.

Conformal Mapping (H.1)

20160630

Mapping $\quad \omega = f(z)$

$\omega = f(z)$

Complex number $\xrightarrow{\quad}$ Complex number

$$\omega = f(z)$$
$$= u + iv$$

$$\omega = f(x + iy)$$
$$= u(x, y) + i\, v(x, y)$$

$$\boxed{z = x + iy} \longrightarrow \boxed{\omega = u + iv}$$
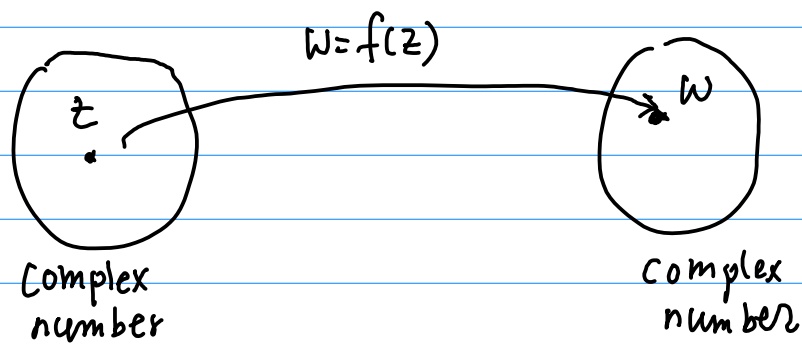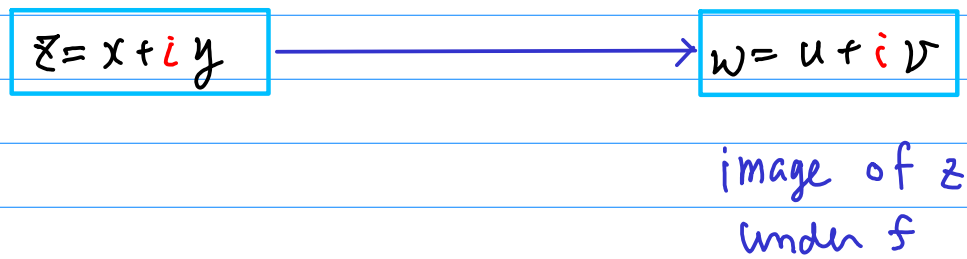
image of z under f

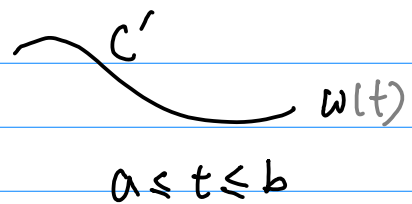# Images of Curves

$z = x + iy$ $\longrightarrow$ $w = u + iv$
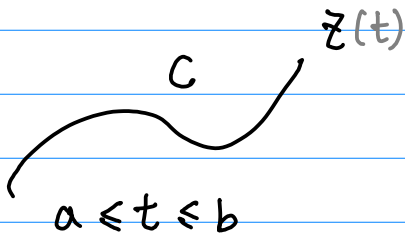
image of z
under f

$z(t) = x(t) + iy(t)$ $\qquad$ $w(t) = u(t) + iv(t)$

$$w = f(z(t))$$
$$= u(x(t), y(t)) + iv(x(t), y(t))$$

$z(t)$

C

$a \le t \le b$

C'

$w(t)$

$a \le t \le b$

$$f(z) = z + z_0 \qquad \text{translation}$$

$$g(z) = e^{i\theta_0} \cdot z \qquad \text{rotation}$$

$$h(z) = \alpha z \qquad \text{magnification}$$

α : fixed positive real number

Real Power Function

$$f(z) = z^{\alpha}$$

α: a fixed positive real number

z : $\quad 0 \leq \text{Arg}(z) \leq \theta_0$

$\omega = f(z)$: $\quad 0 \leq \text{Arg}(\omega) \leq \alpha \theta_0$

Angle Preserving Mappings

A complex mapping $w = f(z)$
defined on domain D

Conformal at $z = z_0$ in D

when f preserves the angle
between any 2 curves in D
that intersects at z_0

C_1 and C_2 intersect in D at z_0
the corresponding images C_1' and C_2'

the angle between C_1 and C_2
the angle between C_1' and C_2'

$$| z_1' - z_2' |^2 = | z_1'|^2 + | z_2'|^2 - 2 | z_1' | | z_2' | \cos \theta$$

$$\theta = \cos^{-1} \left(\frac{| z_1'|^2 + | z_2'|^2 - | z_1' - z_2' |^2}{2 | z_1' | | z_2' |} \right)$$

$$\varphi = \cos^{-1} \left(\frac{| w_1'|^2 + | w_2'|^2 - | w_1' - w_2' |^2}{2 | w_1' | | w_2' |} \right)$$

$f(z)$:   analytic in the domain D

$f'(z_0) \neq 0$ ⟹ f is conformal at $z = z_0$

$z = z(t)$ $w = f(z(t))$

$w' = f'(z(t)) \, z'(t)$

C_1 and C_2 intersect in D at z_0

$$w_1' = f'(z_0) \, z_1'$$
$$w_2' = f'(z_0) \, z_2'$$

$$\varphi = \cos^{-1}\left(\frac{|w_1'|^2 + |w_2'|^2 - |w_1' - w_2'|^2}{2 \, |w_1'| \, |w_2'|} \right)$$

$$= \cos^{-1}\left(\frac{|f'(z_0) z_1'|^2 + |f'(z_0) z_2'|^2 - |f'(z_0) z_1' - f'(z_0) z_2'|^2}{2 \, |f'(z_0) z_1'| \, |f'(z_0) z_2'|} \right)$$

$$= \cos^{-1}\left(\frac{|z_1'|^2 + |z_2'|^2 - |z_1' - z_2'|^2}{2 \, |z_1'| \, |z_2'|} \right)$$

$$= \theta$$

Transform Theorem for Harmonic Functions

f : analytic function $\quad\quad D \rightarrow D'$

u : harmonic in D'

the real-valued function $\quad\quad u(x, y) = u(f(z))$

harmonic in D

Linear Fractional Transformations

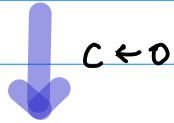$$T(z) = \frac{az + b}{cz + d}$$

$$T'(z) = \frac{ad - bc}{(cz + d)^2}$$

$$\lim_{z \to z_0} T(z) = \infty \qquad z_0 = -\frac{d}{c}$$

$$\lim_{|z| \to \infty} T(z) = \lim_{|z| \to \infty} \frac{a + b/z}{c + d/z} = \frac{a}{c}$$

Circle Preserving Property

$$T(z) = \frac{az + b}{cz + d}$$ Linear Fractional Trans.

