

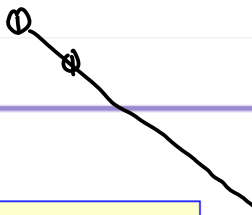
Row Reduction (H1)

20160105

Copyright (c) 2015 Young W. Lim.

Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

REF: Row Echelon Forms (1)



zero rows	→	Should be grouped at the <u>bottom</u>
non-zero row	→	A leading <u>1</u> The 1 st non-zero element should be one
Any successive non-zero rows	→	The leading 1 of the lower row should be farther to the right than the leading 1 of the higher row



REF: Row Echelon Forms (2)

zero rows → Should be grouped at the bottom

0	0	0	0	...	0
0	0	0	0	...	0

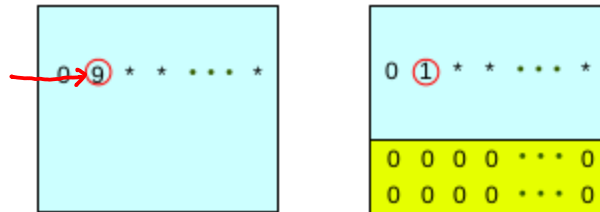
0	0	0	0	...	0
0	0	0	0	...	0

REF: Row Echelon Forms (3)

non-zero row →

A leading one

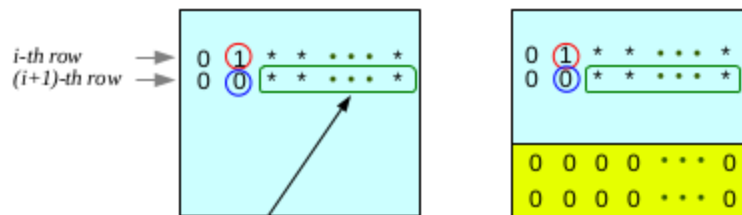
The 1st non-zero element should be one



REF: Row Echelon Forms (4)

Any successive non-zero rows →

The leading **1** of the **lower row** should be farther to the **right** than the leading **1** of the **higher row**



The possible location of the leading one

Could be like this $0 \ 0 \ 1 \ * \ \dots \ *$

Or like this $0 \ 0 \ 0 \ 1 \ \dots \ *$

Or like this $0 \ 0 \ 0 \ 0 \ \dots \ 1$

REF: Row Echelon Forms (3)

non-zero row \rightarrow

A leading one

The 1st non-zero element should be one

$$\begin{array}{cccccc} 0 & \textcircled{9} & * & * & \dots & * \end{array}$$

$$\begin{array}{cccccc} 0 & \textcircled{1} & * & * & \dots & * \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{array}$$

REF: Row Echelon Forms (4)

Any successive non-zero rows \rightarrow

The leading **1** of the **lower row** should be farther to the **right** than the leading **1** of the **higher row**

$$\begin{array}{l} i\text{-th row} \rightarrow \\ (i+1)\text{-th row} \rightarrow \end{array} \begin{array}{cccccc} 0 & \textcircled{1} & * & * & \dots & * \\ 0 & \textcircled{0} & * & * & \dots & * \end{array}$$

$$\begin{array}{cccccc} 0 & \textcircled{1} & * & * & \dots & * \\ 0 & \textcircled{0} & * & * & \dots & * \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{array}$$

The possible location of the leading one

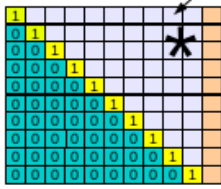
Could be like this $0 \textcircled{0} \textcircled{1} * \dots *$

Or like this $0 \textcircled{0} \textcircled{0} \textcircled{1} \dots *$

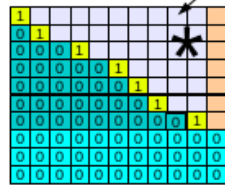
Or like this $0 \textcircled{0} \textcircled{0} \textcircled{0} \dots \textcircled{1}$

Examples

Row Echelon Form

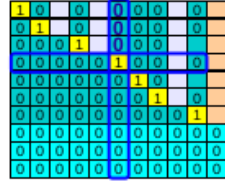
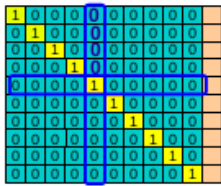


