Row Reduction (H1)

20160105

REF: Row Echelon Forms (1)

zero rows	⇨	Should be grouped at the bottom
non-zero row	⇨	A leading 1 The 1st non-zero element should be one
Any successive non-zero rows	⇨	The leading **1** of the lower row should be farther to the **right** than the leading **1** of the higher row

REF: Row Echelon Forms (2)

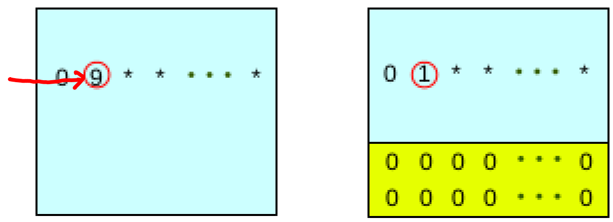
zero rows ⇨ Should be grouped at the bottom

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad \begin{bmatrix} & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

REF: Row Echelon Forms (3)

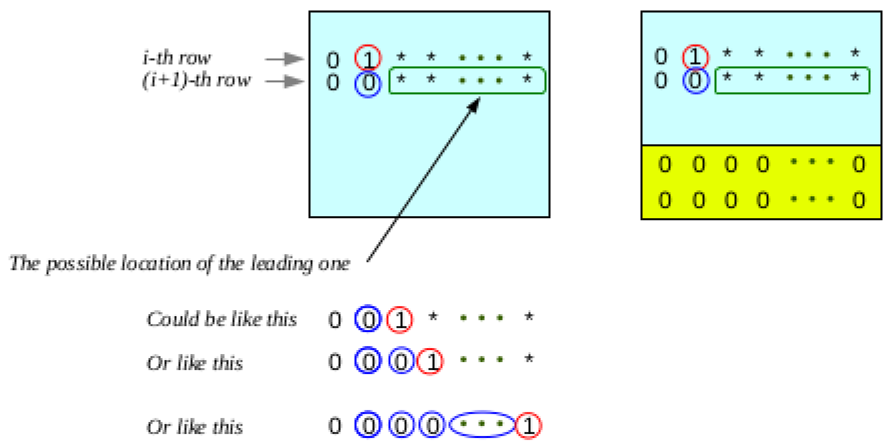
non-zero row ⇨ A leading one

The 1st non-zero element should be one



REF: Row Echelon Forms (4)

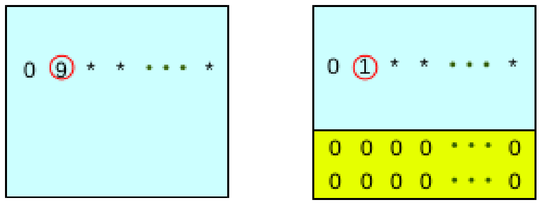
Any successive non-zero rows ⇨ The leading **1** of the lower row should be farther to the **right** than the leading **1** of the higher row



REF: Row Echelon Forms (3)

