Complex Series (3A)

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2/21/14

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Power and Taylor Series

Power series

$$\sum_{n=0}^{\infty} c_n (z-a)^n$$

$$= c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots$$

always converges if $|z - a| < R$

➡ can also be differentiated

Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

$$= f(a) + f'(a)(z-a) + \frac{f''(a)}{2}(z-a)^2 + \cdots$$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

only valid if the series converges

Cauchy's Formula and Taylor Series

Cauchy's Formula

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(w)}{w-z} dw \quad \text{(I)}$$

if $f'(z)$ exists
in the neighborhood of a point $z=a$

➡ $f(z)$ is *infinitely differentiable*
in that neighborhood

➡ $f(z)$ can be expanded
in a Taylor series about a
that converges inside a disk
whose radius is equal to the
distance between a and
the *nearest* *singularity* of $f(z)$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} dw \quad \text{(II)}$$

Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

$$= f(a) + f'(a)(z-a) + \frac{f''(a)}{2}(z-a)^2 + \cdots$$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

(=) only valid if the series converges

the region of convergence

Analyticity

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$$

$$\frac{\Delta f}{\Delta z} = \frac{f(z+\Delta z) - f(z)}{\Delta z}, \quad \Delta z = \Delta x + i\Delta y$$

$f(z)$: **analytic in a region** ⟺

$f(z)$ has a (unique) derivative at every point of the region

$f(z)$: **analytic at a point** $z = a$ ⟺

$f(z)$ has a (unique) derivative at every point of some small circle about $z = a$

Regular point of $f(z)$ — a point at which $f(z)$ is analytic

Singular point of $f(z)$ — a point at which $f(z)$ is <u>not</u> analytic

Isolated singular point of $f(z)$ — a point at which $f(z)$ is analytic everywhere else inside some small circle about the singular point

Isolated Singularities

$f(z)$ **: singular** at $z = z_0$

$f(z)$ is <u>not</u> analytic at $z = z_0$
but every neighborhood of $z = z_0$
contains points at which $f(z)$ is analytic

$z = z_0$ **: isolated singularity**

$f(z)$ has neighborhood without
further singularities of $f(z)$

$\tan(z)$ $z = \frac{\pi}{2} + n\pi$ $n = \pm 1, \pm 2, \pm 3, \cdots$

:isolated singularities

$\tan\left(\frac{1}{z}\right)$ $z = \frac{1}{\left(\frac{\pi}{2} + n\pi\right)}$ $\frac{1}{z} = \frac{\pi}{2} + n\pi$

~~:isolated singularities~~

Infinitely Differentiable

$f(z) = u(x,y) + i v(x,y)$: **analytic** in a region R

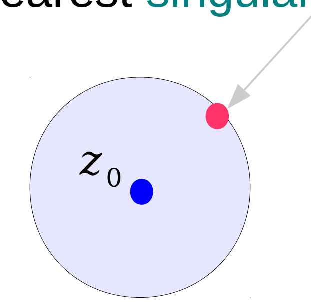
➡ derivatives of all orders at points inside region

$f'(z_0),\ f''(z_0),\ f^{(3)}(z_0),\ f^{(4)}(z_0),\ \cdots$

➡ **Taylor series expansion**
about any point z_0 inside the region

The power series **converges**
inside the circle about z_0

This circle extends to the nearest singular point



if $f'(z)$ exists
in the neighborhood of a point a

➡ $f(z)$ is *infinitely differentiable*
in that neighborhood

➡ $f(z)$ can be expanded
in a Taylor series about z_0
that converges inside a disk
whose radius is equal to the
distance between z_0 and
the *nearest singularity* of $f(z)$

Power Series

A **power series** in powers of $(z - z_0)$

non-negative powers

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots$$

converges for $|z - z_0| < R$

- termwise differentiation
- termwise integration

the same radius of convergence R

A **power series** in powers of $z = (z - 0)$

non-negative powers

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$$

Cauchy's Integral Formula

$f(z)$: **analytic on** and **inside** simple close curve C

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}\,dz$$

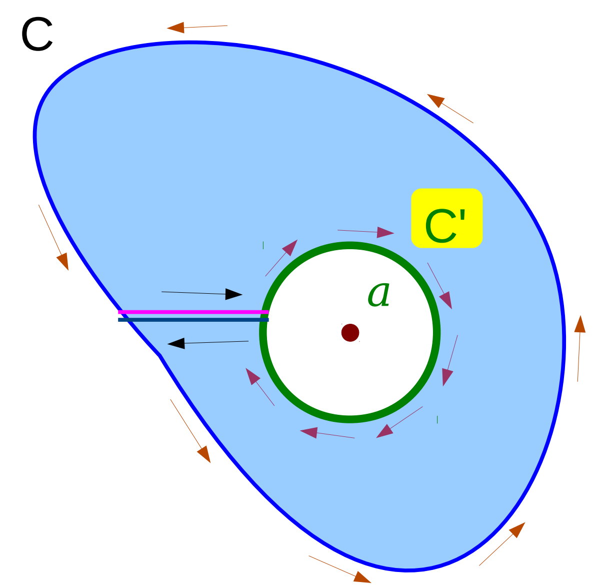
the value of $f(z)$ at a point $z = a$ inside C