Row Echelon Form



zero rows

Reduced Row Echelon Form



zero rows

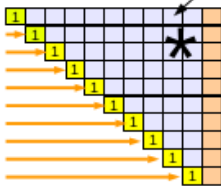
Row Reduciton (1A)

23

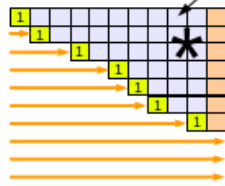
Young Won Lim
03/09/2015

Examples

Row Echelon Form

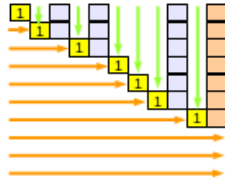
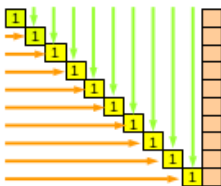


Row Echelon Form



zero rows

Reduced Row Echelon Form



zero rows

Row Reduciton (1A)

24

Young Won Lim
03/09/2015



```
octave:1> A = [ 2 0 0; 0 3 0; 0 0 4]
A =
```

```
 2 0 0
 0 3 0
 0 0 4
```

```
octave:2>
octave:2>
octave:2> inv(A)
ans =
```

```
 0.50000 -0.00000 -0.00000
 0.00000  0.33333 -0.00000
 0.00000  0.00000  0.25000
```

```
octave:3>
octave:3> rref(A)
ans =
```

```
 1 0 0
 0 1 0
 0 0 1
```

```
octave:4>
octave:4>
octave:4>
```

```
octave:4> B = [ 2 0 0; 4 0 0; 0 0 4]
B =
```

```
 2 0 0
 4 0 0
 0 0 4
```

```
octave:5> rref(B)
ans =
```

```
 1 0 0
 0 0 1
 0 0 0
```

```
octave:6> det(B)
ans = 0
```

```
octave:7>
octave:7>
```

```
octave:7> C = [ 2 0 0; 2 0 4; 0 0 4]
C =
```

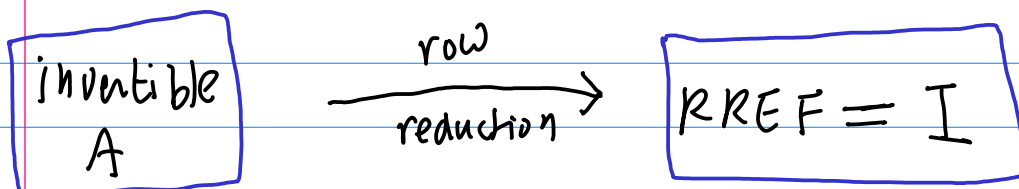
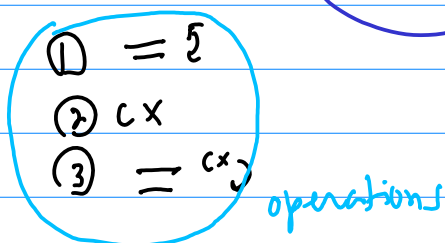
```
 2 0 0
 2 0 4
 0 0 4
```

```
octave:8> rref(C)
ans =
```

```
 1 0 0
 0 0 1
 0 0 0
```

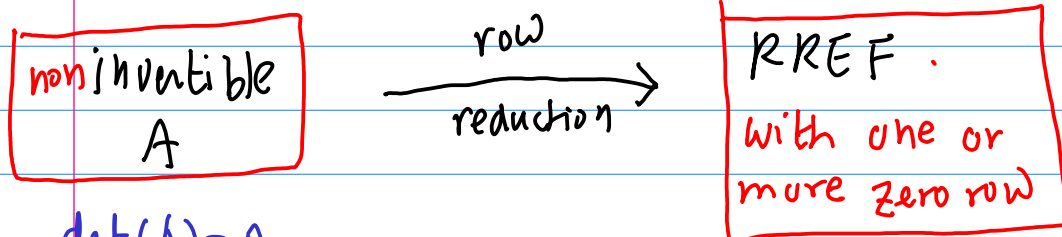
```
octave:9>
```

$n \times n$ matrix A



$\det(A) \neq 0$
full rank n

all rows are
linearly independent



$\det(A) = 0$

rank $< n$

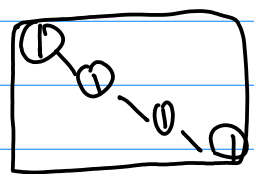
some linearly
dependent rows

A $n \times n$

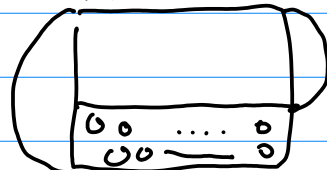
rank 는 linearly independent한
row의 개수 (column의 개수)