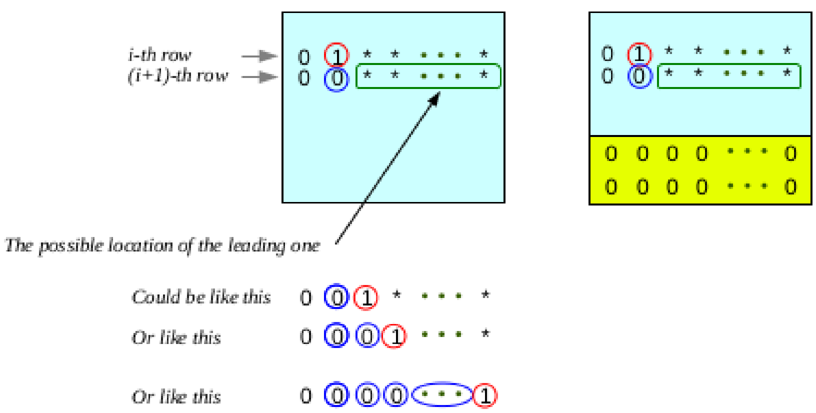
non-zero row ⇒ A leading one

The 1st non-zero element should be one

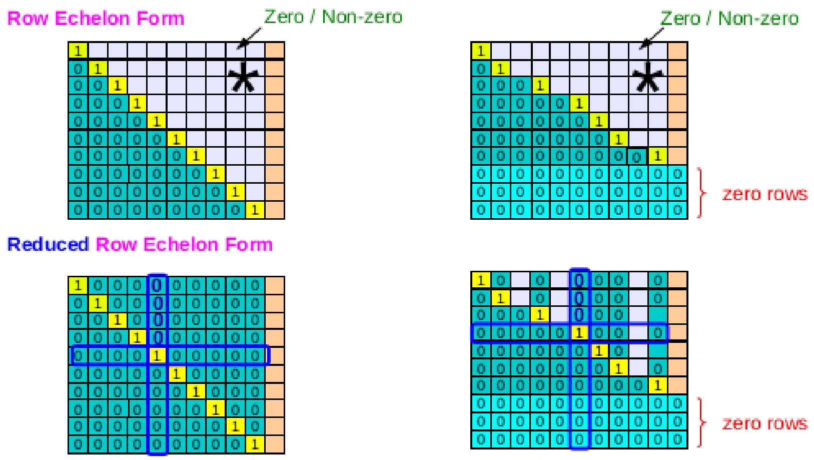


REF: Row Echelon Forms (4)

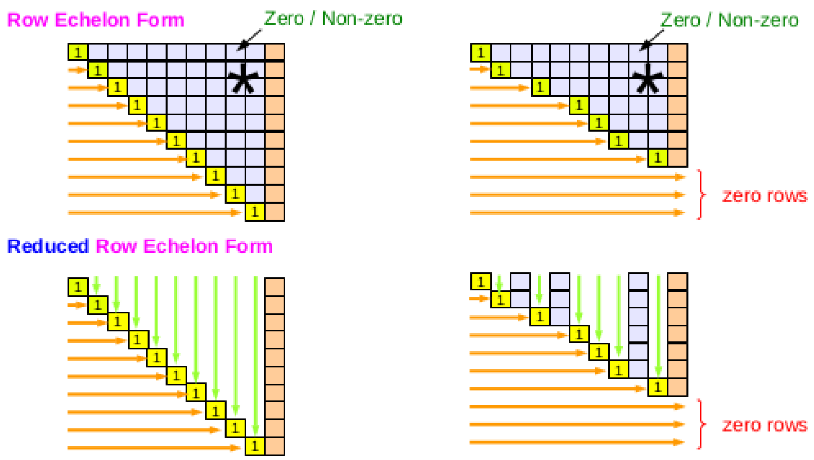
Any successive non-zero rows ⇒ The leading **1** of the lower row should be farther to the **right** than the leading **1** of the higher row



Examples



Examples



```
octave:1> A = [ 2  0 0 ; 0 3 0 ; 0 0 4]
A =

   2   0   0
   0   3   0
   0   0   4

octave:2>
octave:2>
octave:2> inv(A)
ans =

   0.50000  -0.00000  -0.00000
   0.00000   0.33333  -0.00000
   0.00000   0.00000   0.25000

octave:3>
octave:3> rref(A)
ans =

   1   0   0
   0   1   0
   0   0   1

octave:4>
octave:4>
octave:4>
```

```
octave:4> B = [ 2 0 0; 4 0 0; 0 0 4]
B =

   2   0   0
   4   0   0
   0   0   4

octave:5> rref(B)
ans =

   1   0   0
   0   0   1
   0   0   0

octave:6> det(B)
ans = 0
octave:7>
octave:7>
```

```
octave:7> C = [ 2 0 0; 2 0 4; 0 0 4]
C =

   2   0   0
   2   0   4
   0   0   4

octave:8> rref(C)
ans =

   1   0   0
   0   0   1
   0   0   0

octave:9>
```

$n \times n$ matrix A

$A \xrightarrow[\text{reduction}]{\text{row}}$ REF / RREF

Row Echelon Form

Reduced Row Echelon Form

① $\leftrightarrow$
② $c\times$
③ $=$ $c\times$

operations

$\boxed{\text{invertible } A} \xrightarrow[\text{reduction}]{\text{row}} \boxed{RREF = I}$

$\det(A) \neq 0$

full rank n

all rows are linearly independent

$\boxed{\text{noninvertible } A} \xrightarrow[\text{reduction}]{\text{row}}$ RREF with one or more zero row

$\det(A) = 0$

rank $< n$

some linearly dependent rows

A $\quad n \times n$

rank는 linearly independent한 row의 개수 (col의 개수)

$\Rightarrow$ RREF에서 non-zero row의 개수

$\det(A) \neq 0$ $\quad$ RREF $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = I$

