Initial Value Problems (4A)

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Initial Value Problem

$$\frac{d^n y}{d x^n} = f(x, y, y', \cdots, y^{(n-1)})$$

on some interval *I* containing x_0

General Form

$$\frac{d^{n-1}}{d x^{n-1}} y(x_0) = k_{n-1}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\frac{d}{d x} y(x_0) = y'(x_0) = k_1$$

$$y(x_0) = y(x_0) = k_0$$

n **Initial Conditions**

at $x = x_0$

IVP

Initial Value Problem – variable coefficients

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)\, y = g(x)$$

Linear Equation
with <u>variable</u> coefficients

$$\frac{d^{n-1}}{dx^{n-1}} y(x_0) = k_{n-1}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\frac{d}{dx} y(x_0) = y'(x_0) = k_1$$

$$y(x_0) = y(x_0) = k_0$$

n Initial Conditions

at $x = x_0$

IVP

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Initial Value Problem – constant coefficients

n +1

$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

n

Linear Equation
with constant coefficients

n

$$\frac{d^{n-1}}{dx^{n-1}} y(x_0) = y^{(n-1)}(x_0) = k_{n-1}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\frac{d}{dx} y(x_0) = y'(x_0) = k_1$$

$$y(x_0) = y(x_0) = k_0$$

n Initial Conditions at $x = x_0$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x} + y_p(x)$$

n Parameters c_i

Boundary Value Problem

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$\begin{cases} y(a) = y_0 \\ y(b) = y_1 \end{cases} \quad \begin{cases} y'(a) = y_0 \\ y(b) = y_1 \end{cases} \quad \begin{cases} y(a) = y_0 \\ y'(b) = y_1 \end{cases} \quad \begin{cases} y'(a) = y_0 \\ y'(b) = y_1 \end{cases}$$

Various Boundary Conditions

1st Order 2nd Order IVP's

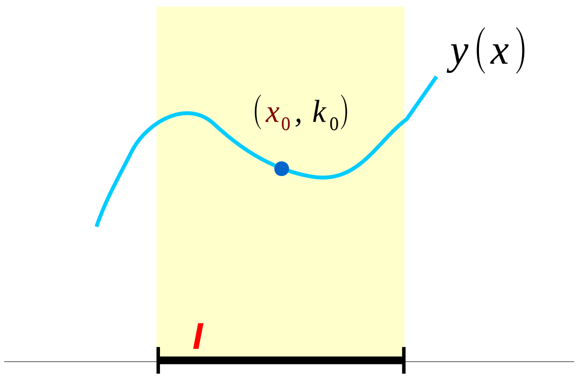
$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = k_0$$