↓ $c \leftarrow 0$

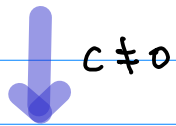$$T(z) = Az + B$$ Linear Trans

Linear Complex Mapping   Composite of $\begin{cases} \text{rotation} \\ \text{magnification} \\ \text{translation} \end{cases}$

circle in the z-plane $\longrightarrow$ circle in the w-plane

$$T(z) = \frac{az + b}{cz + d} \qquad \text{Linear Fractional Trans.}$$

$$\downarrow \quad c \neq 0$$

$$T(z) = \frac{bd - ad}{c} \, \frac{1}{cz + d} + \frac{a}{c}$$

$$= A \, \frac{1}{cz + d} + B \qquad\qquad z_1 = cz + d$$

$$= A \, \frac{1}{z_1} + B$$

$$= A \, z_2 + B \qquad\qquad z_2 = \frac{1}{z_1}$$

General Linear Fractional Trans.

composite of two linear functions
the inversion $w = 1/z$

$$|z - z_1| = r \qquad\qquad w = \frac{1}{z}$$

$$\left|\frac{1}{w} - \frac{1}{w_1}\right| = \frac{|w - w_1|}{|w||w_1|} = r$$

$$|w - w_1| = r|w_1||w - 0|$$

$$|w - w_1| = \lambda|w - w_2|$$

$\lambda = 1$ line

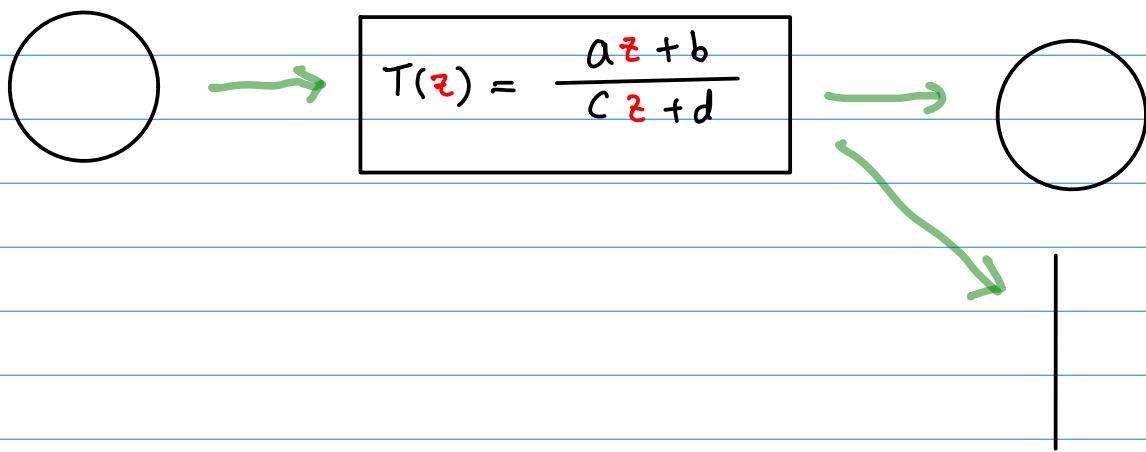$\lambda \neq 1, \lambda > 0$ circle

the image of the circle $|z - z_1| = r$
under the inversion $w = 1/z$
a circle except when $r = 1/|w_1| = |z_1|$

Circle - Preserving Property

A Linear Fractional Transformation

$$T(z) = \frac{az + b}{cz + d}$$

to solve   Dirichelet   problems
   construct special functions
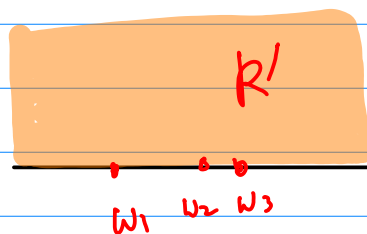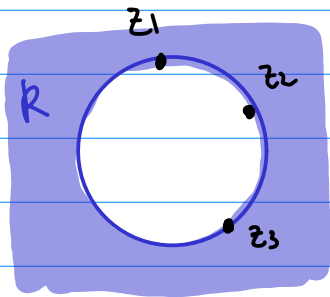   that map a given circular region R
         to a target region R'
            where the corresponding Dirichlet problem
               is solvable.

A Circular Boundary — by 3 points

Find   a linear frac trans $w = T(z)$

z_1, z_2, z_3 in R $\longrightarrow$ w_1, w_2, w_3 in R'

interior of R $\longrightarrow$ interior of R'

$$T(z) = \frac{az + b}{cz + d} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$T_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \qquad A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

$$T_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2} \qquad A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

$$\omega = T(z) = T_2(T_1(z)) \qquad A = A_2 A_1$$

$$z = T^{-1}(\omega) \qquad \qquad adj(A)$$

Triples of Triples

$$T(z) = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1} \qquad \begin{cases} T(z_1) = 0 \\ T(z_2) = 1 \\ T(z_3) = \infty \end{cases}$$

cross ratio of the complex numbers z, z_1, z_2, z_3

$$S(w) = \frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1} \qquad \begin{cases} S(w_1) = 0 \\ S(w_2) = 1 \\ S(w_3) = \infty \end{cases}$$

$$\begin{cases} S^{-1}(0) = w_1 \\ S^{-1}(1) = w_2 \\ S^{-1}(\infty) = w_3 \end{cases}$$

$$\begin{cases} T(z_1) = 0 \\ T(z_2) = 1 \\ T(z_3) = \infty \end{cases} \qquad \begin{cases} S^{-1}(0) = w_1 \\ S^{-1}(1) = w_2 \\ S^{-1}(\infty) = w_3 \end{cases}$$