A^{-1}

$\det(A) = 0$ $\quad$ RREF $n \left\{ \begin{bmatrix} & & & \\ & & & \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right.$ $\rightarrow$ non-zero row의 개수

rank

$\cancel{A^{-1}}$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \longrightarrow \overset{\text{RREF}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \Big\} 3$$

rank = 3

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ \boxed{0 \quad 0 \quad 0} \end{bmatrix}$$

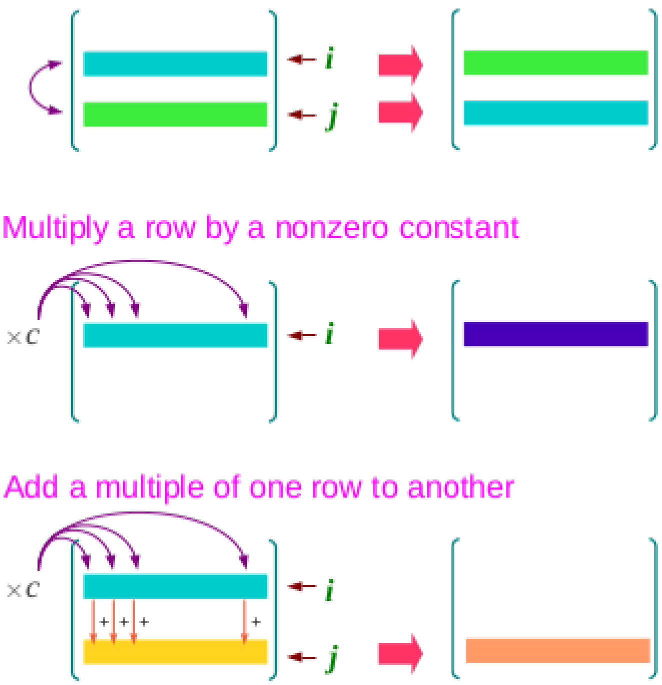
$$C = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 4 \\ 0 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ \boxed{0 \quad 0 \quad 0} \end{bmatrix}$$

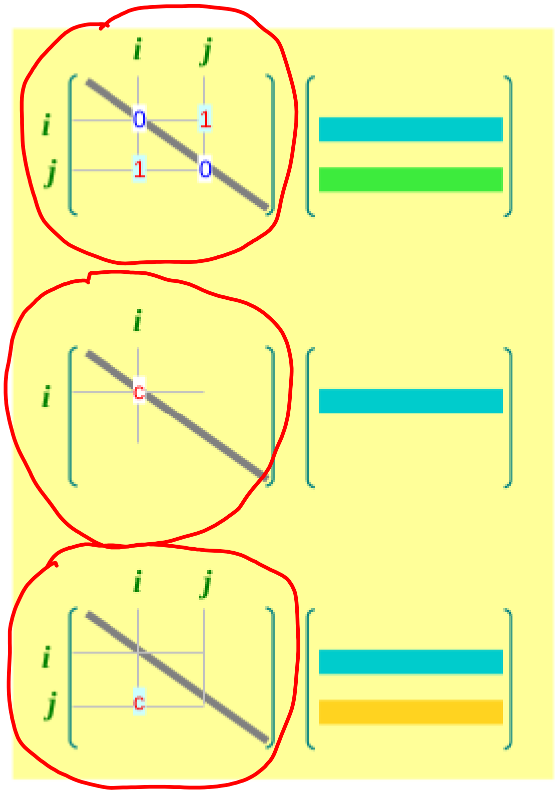
Elementary Matrix

Interchange two rows

Multiply a row by a nonzero constant

Add a multiple of one row to another





Product of Elementary Matrices

$$\overset{E_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}} \cdot \overset{E_2}{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \overset{E_1}{\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \left[\begin{array}{ccc|c} \multicolumn{3}{c|}{A} & \\ +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right]$$

$$\overset{E_6}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}} \cdot \overset{E_5}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}} \cdot \overset{E_4}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\overset{E_9}{\begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \overset{E_8}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \overset{E_7}{\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \Rightarrow \left[\begin{array}{ccc|c} +1 & 0 & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right]$$

$$\boxed{E_k \cdots E_2 E_1 A = I_n}$$

$$E_k^{-1} E_k E_{k-1} \cdots E_2 E_1 A = E_k^{-1} I_n$$

$$E_{k-1} \cdots E_2 E_1 A = E_k^{-1}$$

$$E_{k-1}^{-1} E_{k-1} \cdots E_2 E_1 A = E_{k-1}^{-1} E_k^{-1}$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

(Elementary Matrices)