*on some interval **I** containing x_0*

1 Initial Condition at $x = x_0$

1st Order IVP



$$\frac{dy}{dx} = f(x, y, y')$$

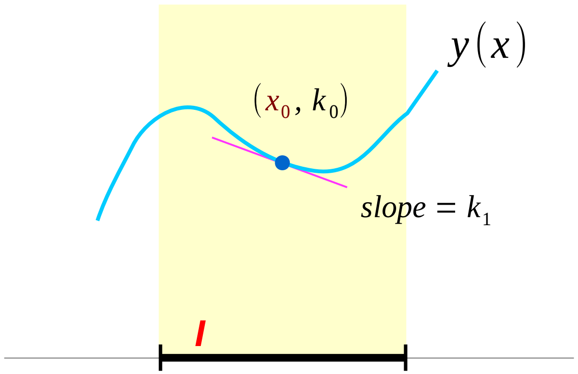
$$y(x_0) = k_0$$

$$y'(x_0) = k_1$$

*on some interval **I** containing x_0*

2 Initial Conditions at $x = x_0$

2nd Order IVP



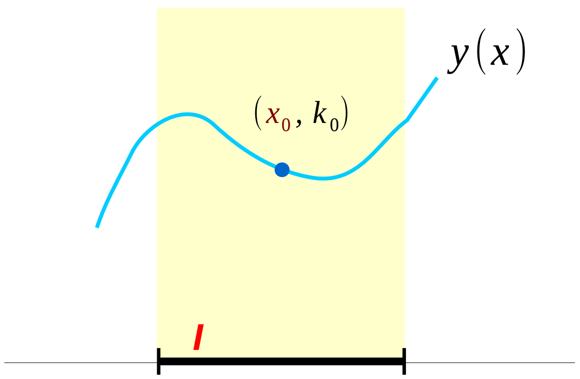
$$\frac{dy}{dx} = f(x, y)$$

on some interval ***I*** containing x_0

$$y(x_0) = k_0$$

1 Initial Condition at $x = x_0$

1st Order IVP



$f(x, y)$ **and** $\frac{\partial f}{\partial y}$ ***are continuous on R***

R
$$a \le x \le b$$
$$c \le y \le d$$

⇒

The solution y(x) of the IVP
1) exists on the interval I0
2) is unique

I0 $x_0 - h \le x \le x_0 + h \quad (h > 0)$

contained in [a, b]

Existence of a unique solution : Linear 1st Order IVPs

$$a_1(x)\frac{dy}{dx} + a_0(x)\, y = g(x)$$

on some interval ***I*** containing x_0

$$y(x_0) = k_0$$

1 Initial Condition at $x = x_0$

1st Order IVP

Non-homogeneous Equation with variable coefficients

$$a_1(x),\ a_0(x),\ g(x)$$

are all continuous on the interval I

and $a_n(x) \neq 0$

→

The solution y(x) of the IVP

1) exists on the interval I

2) is unique

Existence of a unique solution : Linear 1st Order IVPs

$$\frac{dy}{dx} + p(x)\, y = g(x)$$

*on some interval **I** containing **x_0***

$$y(x_0) = k_0$$

1 Initial Condition at $x = x_0$

1st Order IVP

Non-homogeneous Equation with variable coefficients

$p(x)$, $g(x)$

are all continuous on the interval I

→

The solution y(x) of the IVP

1) exists on the interval I

2) is unique

Existence : Proof

$$y' + p(x)\ y = g(x) \qquad y(x_0) = k_0$$

$p(x)$ **continuous on the interval I**

$\Rightarrow$ $\int_{x_0}^{x} p(s)ds$ **differentiable**

$$\frac{d}{dx}\int_{x_0}^{x} p(s)ds = p(x)$$

$$\mu(x) = e^{\int_{x_0}^{x} p(s)ds}$$

$$\mu'(x) = e^{\int_{x_0}^{x} p(s)ds}\, p(x)$$

$$(\mu(x)y)' = \mu'(x)y + \mu(x)y'$$

$$= \mu(x)p(x)y + \mu(x)y'$$

$$(\mu(x)y)' = \mu(x)g(x)$$

$$[\mu(x)y]_{x_0}^{x} = \int_{x_0}^{x} \mu(s)g(s)ds$$

$$\mu(x)y(x) - \mu(x_0)y(x_0) = \int_{x_0}^{x} \mu(s)g(s)ds$$

$$\mu(x)y(x) - y(x_0) = \int_{x_0}^{x} \mu(s)g(s)ds$$

$$y(x) = \frac{1}{\mu(x)}\left\{y(x_0) + \int_{x_0}^{x} \mu(s)g(s)ds\right\}$$

http://faculty.atu.edu/mfinan/3243/diffq1book.pdf

Uniqueness : Proof

$$y_1' + p(x)\ y_1 = g(x) \qquad y_1(x_0) = k_0$$

$$y_2' + p(x)\ y_2 = g(x) \qquad y_2(x_0) = k_0$$

$$w(x) = y_1(x) - y_2(x)$$

$$w' + p(x)\ w = 0$$

$$\mu(x) = e^{\int_{x_0}^{x} p(s)\,ds}$$

$$\mu(x) w' + \mu(x) p(x)\ w = 0$$

$$(\mu(x) w)' = 0$$

$$\mu(x) w(x) = C$$

$$w(x) = C/\mu(x)$$

$$w(x) = C e^{-\int_{x_0}^{x} p(s)\,ds}$$

$$w(x_0) = y_1(x_0) - y_2(x_0) = k_0 - k_0 = 0$$

$$C = 0$$

$$w(x) = 0$$

http://faculty.atu.edu/mfinan/3243/diffq1book.pdf

1^{st} Order IVP Counter examples (1)