\Rightarrow RREF에서 non-zero row의 개수

$\det(A) \neq 0$
 A^{-1}

RREF
 = I

$\det(A) = 0$
 ~~A^{-1}~~

RREF
 n  \rightarrow non-zero row의 개수
rank

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$



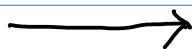
RREF

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Bigg) 3$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

rank=3

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

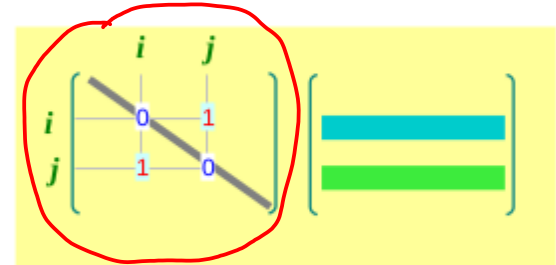
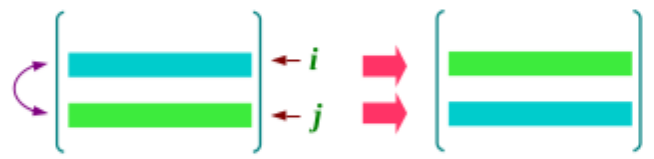
$$C = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$



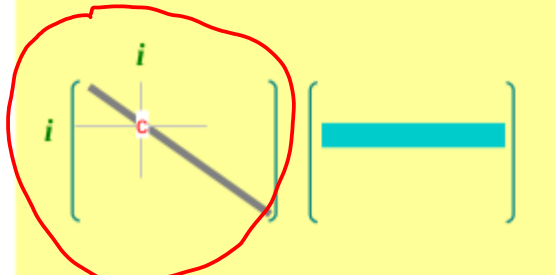
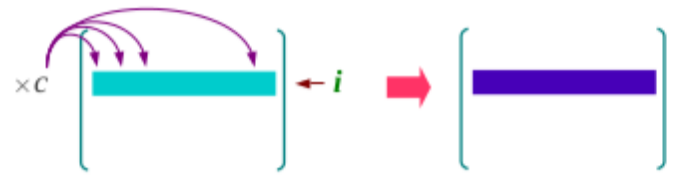
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Elementary Matrix

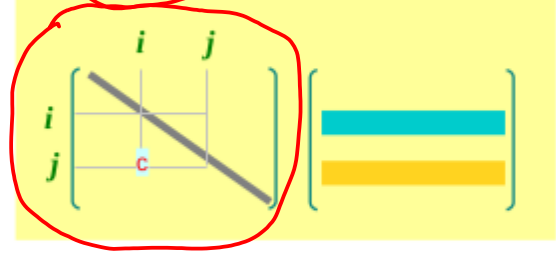
Interchange two rows



Multiply a row by a nonzero constant



Add a multiple of one row to another



Product of Elementary Matrices

$E_3 \cdot E_2 \cdot E_1 \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{bmatrix}$

$E_6 \cdot E_5 \cdot E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E_9 \cdot E_8 \cdot E_7 = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} +1 & 0 & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{bmatrix}$

Elementary Matrix (2A)

18

Young Won Lim
03/09/2015

$$E_k \cdots E_2 E_1 A = I_n$$

$$\Downarrow$$

$$E_k^{-1} E_k E_{k-1} \cdots E_2 E_1 A = E_k^{-1} I_n$$

$$E_{k-1} \cdots E_2 E_1 A = E_k^{-1}$$

$$\Downarrow$$

$$E_{k-1}^{-1} E_{k-1} \cdots E_2 E_1 A = E_{k-1}^{-1} E_k^{-1}$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} \quad \text{(Elementary Matrices)}$$

|| A^{-1}

$$E_9 E_8 E_7 E_6 E_5 E_4 E_3 E_2 E_1 A = I$$

$$E_8 E_7 E_6 E_5 E_4 E_3 E_2 E_1 A = E_9^{-1} I$$

$$E_7 E_6 E_5 E_4 E_3 E_2 E_1 A = E_8^{-1} E_9^{-1} I$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1} E_7^{-1} E_8^{-1} E_9^{-1}$$