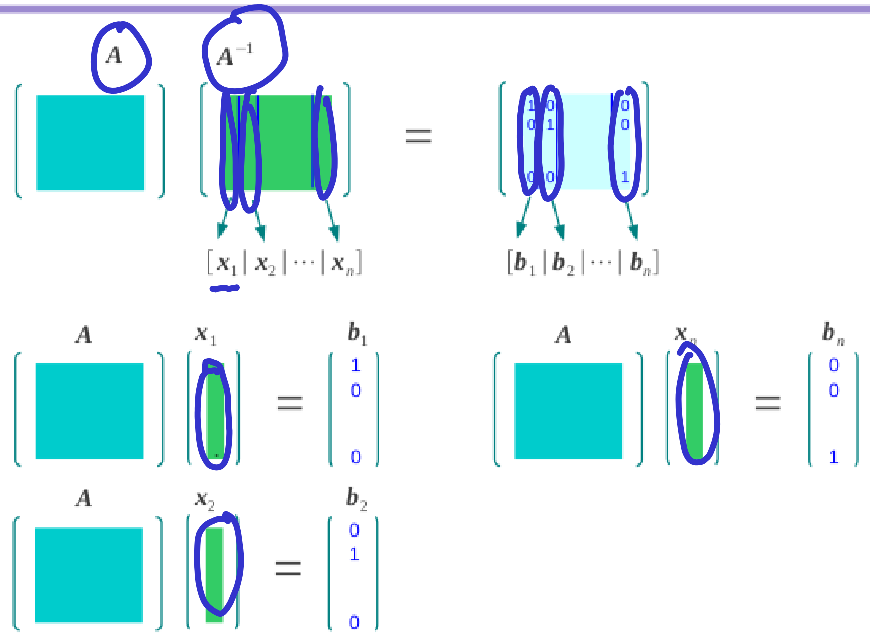
$$\overbrace{E_9 E_8 E_7 E_6 E_5 E_4 E_3 E_2 E_1}^{A^{-1}} \boxed{A} = I$$

$$E_8 E_7 E_6 E_5 E_4 E_3 E_2 E_1 A = E_9^{-1} I$$

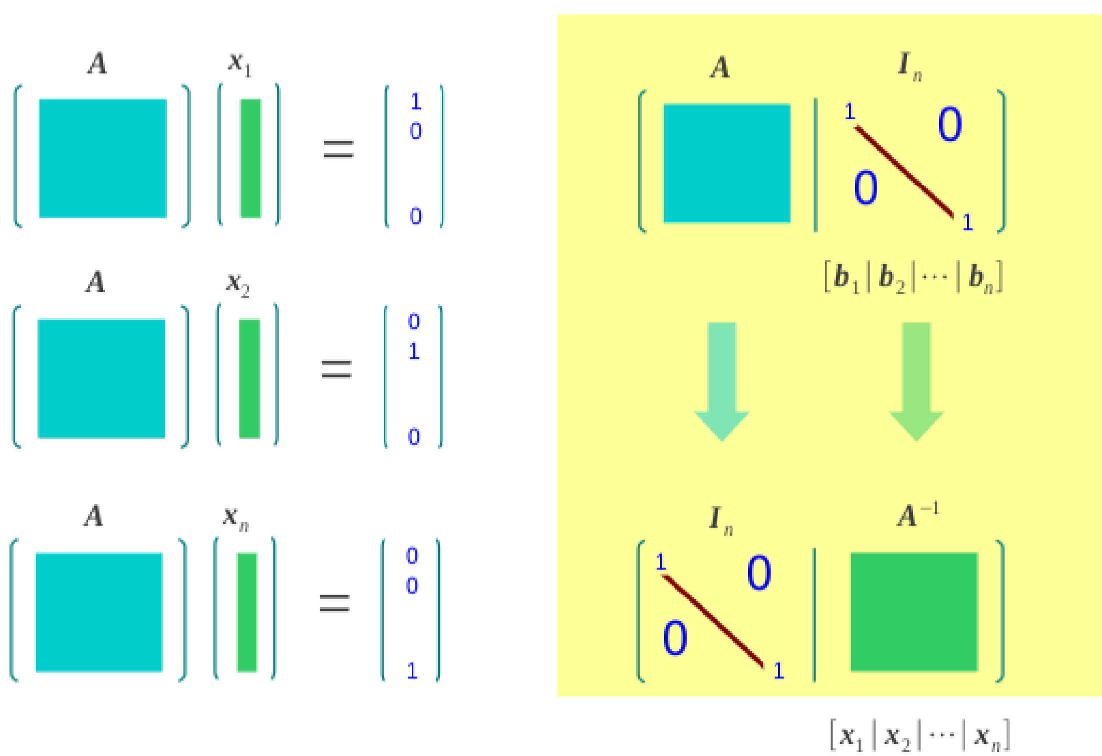
$$E_7 E_6 E_5 E_4 E_3 E_2 E_1 A = E_8^{-1} E_9^{-1} I$$

$$\boxed{A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1} E_7^{-1} E_8^{-1} E_9^{-1}}$$