$w = S^{-1}(T(z))$ maps the triple $z_1, z_2,$ and z_3
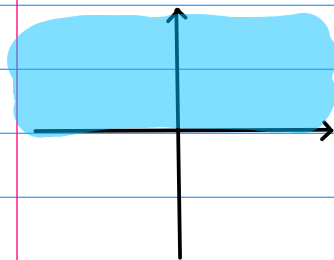to the triple $w_1, w_2,$ and w_3

$$S(w) = T(z)$$

$$\frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1} = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}$$

Riemann Mapping Theorem
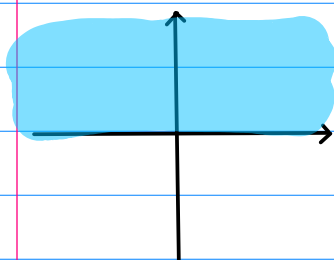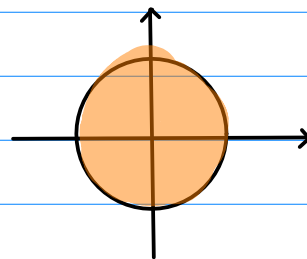
D' a simply connected domain
with at least one boundary point

the existence of an analytic function g
which conformaly maps
the unit open disk $|z| < 1$
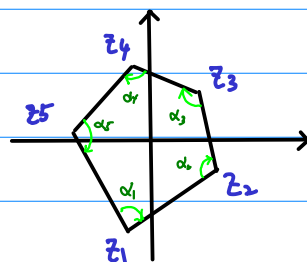onto D'

Lin. Frac. Trans

conformal map f

Lin. Frac. Trans

conformal map f

Schwarz - Christoffel Transformation

Unlike the Riemann mapping theorem

Can specify
$\qquad$ a form of for the **derivative $f'(z)$**
$\qquad\qquad$ of a conformal mapping **$f(z)$**
$\qquad\qquad$ from the upper half plane
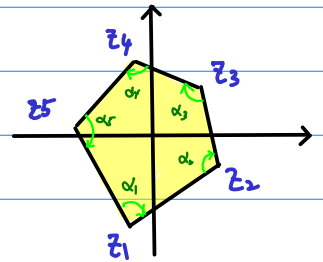$\qquad\qquad$ to a <u>bounded</u> or <u>unbounded</u> polygonal region
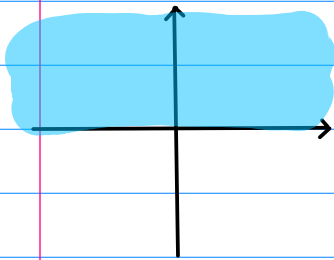
Lin. Frac. Trans

conformal map f

Bounded
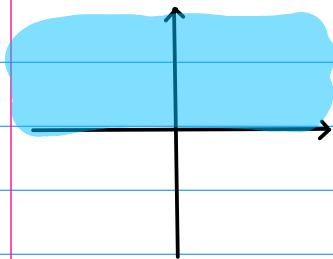
Lin. Frac. Trans

conformal map f
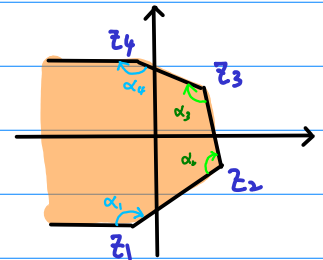
unbounded

mapping

$$f(z) = (z - x_1)^{\alpha/\pi} \quad , \quad 0 \le \alpha < 2\pi$$

on the upper half plane $\quad y \ge 0$

$$\zeta = (z - x_1)$$

$$\omega = \zeta^{\alpha/\pi}$$

$$[0, \pi]$$
$$\downarrow \quad \downarrow$$
$$[0, \alpha]$$

$$f'(z) = \frac{\alpha}{\pi} (z - x_1)^{\alpha/\pi - 1}$$
$$= A (z - x_1)^{\alpha/\pi - 1}$$