$y' = \lvert y \rvert$	$y(0) = y_0$	***IVP***

$y > 0$	$y < 0$
$y' = y$	$y' = -y$
$\int \frac{1}{y} dy = \int dx$	$\int \frac{1}{y} dy = -\int dx$
$\ln y = x + c$	$\ln y = -x + c$
$y = e^{x+c}$	$y = e^{-x+c}$
$y = Ce^{x}$	$y = Ce^{-x}$

$f(x,y) = f(y) = \lvert y \rvert$ ***continuous***

$$\frac{\partial f}{\partial y} = \frac{df}{dy}$$

discontinuous
over any interval containing y = 0

a unique solution
for [y > 0] , [y = 0], [y < 0]

$f(x,y)$ ***and*** $\frac{\partial f}{\partial y}$ ***are continuous on R***

R $a \le x \le b$, $c \le y \le d$

The solution y(x) of the IVP
1) exists on the interval I0
2) is unique

I0 $x_0 - h \le x \le x_0 + h \quad (h>0)$

contained in [a, b]

1^{st} Order IVP Counter examples (2)

$y' = y^{1/3}$ $\quad y(0) = 0$ ***IVP***

$\int y^{-1/3}\,dy = \int dx$	$y^{2/3} = \frac{2}{3}x$
$\frac{3}{2}y^{2/3} = x + c$	$y^2 = \left(\frac{2}{3}x\right)^3$
$y = 0 \rightarrow$ $c = 0$	$y = \pm\left(\frac{2}{3}x\right)^{3/2}$

$f(x, y) = f(y) = y^{1/3}$ ***continuous***

$\frac{\partial f}{\partial y} = \frac{df}{dy} = \frac{1}{3y^{2/3}}$ ***discontinuous***

two possible solutions + {y = 0}

$f(x, y)$ ***and*** $\frac{\partial f}{\partial y}$ ***are continuous on R***

R $a \le x \le b$, $c \le y \le d$

$\Rightarrow$

The solution y(x) of the IVP
1) exists on the interval I0
2) is unique

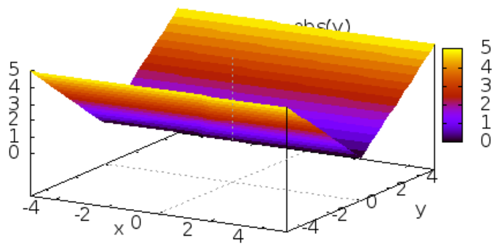
I0 $x_0 - h \le x \le x_0 + h \quad (h > 0)$

contained in [a, b]

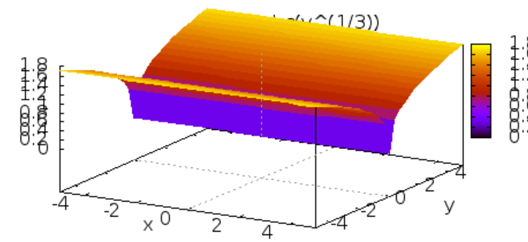
1st Order IVP Counter examples (3)