$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a} = \oint_{ccw\ C'} \frac{f(z)\,dz}{z-a}$$

$$z = a - \rho e^{i\theta}$$

$$dz = i\rho e^{i\theta} d\theta \qquad \frac{dz}{z-a} = \frac{i\rho e^{i\theta} d\theta}{\rho e^{i\theta}}$$

$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a} = \int_0^{2\pi} f(z)\, i\, d\theta = 2\pi i\ f(a)$$



$$\frac{d}{dz} f(z) = \frac{d}{dz}\left\{\frac{1}{2\pi i}\oint \frac{f(w)}{w-z}\,dw\right\}$$

$$f'(z) = \frac{1}{2\pi i}\oint \frac{f(w)}{(w-z)^2}\,dw$$

Taylor Series

A **power series** in powers of $(z - z_0)$

non-negative powers

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

Conversely, **every** analytic function $f(z)$ can be represented by power series.

The **Taylor series** of a function $f(z)$

non-negative powers

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad a_n = \frac{1}{n!} f^{(n)}(z_0)$$

converges for all z in the open disk with center z_0 and radius
generally equal to the distance from z_0 to the nearest singularity of $f(z)$

Taylor Series Coefficients

A **power series** in powers of $(z-z_0)$

non-negative powers

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

The **Taylor series** of a function $f(z)$

non-negative powers

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} dw$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z_0)^{n+1}} dw$$

Maclaurin Series Coefficients

A **power series** in powers of $z \quad = (z - 0)$

non-negative powers

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$$

The **Maclaurin series** of a function $f(z)$

non-negative powers

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$a_n = \frac{1}{n!} f^{(n)}(0)$$

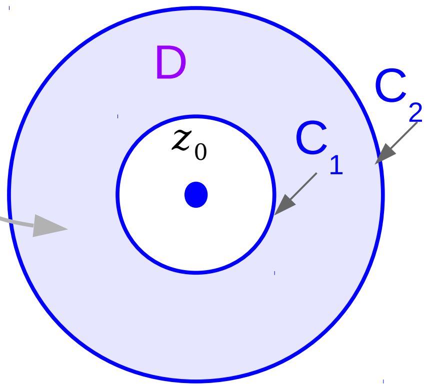
$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} dw$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w^{n+1}} dw$$

Laurent's Theorem

$f(z)$: **analytic** in the annular domain D
between concentric circles C_1 and C_2
centered at z_0

analytic f(z)



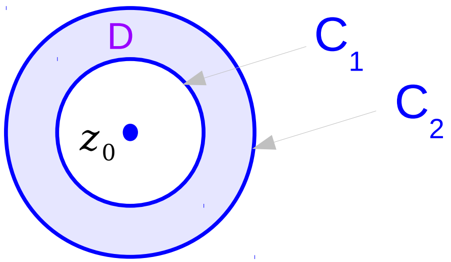
concentric circles,
annular domain

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$
$$+ b_1(z-z_0)^{-1} + b_2(z-z_0)^{-2} + \cdots$$

: **convergent** in the region D

Laurent's Theorem - Region of Convergence

$f(z)$: **analytic** in the annular domain D
between concentric circles C_1 and C_2
centered at z_0

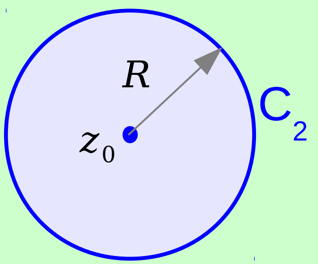


$$a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

For this "a" series to converge,
the ROC must be in the form

$$|z-z_0| < R$$

inside of C_2



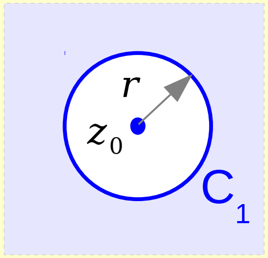
$$+ b_1(z-z_0)^{-1} + b_2(z-z_0)^{-2} + \cdots$$