$$f(z) = (z - x_1)^{\alpha/\pi} \qquad 0 \le \alpha < 2\pi$$

z ζ ω

$$\zeta = (z - x_1) \qquad\qquad \omega = \zeta^{\alpha/\pi}$$

(Arg) $[0, \pi]$

$\downarrow \quad \downarrow$

$[0, \alpha]$
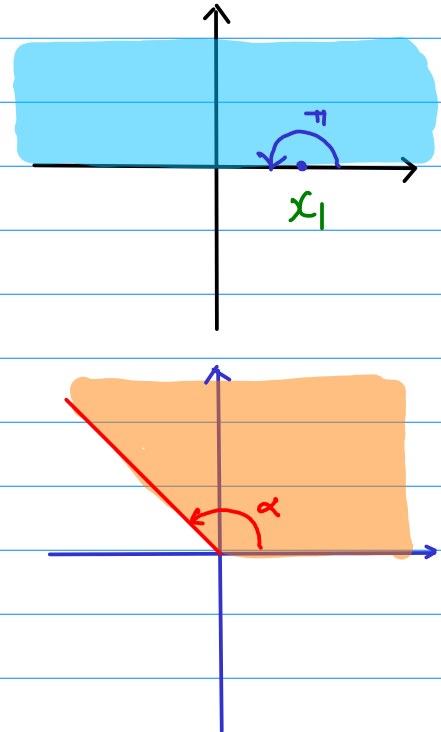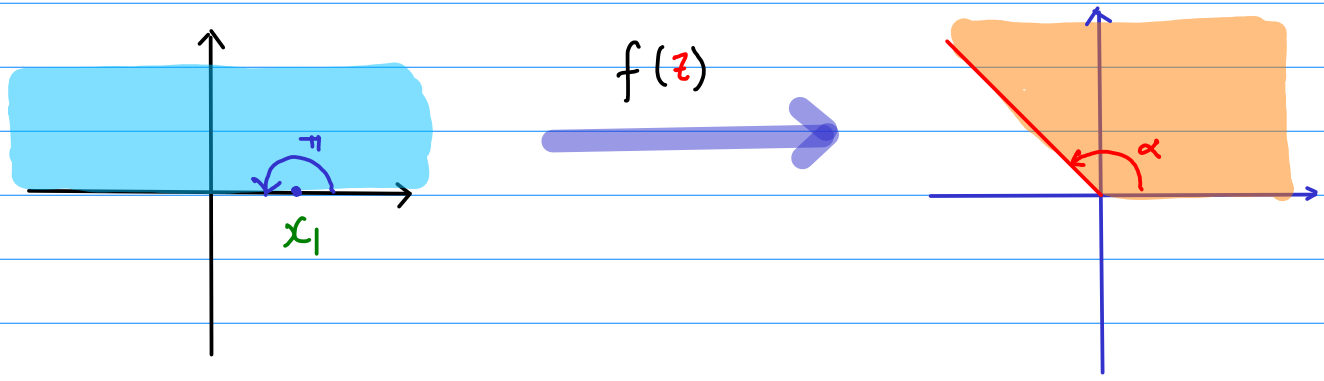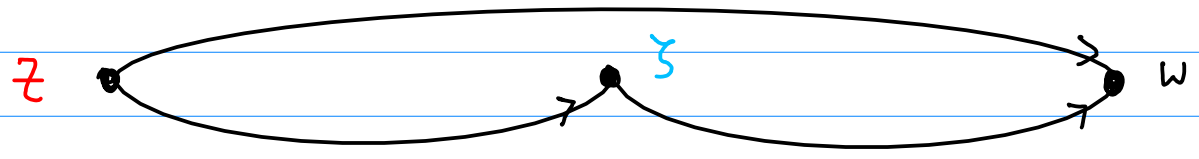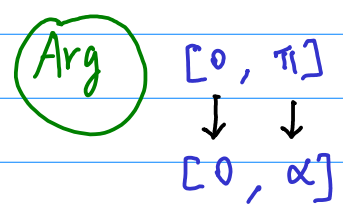
$$f(z) = (z - x_1)^{\alpha/\pi} \qquad 0 \le \alpha < 2\pi$$

$$f'(z) = \frac{\alpha}{\pi} (z - x_1)^{\alpha/1}$$

$$= A (z - x_1)^{\alpha/\pi - 1}$$

Assum $f(z)$ analytic in the upper half plane $(y \geqslant 0)$

$$f'(z) = A \ (z - x_1)^{\frac{\alpha_1}{\pi} - 1} \ (z - x_2)^{\frac{\alpha_2}{\pi} - 1}$$

$$x_1 < x_2$$

$\arg (w'(t)) = \text{const} \quad \Longrightarrow$

a   curve $w = w(t)$: a line
in the   w-plane

$$f'(t) = A \ (t - x_1)^{\frac{\alpha_1}{\pi} - 1} \ (t - x_2)^{\frac{\alpha_2}{\pi} - 1}$$

$$\arg \left\{ f'(t) \right\} = \text{Arg}(A)$$
$$+ \left(\frac{\alpha_1}{\pi} - 1 \right) \text{Arg} (t - x_1)$$
$$+ \left(\frac{\alpha_2}{\pi} - 1 \right) \text{Arg} (t - x_2)$$

(X)

(O)
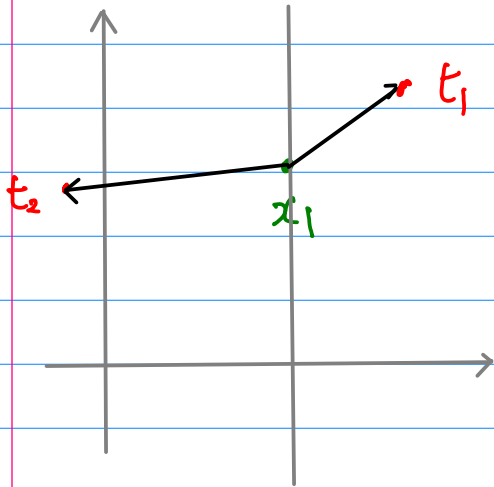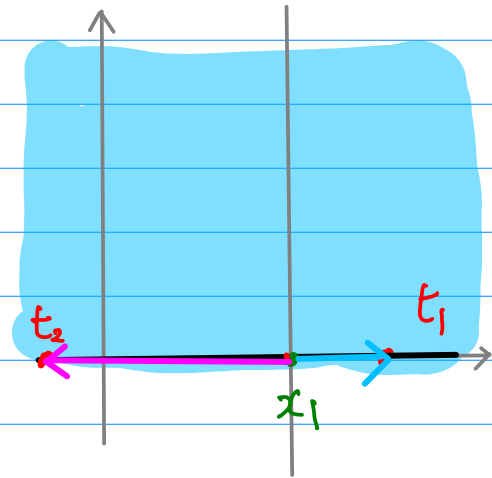
$$\begin{cases} t < x_1 & Arg(t - x_1) = \pi \\ t > x_1 & Arg(t - x_1) = 0 \end{cases}$$

$$\begin{cases} t < x_2 & Arg(t - x_2) = \pi \\ t > x_2 & Arg(t - x_2) = 0 \end{cases}$$

① x_1 ② x_2 ③

$t < x_1$ $x_1 < t < x_2$ $t > x_2$

$\text{Arg}(t - x_1) = \pi$ $\text{Arg}(t - x_1) = 0$ $\text{Arg}(t - x_1) = 0$

$\text{Arg}(t - x_2) = \pi$ $\text{Arg}(t - x_2) = \pi$ $\text{Arg}(t - x_2) = 0$