$y' = |y| \qquad y(0) = y_0$ ***IVP***

$y' = y^{1/3} \qquad y(0) = 0$ ***IVP***



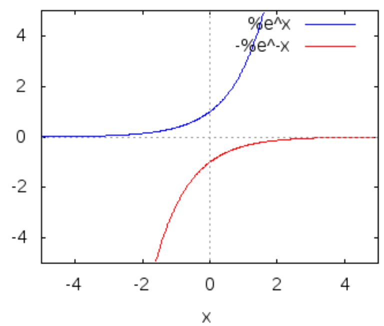
$f(x, y) = |y|$



$f(x, y) = y^{1/3}$

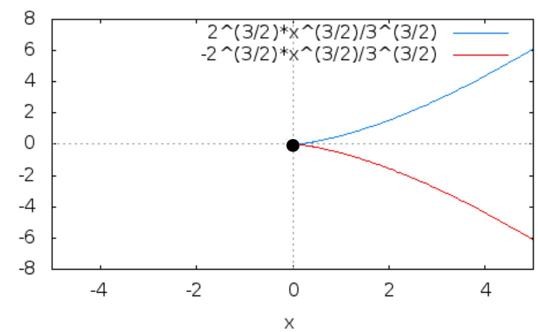
$\frac{\partial f}{\partial y}$ ***discontinuous***
over any interval
containing y = 0

$\frac{\partial f}{\partial y}$ ***discontinuous***
over any interval
containing y = 0



$y = C e^{x}$
$y = C e^{-x}$

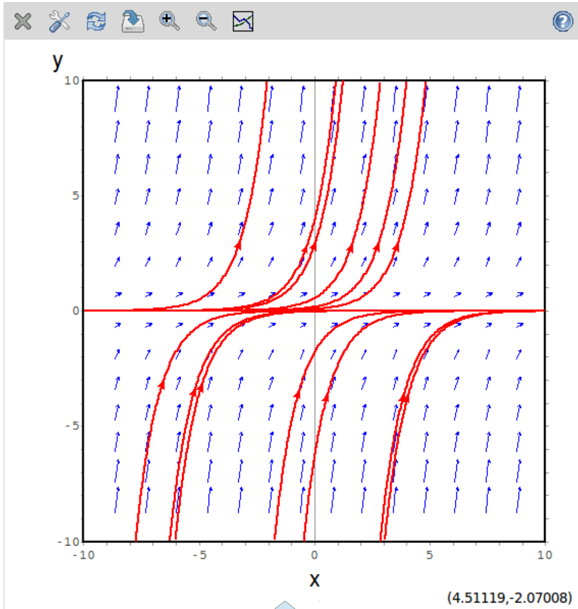
a unique solution
for [y > 0] , [y = 0], [y < 0]



$y = \pm\left(\frac{2}{3}x\right)^{3/2}$

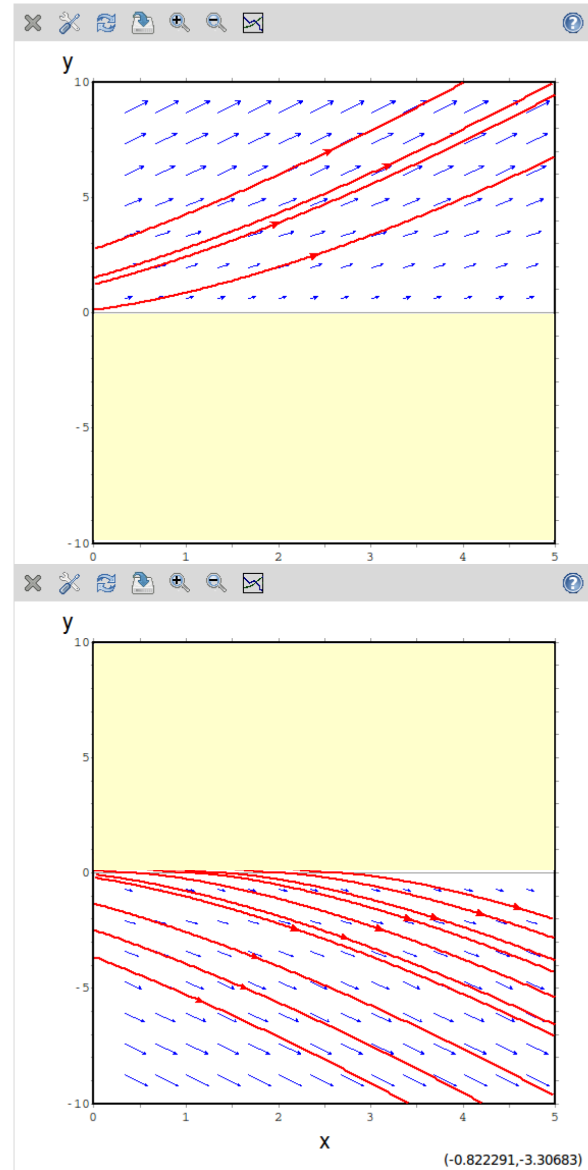
non-unique
solutions

1st Order IVP Counter examples (4)



