrow$$

Forward Phase - Gaussian Elimination

Gauss-Jordan Elimination – Step 6

$$
\begin{aligned}
+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 &= +4 \quad (L_1)\\
0x_1 + 1x_2 + 1x_3 &= +2 \quad (L_2)\\
0x_1 + 0x_2 + 1x_3 &= -1 \quad (L_3)
\end{aligned}
\qquad
\left(\begin{array}{ccc|c}
+1 & +1/2 & \boxed{-1/2} & +4\\
0 & +1 & \boxed{+1} & +2\\
0 & 0 & +1 & -1
\end{array}\right)
$$

$$
\begin{aligned}
0x_1 + 0x_2 + \tfrac{1}{2}x_3 &= -\tfrac{1}{2} \quad \boxed{+\tfrac{1}{2}\times L_3}\\
+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 &= +4 \quad (L_1)
\end{aligned}
\qquad
\begin{array}{cccc}
0 & 0 & +1/2 & -1/2\\
+1 & +1/2 & -1/2 & +4
\end{array}
$$

$$
\begin{aligned}
0x_1 + 0x_2 - 1x_3 &= +1 \quad \boxed{-1\times L_3}\\
0x_1 + 1x_2 + 1x_3 &= +2 \quad (L_2)
\end{aligned}
\qquad
\begin{array}{cccc}
0 & 0 & -1 & +1\\
0 & +1 & +1 & +2
\end{array}
$$

$$
\begin{aligned}
+1x_1 + \tfrac{1}{2}x_2 + 0x_3 &= +\tfrac{7}{2} \quad (+\tfrac{1}{2}\times L_3 + L_1)\\
0x_1 + 1x_2 + 0x_3 &= +3 \quad (-1\times L_3 + L_2)\\
0x_1 + 0x_2 + 1x_3 &= -1 \quad (L_3)
\end{aligned}
\qquad
\left(\begin{array}{ccc|c}
+1 & +1/2 & \boxed{0} & +7/2\\
0 & +1 & \boxed{0} & +3\\
0 & 0 & +1 & -1
\end{array}\right)
$$

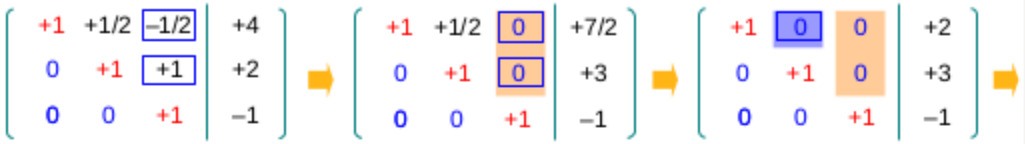
Gauss-Jordan Elimination – Step 7

$$
\begin{aligned}
+1x_1 + \tfrac{1}{2}x_2 + 0x_3 &= +\tfrac{7}{2} \quad (L_1)\\
0x_1 + 1x_2 + 0x_3 &= +3 \quad (L_2)\\
0x_1 + 0x_2 + 1x_3 &= -1 \quad (L_3)
\end{aligned}
\qquad
\left(\begin{array}{ccc|c}
+1 & \boxed{+1/2} & 0 & +7/2\\
0 & +1 & 0 & +3\\
0 & 0 & +1 & -1
\end{array}\right)
$$

$$
\begin{aligned}
0x_1 - \tfrac{1}{2}x_2 + 0x_3 &= -\tfrac{3}{2} \quad \boxed{-\tfrac{1}{2}\times L_2}\\
+1x_1 + 0x_2 - 0x_3 &= +2 \quad (L_1)
\end{aligned}
\qquad
\begin{array}{ccc|c}
0 & -1/2 & 0 & -3/2\\
+1 & +1/2 & 0 & +7/2
\end{array}
$$

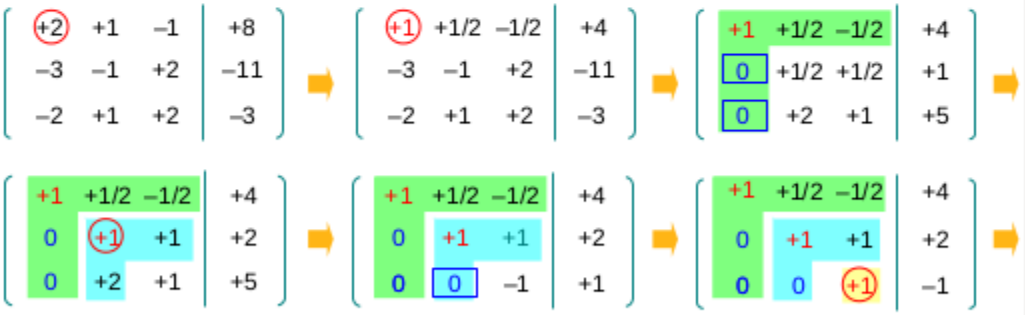
$$
\begin{aligned}
+1x_1 + 0x_2 - 0x_3 &= +2 \quad (-\tfrac{1}{2}\times L_2 + L_1)\\
0x_1 + 1x_2 + 0x_3 &= +3 \quad (L_2)\\
0x_1 + 0x_2 + 1x_3 &= -1 \quad (L_3)
\end{aligned}
\qquad
\left(\begin{array}{ccc|c}
+1 & \boxed{0} & 0 & +2\\
0 & +1 & 0 & +3\\
0 & 0 & +1 & -1
\end{array}\right)
$$

Backward Phase



Gauss-Jordan Elimination

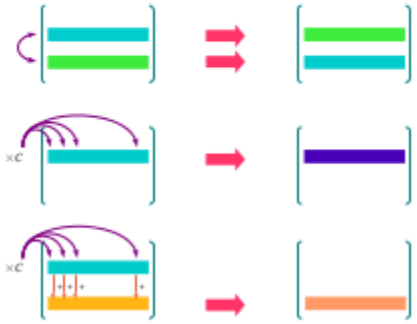
Forward Phase – Gaussian Elimination



Backward Phase – Guass-Jordan Elimination

Gauss-Jordan Elimination

$$\begin{pmatrix} +2 & +1 & -1 \\ -3 & -1 & +2 \\ -2 & +1 & +2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} +8 \\ -11 \\ -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} * \\ * \\ * \end{pmatrix}$$



$$\begin{bmatrix} 0 & 1 & 1 & 5 \\ 0 & 2 & 3 & 13 \\ 1 & 4 & 2 & 15 \end{bmatrix} \xrightarrow{R_{13}} \begin{bmatrix} 1 & 4 & 2 & 15 \\ 0 & 2 & 3 & 13 \\ 0 & 1 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 4 & 2 & 15 \\ 0 & 1 & \frac{3}{2} & \frac{13}{2} \\ 0 & 1 & 1 & 5 \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} 1 & 4 & 2 & 15 \\ 0 & 1 & \frac{3}{2} & \frac{13}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

$$\xrightarrow{-2R_3} \begin{bmatrix} 1 & 4 & 2 & 15 \\ 0 & 1 & \frac{3}{2} & \frac{13}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{-\frac{3}{2}R_3+R_2} \begin{bmatrix} 1 & 4 & 2 & 15 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{-2R_3+R_1} \begin{bmatrix} 1 & 4 & 0 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{-4R_2+R_1} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$