$\text{arg}\{f'(t)\}$

$\text{Arg}(A)$
$+ (\alpha_1 - \pi)$
$+ (\alpha_2 - \pi)$

const

$\text{Arg}(A)$
$+ \; 0$
$+ (\alpha_2 - \pi)$

const

$\text{Arg}(A)$
$+ \; 0$
$+ \; 0$

const

$+ (\pi - \alpha_1)$ $+ (\pi - \alpha_2)$

① x_1 ② x_2 ③

$t < x_1$ $x_1 < t < x_2$ $t > x_2$
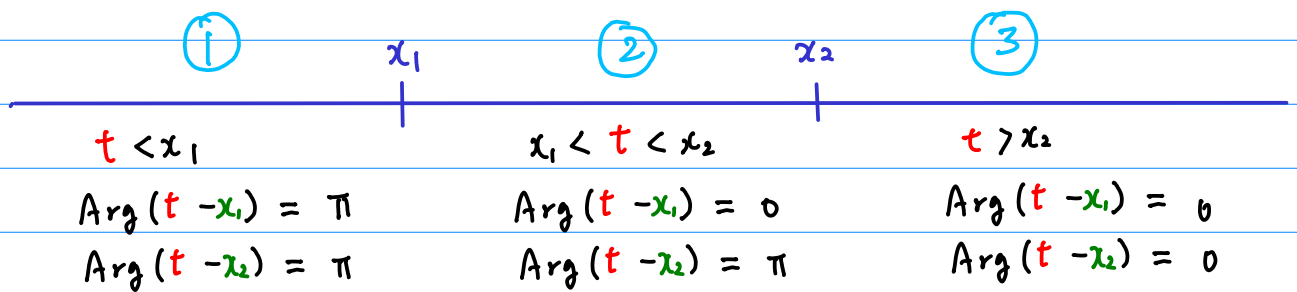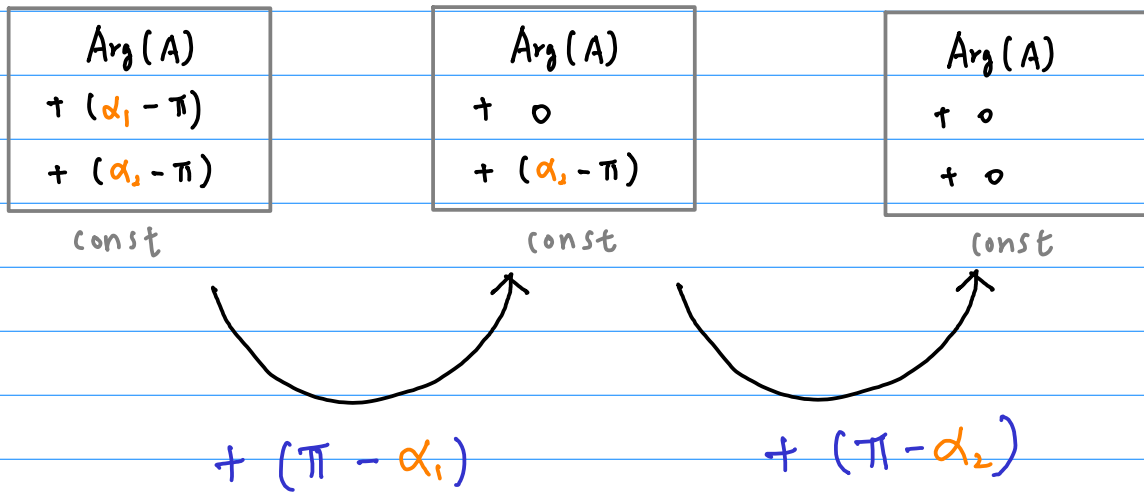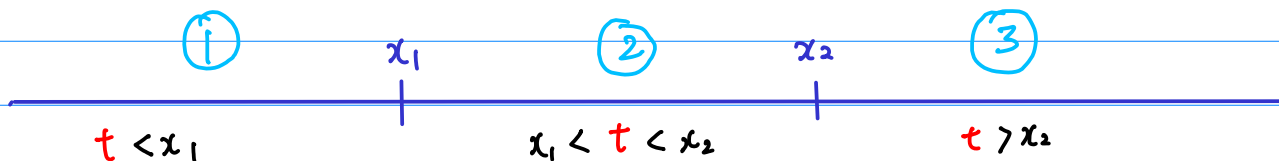
$Arg(A) + (\alpha_1 - \pi) + (\alpha_2 - \pi)$ $Arg(A) + (\alpha_2 - \pi)$ $Arg(A)$

$$arg\{f'(t)\} = \begin{cases} Arg(A) + (\alpha_1 - \pi) + (\alpha_2 - \pi) & [\; t < x_1 \;] \\ Arg(A) + (\alpha_2 - \pi) & [\; x_1 < t < x_2 \;] \\ Arg(A) & [\; t > x_2 \;] \end{cases}$$