$$y' = |y| \qquad y(0) = y_0 \qquad \textbf{IVP}$$

$$y' = y^{1/3} \qquad y(0) = 0 \qquad \textbf{IVP}$$

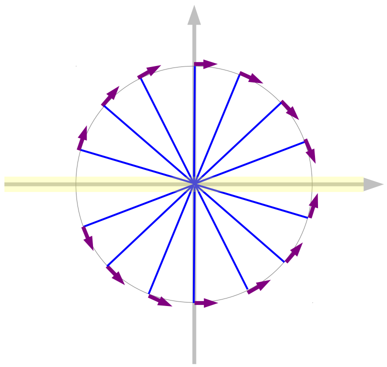


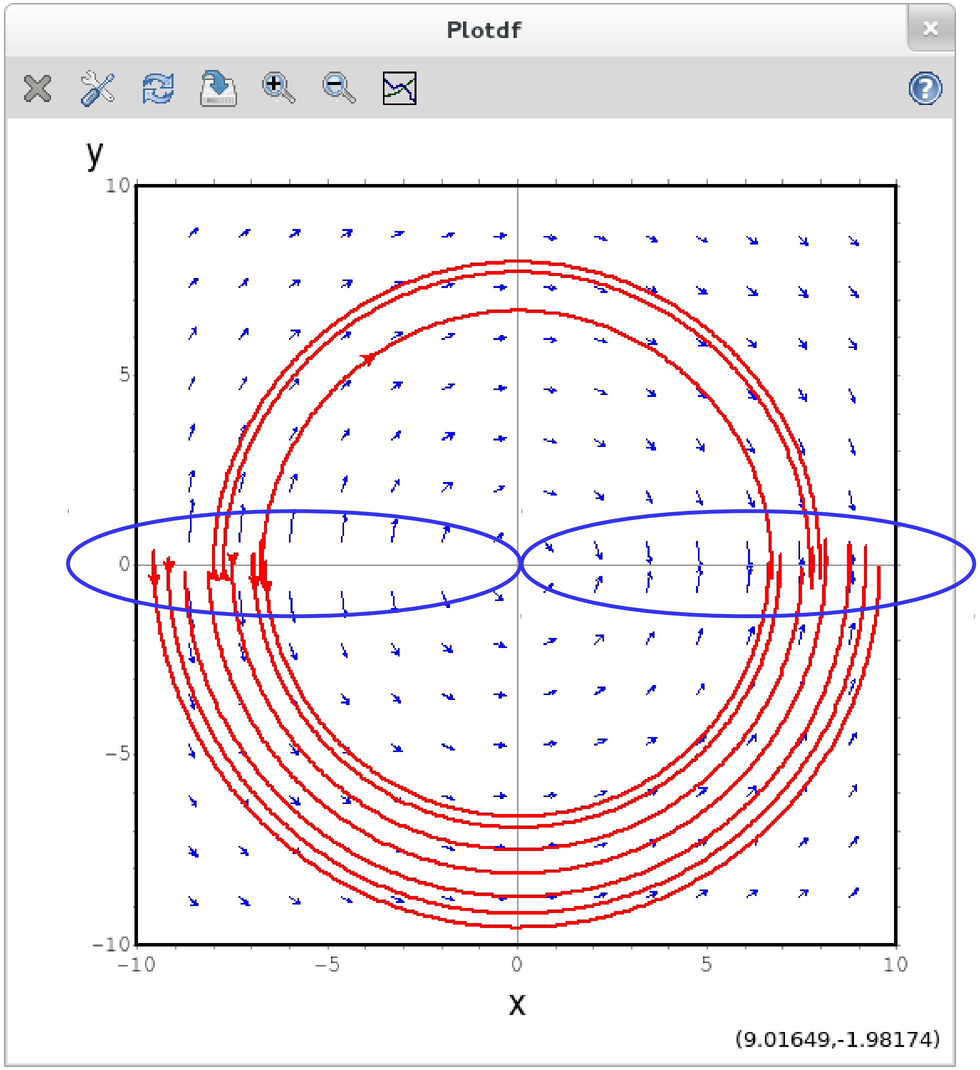
Direction Field of (– x/y)