Inversion Algorithm (1)

$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{matrix} A \\ \left[\begin{array}{c} \square \\ \square \\ \square \end{array} \right] \end{matrix} & \begin{matrix} A^{-1} \\ \left[\begin{array}{c} \square \\ \square \\ \square \end{array} \right] \end{matrix} \\
 \left[\begin{array}{c|c} \square & \square \\ \square & \square \\ \square & \square \end{array} \right] & = & \left[\begin{array}{c|c} \begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \end{array} \right] \\
 \left[x_1 \mid x_2 \mid \cdots \mid x_n \right] & & \left[b_1 \mid b_2 \mid \cdots \mid b_n \right]
 \end{array} \\
 \\
 \begin{array}{ccc}
 \begin{matrix} A \\ \left[\begin{array}{c} \square \\ \square \\ \square \end{array} \right] \end{matrix} & \begin{matrix} x_1 \\ \left[\begin{array}{c} \square \\ \square \\ \square \end{array} \right] \end{matrix} & = & \begin{matrix} b_1 \\ \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right] \end{matrix} \\
 \\
 \begin{matrix} A \\ \left[\begin{array}{c} \square \\ \square \\ \square \end{array} \right] \end{matrix} & \begin{matrix} x_2 \\ \left[\begin{array}{c} \square \\ \square \\ \square \end{array} \right] \end{matrix} & = & \begin{matrix} b_2 \\ \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right] \end{matrix} \\
 \\
 \begin{matrix} A \\ \left[\begin{array}{c} \square \\ \square \\ \square \end{array} \right] \end{matrix} & \begin{matrix} x_n \\ \left[\begin{array}{c} \square \\ \square \\ \square \end{array} \right] \end{matrix} & = & \begin{matrix} b_n \\ \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right] \end{matrix}
 \end{array}
 \end{array}$$

Inversion Algorithm (2)

$$\begin{array}{ccc}
 \begin{matrix} A \\ \left[\begin{array}{c} \square \\ \square \\ \square \end{array} \right] \end{matrix} & \begin{matrix} x_1 \\ \left[\begin{array}{c} \square \\ \square \\ \square \end{array} \right] \end{matrix} & = & \begin{matrix} \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right] \\ b_1 \end{matrix} \\
 \\
 \begin{matrix} A \\ \left[\begin{array}{c} \square \\ \square \\ \square \end{array} \right] \end{matrix} & \begin{matrix} x_2 \\ \left[\begin{array}{c} \square \\ \square \\ \square \end{array} \right] \end{matrix} & = & \begin{matrix} \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right] \\ b_2 \end{matrix} \\
 \\
 \begin{matrix} A \\ \left[\begin{array}{c} \square \\ \square \\ \square \end{array} \right] \end{matrix} & \begin{matrix} x_n \\ \left[\begin{array}{c} \square \\ \square \\ \square \end{array} \right] \end{matrix} & = & \begin{matrix} \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right] \\ b_n \end{matrix}
 \end{array}$$

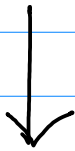
$$\begin{array}{ccc}
 \begin{matrix} A & I_n \\ \left[\begin{array}{c|c} \square & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{array} \right] & & \begin{matrix} \left[\begin{array}{c|c} \square & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{array} \right] \\ \left[b_1 \mid b_2 \mid \cdots \mid b_n \right] \end{matrix} \\
 \downarrow & & \downarrow \\
 \begin{matrix} I_n & A^{-1} \\ \left[\begin{array}{c|c} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \square \end{array} \right] & & \begin{matrix} \left[\begin{array}{c|c} \square & \square \end{array} \right] \\ \left[x_1 \mid x_2 \mid \cdots \mid x_n \right] \end{matrix}
 \end{array}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \mathbf{A} \\ \mathbf{A} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8.6 #1, 3, 5, 7, 9, 11, 13, 15, 17, 19

Inverse Matrix

① find $\det(A) =$

② if $\det(A) \neq 0$ Find A^{-1} using $\text{adj}(A)$

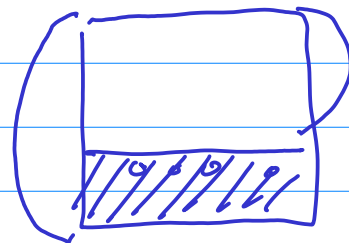
③ $[A] \longrightarrow$ RREF?
rank?