$\boxed{\omega = f(t)}$

②

③ $\pi - \alpha_2$

$\pi - \alpha_1$

①

$\omega = f(t) \quad x_1 < t < x_2$

$\pi - \alpha_2$

$\omega = f(t)$ ③
$t > x_2$

α_2

α_1 $\pi - \alpha_1$

① $\omega = f(t)$ ②

$t < x_1$
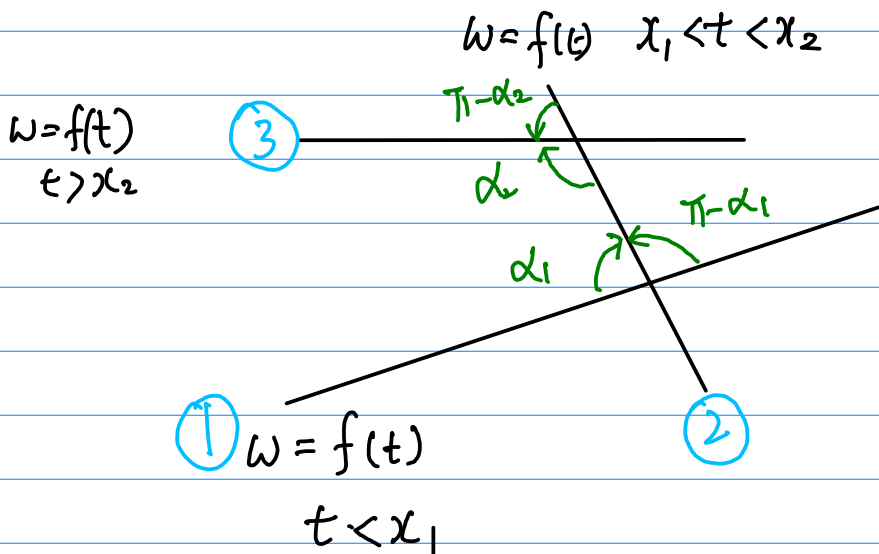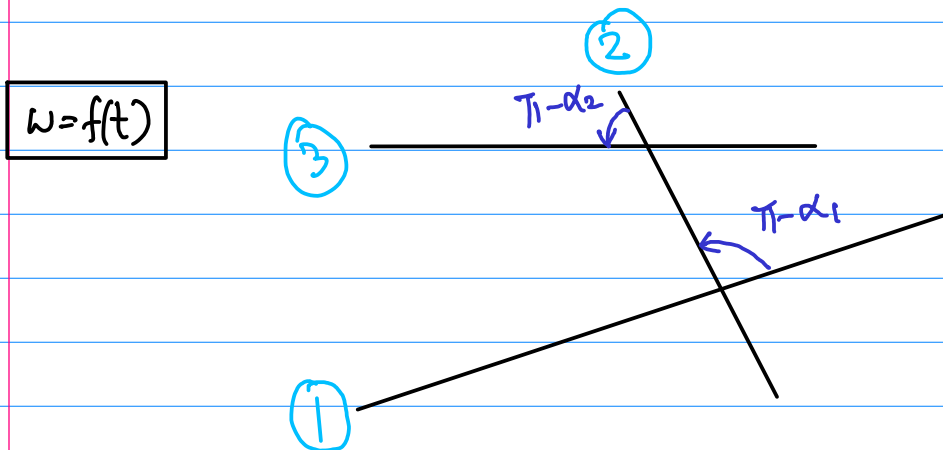
Schwarz - Christoffel Formula

$f(z)$ analytic in the upper half-plane $y > 0$

$$f'(z) = A(z - x_1)^{\alpha_1/\pi - 1}(z - x_2)^{\alpha_2/\pi - 1} \cdots (z - x_n)^{\alpha_n/\pi - 1}$$

$$x_1 < x_2 < \cdots < x_n$$

$$0 < \alpha_i < 2\pi \quad i = 1, \cdots, n$$

$f(z)$ maps the upper half-plane $y > 0$
to a polygonal region
with interior angles $\alpha_1, \alpha_2, \cdots, \alpha_n$

① can select the location of 3 points
x_k on the x-axis

careful selection simplifies the computation of $f(z)$

other remaining points depends on the shape of
the target polygon

② A general formula for $f(z)$

$$f'(z) = A(z-x_1)^{\alpha_1/\pi - 1}(z-x_2)^{\alpha_2/\pi - 1} \cdots (z-x_n)^{\alpha_n/\pi - 1}$$

$$f(z) = A\int \boxed{(z-x_1)^{\alpha_1/\pi - 1}(z-x_2)^{\alpha_2/\pi - 1} \cdots (z-x_n)^{\alpha_n/\pi - 1}} \, dz + B$$

$$= A \; g(z) \; + B$$

$$g(z) = \int \boxed{(z-x_1)^{\alpha_1/\pi - 1}(z-x_2)^{\alpha_2/\pi - 1} \cdots (z-x_n)^{\alpha_n/\pi - 1}} \, dz$$

$$h(z) = Az + B$$
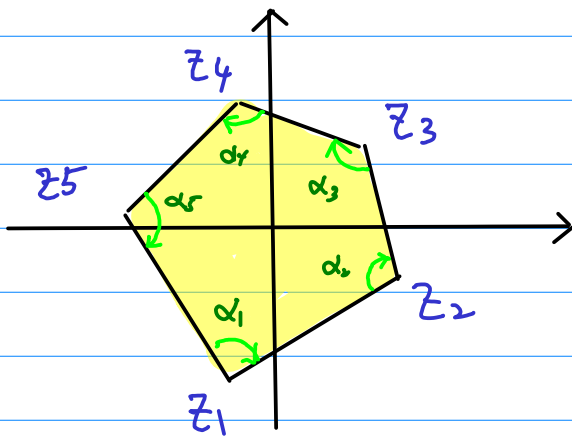
$\begin{cases} \text{magnify} \\ \text{rotate} \\ \text{translate} \end{cases}$ the image polygon produced by $g(z)$

③ If the polygon regions are bounded

only n-1 of the n interior angles
Should be included in the Swartz-Christoffel formula
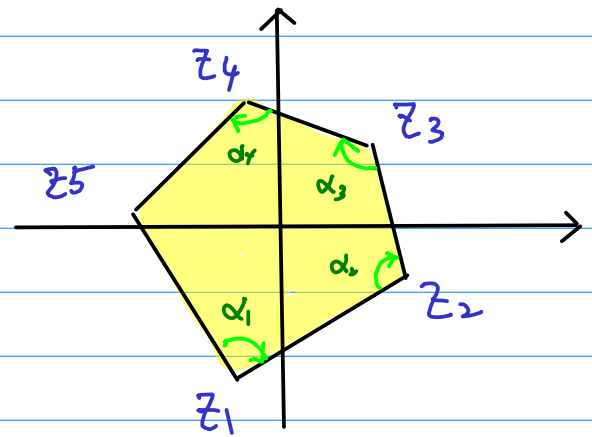
Bounded   Bounded