$$\frac{dy}{dx} = -\frac{x}{y}$$

2-d version of F(x,y)

$$F(x, y) = -\frac{x}{y}$$



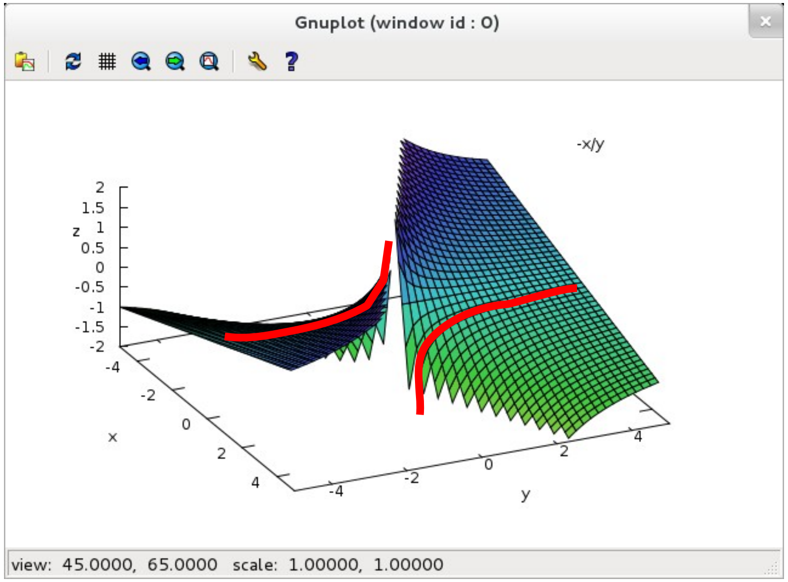


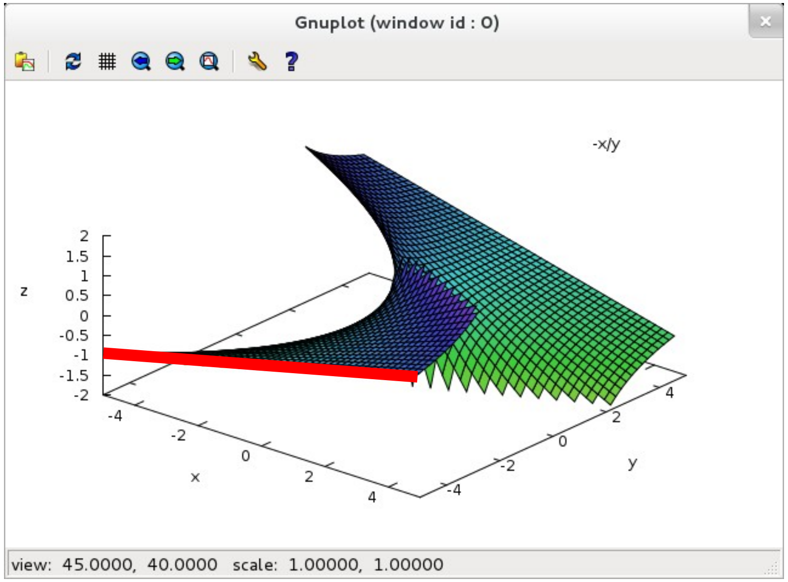
3-d Plot of (– x/y)