principal part

For this "b" series to converge,
the ROC must be in the form

$$\left|\frac{1}{z-z_0}\right| < r$$

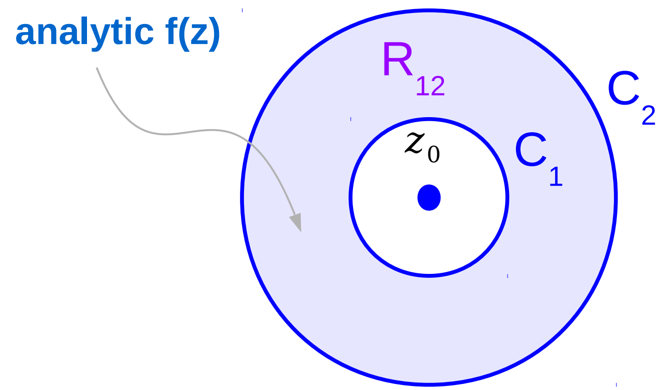
outside of C_1



Expanding, Compressing the Region

: **convergent** also in the enlarged open annulus
expanding C_2 and compressing C_1
until the circles reach a singular point

the previous equation is valid for all z near z_0
in some deleted neighborhood of z_0
(punctured open disk)

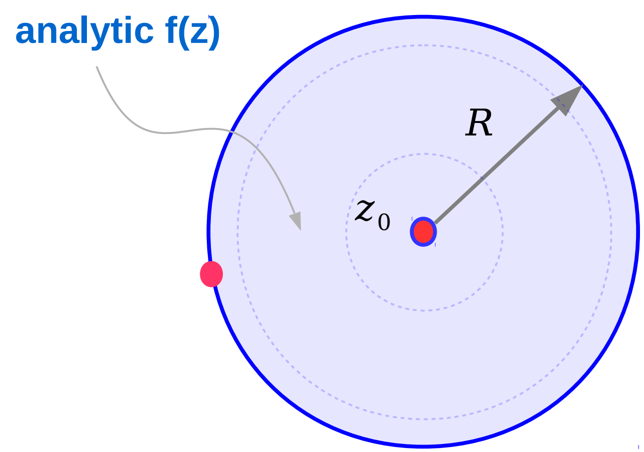


$$r < |z - z_0| < R$$

the special case
z_0 is the only singular point inside C_1
the series is **convergent**
in a disk **except** its center

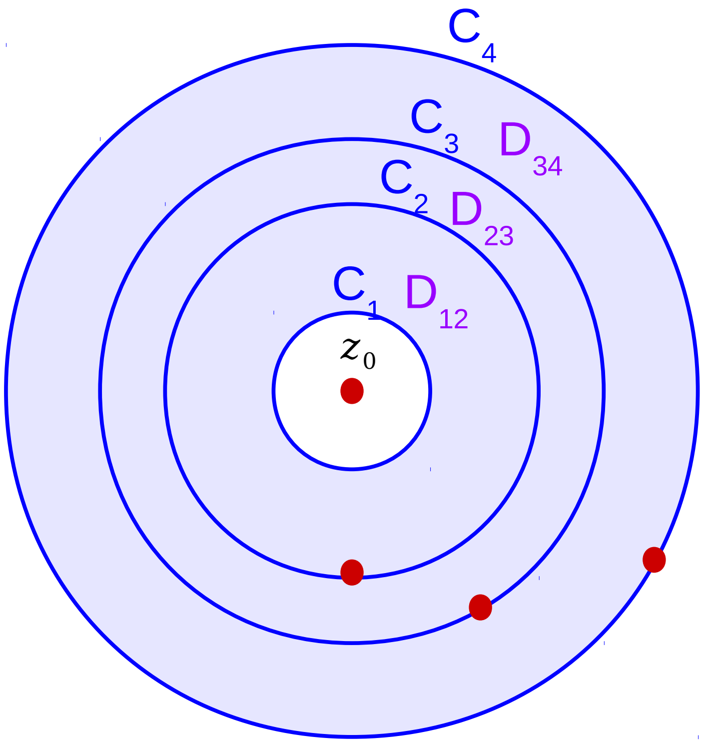
$$\left.\begin{array}{l} a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \\ + b_1(z-z_0)^{-1} + b_2(z-z_0)^{-2} + \cdots \end{array}\right\} = f(z)$$

principal part



$$0 < |z - z_0| < R$$

Different Domains, Different Expansions



$$f(z) = \sum_{n=0}^{+\infty} a_n (z-z_0)^n + \sum_{n=1}^{+\infty} b_n (z-z_0)^{-n} \quad \text{in } D_{12}$$

$$f(z) = \sum_{n=0}^{+\infty} c_n (z-z_0)^n + \sum_{n=1}^{+\infty} d_n (z-z_0)^{-n} \quad \text{in } D_{23}$$

$$f(z) = \sum_{n=0}^{+\infty} e_n (z-z_0)^n + \sum_{n=1}^{+\infty} f_n (z-z_0)^{-n} \quad \text{in } D_{34}$$