Inversion Algorithm (1)



Inversion Algorithm (2)



$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} A^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A \quad A^{-1} = I$$

8.6 #1, 3, 5, 7, 9, 11, 13, 15, 17, 19

Inverse Matrix

① find $\det(A) =$

② if $\det(A) \neq 0$ Find A^{-1} using $\operatorname{adj}(A)$

③ $\begin{bmatrix} A \end{bmatrix} \longrightarrow$ RREF?

rank?

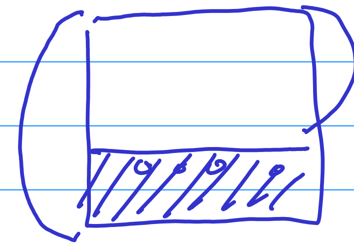
③ if rank $= n$ $\begin{bmatrix} A & | & I \end{bmatrix}$ row reduction

$$\Downarrow \qquad \Downarrow$$

$$\begin{bmatrix} I & | & A^{-1} \end{bmatrix}$$

row reduction

zero row →



1

2 cx

3 cx

REF

RREF →

$$A \longrightarrow RREF(A)$$

⑥ Gauss-Jordan Elimination

연립방정식 $Ax = b$

invertible A $\quad [A \mid b] \longrightarrow \left[I \;\middle|\; \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \right]$

augmented matrix

non-invertible A $\quad [A \mid b] \longrightarrow \left[\begin{matrix} \cdot \\ \boxed{0 \;\; 0 \cdots 0} \end{matrix} \;\middle|\; \boxed{\cdot} \right]$

① inconsistent
$\Rightarrow$ no solution

② consistent
with many solutions

⑥ Find the inverse matrix A^{-1}

$[A \mid I] \longrightarrow [I \mid A^{-1}]$

연립방정식

$$Ax = b$$ Non-homogeneous

$$Ax = 0$$: Homogeneous Linear System

Finding EigenVectors

A

$$(A-\lambda I)p=0$$

$$Bp=0$$ Homogeneous System

Solution p eigenvector

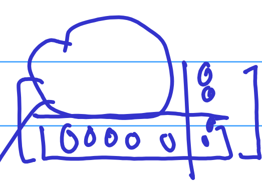
~~$p=0$~~ non zero : meaningful

$\det(B)=0 \Rightarrow$ number $\lambda \Rightarrow B$

eigenvalues

Gauss elimination

$$\left[\, B \;\middle|\; \begin{matrix}0\\0\\\vdots\end{matrix} \,\right] \longrightarrow$$



leading variables

free variables

↳ parameter

s, t, a ....

(497p (예제2)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

$\det(A - \lambda I) = 0$ $\quad \begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix} = 0$

$$\lambda(\lambda+4)(\lambda-3) = 0$$

① $\lambda = 0$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 6 & -1 & 0 & 0 \\ -1 & -2 & -1 & 0 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{13} & 0 \\ 0 & 1 & \frac{6}{13} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + \frac{1}{13}x_3 = 0$ $\qquad x_3 = 13$ $\qquad x_1 = -1$

$x_2 + \frac{6}{13}x_3 = 0$ $\qquad x_2 = -6$

$(-1, -6, 13) \rightarrow (1, 6, -13)$

② $\lambda = -4$

$$\left[\begin{array}{ccc|c} 1+4 & 2 & 1 & 0 \\ 6 & -1+4 & 0 & 0 \\ -1 & -2 & -1+4 & 0 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_3 = 0$ $\quad x_3 = 1 \quad x_1 = -1$

$x_2 - 2x_3 = 0$ $\quad x_2 = 2$

$(-1, 2, 1) \rightarrow (1, -2, -1)$

③ $\lambda = 3$

$$\left[\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 6 & -4 & 0 & 0 \\ -1 & -2 & -4 & 0 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_3 = 0$ $\quad$ $x_3 = 2$ $\quad$ $x_1 = -1$

$x_2 + \frac{3}{2} x_3 = 0$ $\quad$ $x_2 = -3$

$(-1, -3, 2) \rightarrow (1, 3, -2)$