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$F(x, y) = -\frac{x}{y}$$

3-d plot of F(x,y)





$$-\frac{1}{y}$$

$$x$$

Existence of a unique solution

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)\,y = g(x)$$

Non-homogeneous Equation with <u>variable</u> coefficients

$$\frac{d^{n-1}}{dx^{n-1}}y(x_0) = k_{n-1}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\frac{d}{dx}y(x_0) = y'(x_0) = k_1$$

$$y(x_0) = y(x_0) = k_0$$

n Initial Conditions

at $x = x_0$

IVP

$a_n(x),\ a_{n-1}(x),\ \cdots\ a_1(x),\ a_0(x),\ g(x)$

are all continuous on the interval I

and $a_n(x) \neq 0$

⇒

The solution y(x) of the IVP

1) exists on the interval I

2) is unique

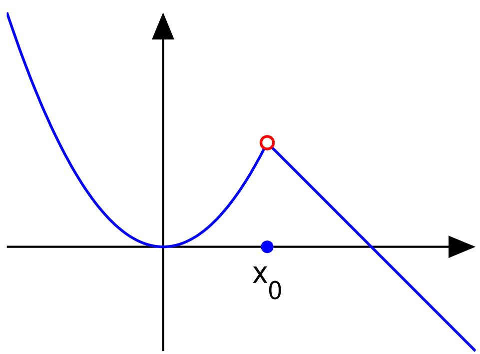
Continuous Function

a continuous function is a function
for which, intuitively, "small" changes in the input
result in "small" changes in the output.

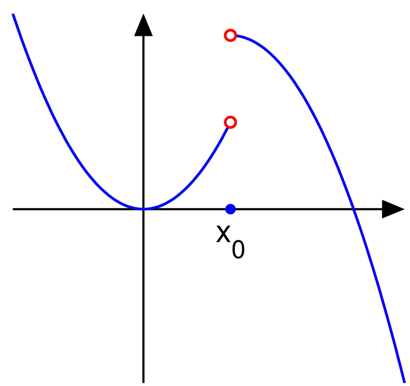
Otherwise, a function is said to be a "discontinuous function".

A continuous function with a continuous inverse function is called a homeomorphism.

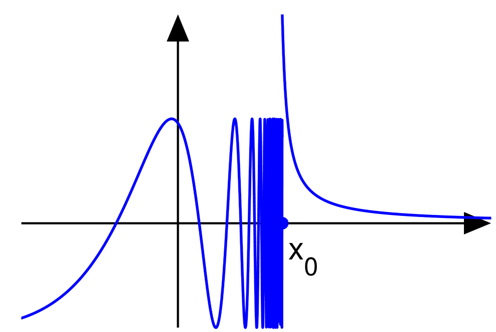
$$f(x) = \begin{cases} \sin \frac{5}{x-1} & \text{for } x < 1 \\ 0 & \text{for } x = 1 \\ \frac{1}{x-1} & \text{for } x > 1 \end{cases}$$