③ if $\text{rank} = n$

$[A \mid I]$ row reduction
 $[I \mid A^{-1}]$

row reduction

zero row \rightarrow



$$\textcircled{1} \quad \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} \rightarrow$$

$$\textcircled{2} \quad c \times \longrightarrow$$

$$\textcircled{3} \quad \begin{array}{c} c \times \longrightarrow \\ \hline r + r \rightarrow \end{array}$$

REF

KREF \rightarrow

$$A \longrightarrow \text{KREF}(A)$$

⑥ Gauss-Jordan Elimination

연립방정식 $Ax = b$

invertible A $[A | b] \longrightarrow [I | \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix}]$
Augmented matrix

non-invertible A $[A | b] \longrightarrow [\dots | \dots]$

① inconsistent
 \Rightarrow no solution

② consistent
with many solutions

⑦ Find the inverse matrix A^{-1}

$$[A | I] \longrightarrow [I | A^{-1}]$$

연립방정식

$$Ax = b \quad \text{Non-homogeneous}$$

$$Ax = 0 \quad : \text{Homogeneous Linear System}$$

Finding Eigen Vectors

A

$$(A - \lambda I) p = 0$$

$$B p = 0$$

Homogeneous System

solution p eigenvector

~~$p = 0$~~ non zero : meaningful

$$\det(B) = 0 \Rightarrow \text{number } \lambda \Rightarrow (B)$$

|
eigenvalues

Gauss elimination

$$[B | \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix}] \rightarrow [\begin{array}{cccc|c} & & & & 0 \\ & & & & 0 \\ & & & & 0 \\ & & & & 0 \\ & & & & 0 \\ & & & & 0 \end{array}]$$

leading variables
free variables

↳ parameter

s, t, a, \dots

свободная переменная

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & 7 & 0 \\ 7 & -2 & -1 \end{bmatrix}$$

$$\det(A - \lambda I) = 0 \quad \left| \begin{array}{ccc|c} 1-\lambda & 2 & 1 & 0 \\ 6 & 7-\lambda & 0 & 0 \\ 7 & -2 & -1-\lambda & 0 \end{array} \right| = 0$$

$$\lambda(\lambda + 4)(\lambda - 3) = 0$$

① $\lambda = 0$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 6 & 7 & 0 & 0 \\ 7 & -2 & -1 & 0 \end{array} \right] \longrightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{13} & 0 \\ 0 & 1 & \frac{6}{13} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_1 + \frac{1}{13}x_3 = 0$$

$$x_3 = 13$$

$$x_1 = -1$$

$$x_2 + \frac{6}{13}x_3 = 0$$

$$x_2 = -6$$

$$(-1, -6, 13) \rightarrow (1, 6, -13)$$

$$\textcircled{2} \quad \lambda = -4$$

$$\left[\begin{array}{ccc|c} 1+4 & 2 & 1 & 0 \\ 6 & -4 & 0 & 0 \\ -1 & -2 & -4 & 0 \end{array} \right] \longrightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_1 + x_3 = 0$$

$$x_2 - 2x_3 = 0$$

$$x_3 = 1 \quad x_1 = -1$$

$$x_2 = 2$$

$$(-1, 2, 1) \rightarrow (1, -2, -1)$$

$$(3) \lambda = 3$$

$$\left[\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 6 & -4 & 0 & 0 \\ -1 & -2 & -4 & 0 \end{array} \right] \longrightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_1 + x_3 = 0$$

$$x_3 = 2$$

$$x_1 = -1$$

$$x_2 + \frac{3}{2}x_3 = 0$$

$$x_2 = -3$$

$$(-1, -3, 2) \rightarrow (1, 3, -2)$$