$\alpha_1, \alpha_2, \alpha_3, \alpha_4$
are sufficient to
determine Schwartz-
Christoffel formula

Laplace Equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 < x < a \qquad 0 < y < b$$

Neuman Condition | (BC) $\quad \dfrac{\partial u}{\partial x}\Big|_{x=0} = 0, \qquad \dfrac{\partial u}{\partial x}\Big|_{x=a} = 0 \qquad 0 < y < b$

Dirichlet Condition | (BC) $\quad u(x, 0) = 0, \qquad u(x, b) = f(x) \qquad 0 < x < a$

Dirichlet Condition | (BC) $\quad u(0, y) = 0, \qquad u(a, y) = 0 \qquad 0 < y < b$

Dirichlet Condition | (BC) $\quad u(x, 0) = 0, \qquad u(x, b) = f(x) \qquad 0 < x < a$

$$u(x, b) = f(x)$$

$$u(0, y) = 0 \qquad\qquad\qquad u(a, y) = 0$$

$$u(x, 0) = 0$$

Dirichlet Problem: Superposition

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 < x < a, \qquad 0 < y < b$$

$$u(0, y) = F(y) \qquad\qquad u(a, y) = G(y) \qquad\qquad 0 < y < b$$

$$u(x, 0) = f(x) \qquad\qquad u(x, b) = g(x) \qquad\qquad 0 < x < a$$

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0 \qquad 0 < x < a, \qquad 0 < y < b$$

$$u_1(0, y) = 0 \qquad\qquad u_1(a, y) = 0 \qquad\qquad 0 < y < b$$

$$u_1(x, 0) = f(x) \qquad\qquad u_1(x, b) = g(x) \qquad\qquad 0 < x < a$$

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0 \qquad 0 < x < a, \qquad 0 < y < b$$

$$u_2(0, y) = F(y) \qquad\qquad u_2(a, y) = G(y) \qquad\qquad 0 < y < b$$

$$u_2(x, 0) = 0 \qquad\qquad u_2(x, b) = 0 \qquad\qquad 0 < x < a$$

Harmonic Functions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad\qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

u v

the (real) & (imaginary) parts of an analytic function
cannot be chosen arbitrarily

both u and v must satisfy Laplace eq

the real-valued function $\phi(x, y)$
  — has continuous
        2nd order partial derivatives
                  in a domain D

  — satisfies Laplace eq

⟷ $\phi(x, y)$: harmonic in D

$$f(z) = u(x, y) + i\,v(x, y) \qquad \underrightarrow{\qquad} \atop \overleftarrow{\qquad}$$

$\boxed{\text{analytic}}$ in domain D

the function $u(x, y)$ & $v(x, y)$ are $\boxed{\text{harmonic}}$ functions

Harmonic Conjugate Functions

⑥ $f(z) = u(x,y) + iv(x,y)$ analytic in D

$\Rightarrow$ u, v: harmonic in D

⑥   given $u(x,y)$: harmonic in D

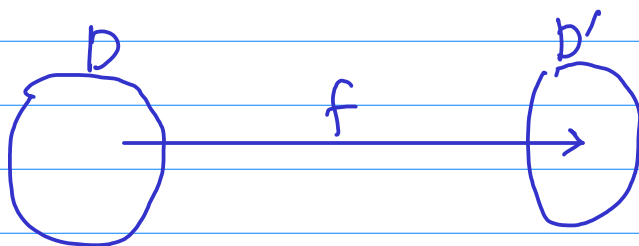find $v(x,y)$: harmonic in D

such that $u(x,y) + iv(x,y)$: analytic in D

   u, v: harmonic conjugate function
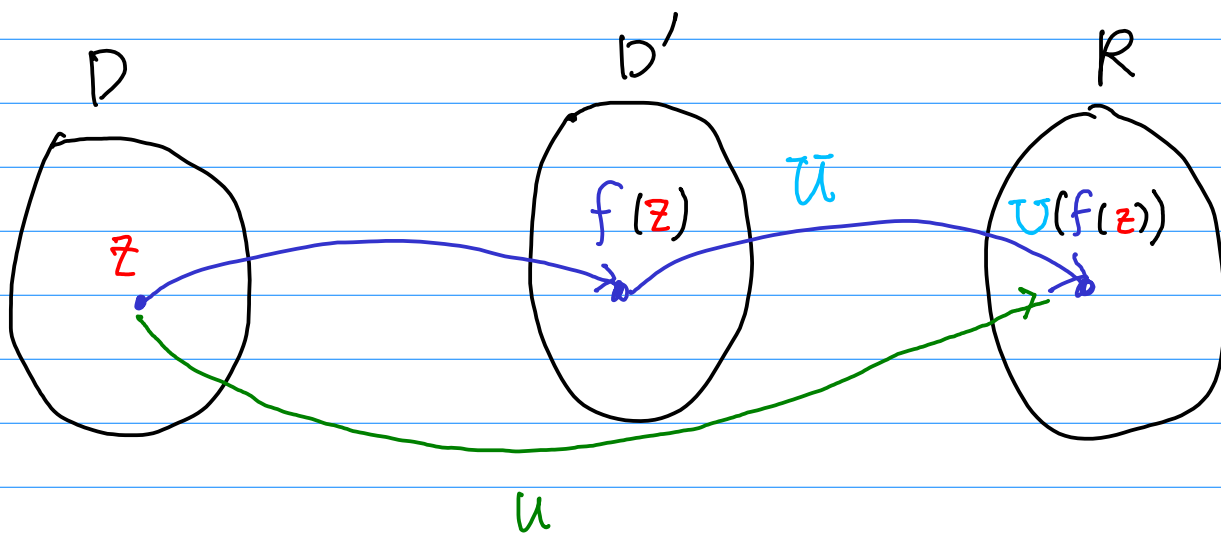
Harmonic Functions & the Dirichlet Problem