Removable discontinuity



Jump discontinuity

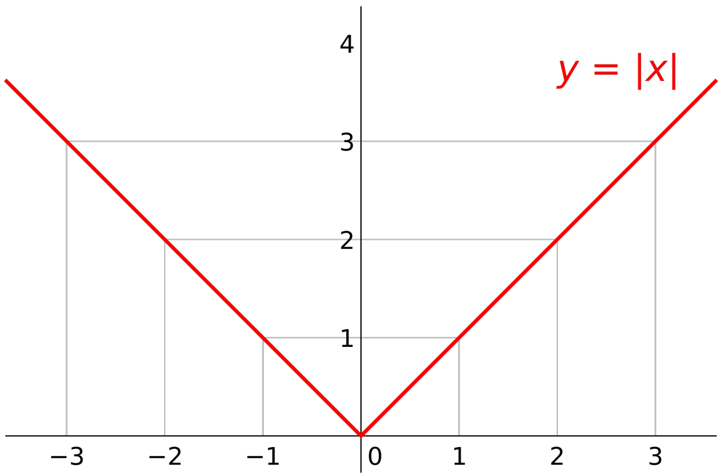


Essential discontinuity

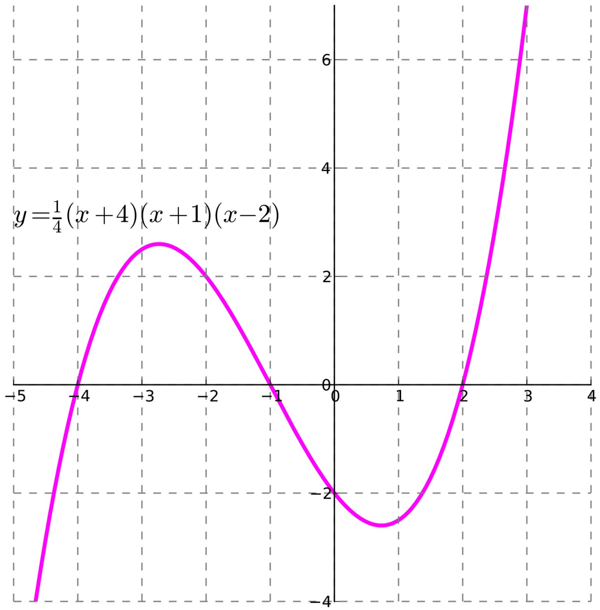
Differentiable Function

a differentiable function of one real variable is
a function whose derivative exists at each point in its domain.

the graph of a differentiable function
must have a non-vertical tangent line at each point in its domain,
be relatively smooth,
and cannot contain any breaks, bends, or cusps.



not differentiable at x=0



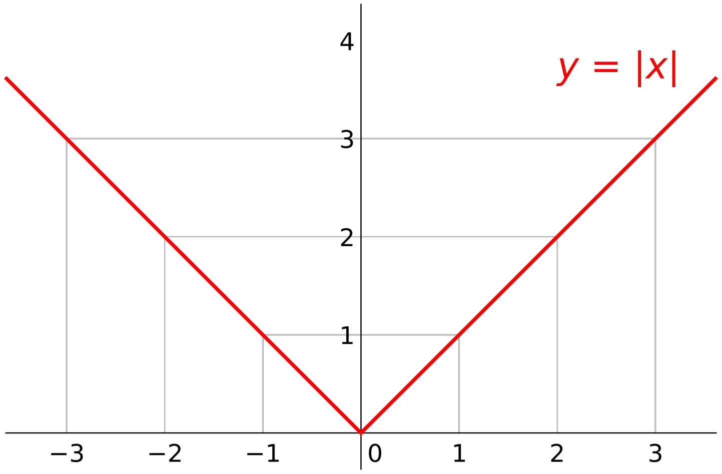
a differentiable function

Differentiability and Continuity

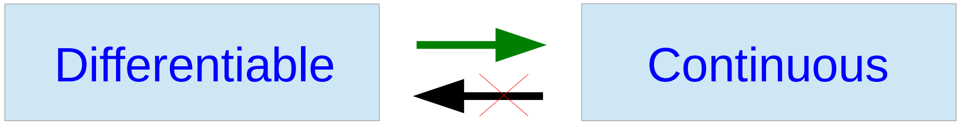
If f is differentiable at a point x0,
then f must also be continuous at x0.

any differentiable function
must be continuous at every point in its domain.

The converse does not hold:
a continuous function need not be differentiable.



not differentiable at x=0



Check for Linear Independent Solutions

Homogeneous Linear n-th order differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

n-th order ***Homogeneous***

$y_1, \quad y_2, \quad \cdots \quad , y_n$

n linearly independent solutions

⟺

$W(y_1, y_2, \cdots, y_n) \neq 0$

$\{y_1, \quad y_2, \quad \cdots \quad , y_n\}$

fundamental set of solutions

$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$

general solution

The general solution for a homogeneous linear n-th order differential equation

References

[1] http://en.wikipedia.org/
[2] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[3] E. Kreyszig, “Advanced Engineering Mathematics”
[4] D. G. Zill, W. S. Wright, “Advanced Engineering Mathematics”
[5] www.chem.arizona.edu/~salzmanr/480a