Transformation Theorem for Harmonic Functions

f : analytic function
that maps a domain D onto a domain D'

real valued function $u(x,y) = U(f(z))$

u is harmonic in D ⬅ U is harmonic in D'

① find conformal mapping $\quad w = f(z)$

the original region Ⓡ $\longrightarrow$ the image region Ⓡ′
$$\downarrow$$
many explicit solutions
to the Dirichlet
problems are known

② Transfer the boundary conditions

the boundary of Ⓡ $\longrightarrow$ the boundary of Ⓡ′

the value of u
at a boundary
point ξ

the value of U
at a boundary
point $f(\xi)$

③ Solve the corresponding Dirichlet problem in R′

④ the solution to the original Dirichlet problem

$$u(x,y) = U(f(z))$$

Poisson Integral Formulas

Dirichlet problems

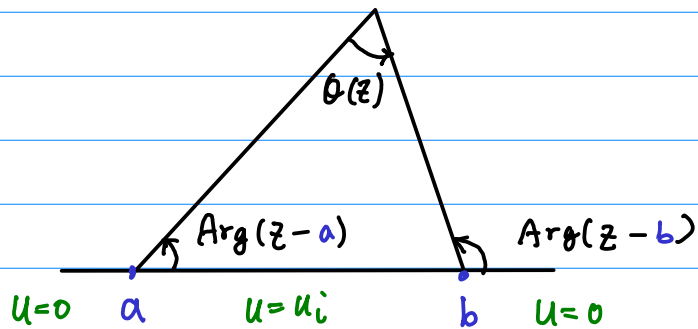in the upper half plane $y \geqslant 0$

in the unit disk $|z| \leqslant 1$

The Poisson integral formula
for the upper half plane

a solution by expressing
the value of a harmonic function $u(x, y)$
at a point in the interior of
the upper half plane
in terms of on the boundary $y = 0$

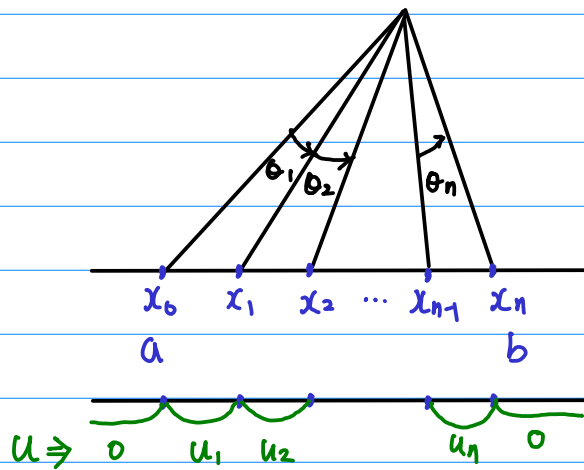# Formulas for the upper half plane

Boundary conditions on $y = 0$

$U = 0 \quad a \qquad U = u_i \qquad b \quad U = 0$

$$U(x, y) = \frac{u_i}{\pi} \left[Arg(z - b) - Arg(z - a) \right]$$

$$u(x, y) = \frac{u_i}{\pi} Arg\left(\frac{z - b}{z - a} \right)$$

$$U(x, 0) = U_i \qquad x_{i-1} \leqslant x \leqslant x_i$$

$$U(x, 0) = 0 \qquad x < x_{i-1}, \quad x_i < x$$

$$U(x, y) = \sum_{i=1}^{n} \frac{U_i}{\pi} \left[\text{Arg}(z - x_i) - \text{Arg}(z - x_{i-1}) \right]$$

$$= \frac{1}{\pi} \sum_{i=1}^{n} U_i \, \theta_i(z)$$

$$\text{Arg}(z - t) = \tan^{-1}\left(\frac{y}{x - t}\right) \qquad [0, \pi]$$

$$\frac{d}{dt} \text{Arg}(z - t) = \frac{y}{(x-t)^2 + y^2}$$

$$U(x, y) = \frac{1}{\pi} \sum_{i=1}^{n} U_i \left[\text{Arg}(z - x_i) - \text{Arg}(z - x_{i-1}) \right]$$

$$= \frac{1}{\pi} \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} U_i \left[\frac{d}{dt} \text{Arg}(z - t) \right] dt$$

$$= \frac{1}{\pi} \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} U_i \left[\frac{y}{(x-t)^2 + y^2} \right] dt$$

$$U(x, y) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \left[\frac{U(t, 0)}{(x-t)^2 + y^2} \right] dt$$

Poisson Integral Formula for the UHP

$U(x, 0)$ piecewise   continuous function on every finite interval bounded on $-\infty < x < +\infty$

$$U(x, y) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{U(t, 0)}{(x-t)^2 + y^2} \, dt$$

$\Rightarrow$ the solution of the corresponding Dirichlet problem on the upper half-plane $y \geqslant 0$

Formula for the Unit Disc

$u(e^{i\theta})$ bounded and piecewise continuous
for $-\pi \le \theta \le +\pi$

Then the solution to the corresponding Dirichlet problem
on the open unit disk $|z| < 1$

$$u(x, y) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} u(e^{it}) \frac{1 - |z|^2}{|e^{it} - z|^2} \, dt$$

Geometric Interpretation

$$a \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial^2 u}{\partial t^2}$$

Fourier Series Form

$$U_n(r, \theta) = r^n \cos n\theta$$
$$U_n(r, \theta) = r^n \sin n\theta$$

a_0, a_n, b_n : the Fourier coefficients of $u(e^{i\theta})$
$$-\pi \leq \theta \leq +\pi$$

Superposition principle

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n r^n \cos n\theta + b_n r^n \sin n\theta \right) \quad \text{harmonic}$$

$$u(1, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta \right) = u(e^{i\theta})$$

the solution of the Dirichlet problem

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} u(e^{i\theta}) \frac{1 - |z|^2}{|e^{it} - z|^2} \, dt$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n r^n \cos n\theta + b_n r^n \sin n\theta \right)$$