different Laurent expansions of a the same function f(z) for different domains

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

principal part $+ b_1(z-z_0)^{-1} + b_2(z-z_0)^{-2} + \cdots$

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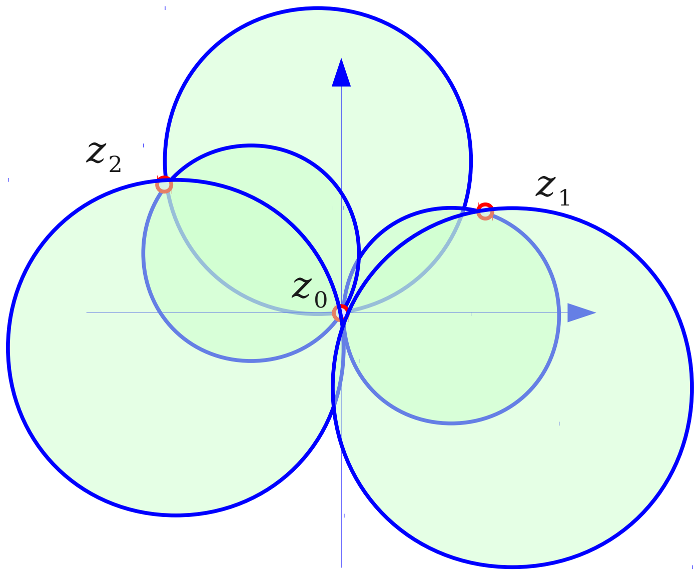
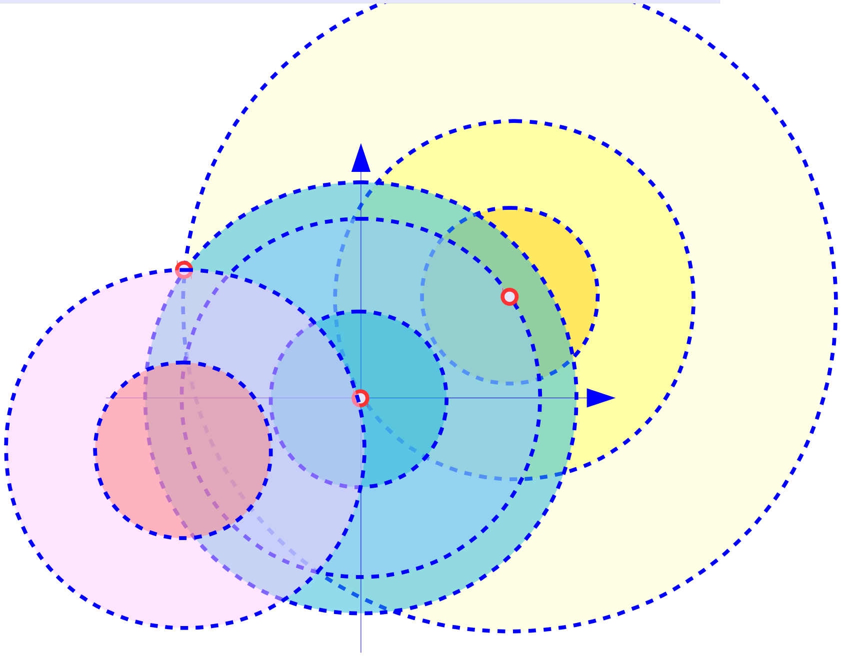
Several Isolated Singularities

Taylor Series



Laurent Series



A Laurent series converges between two concentric circles, if it converges at all.

Several isolated singularites ➡ Several annular rings

➡ Several different Laurent series for each rings

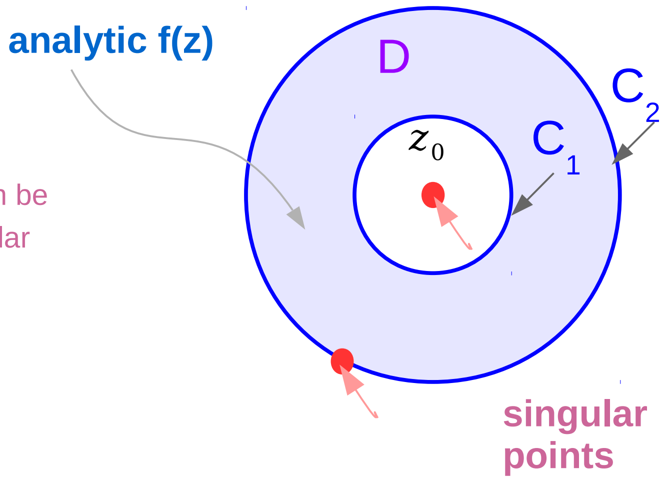
The **Laurent series** of a function $f(z)$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$

z_0 can be singular

converges in the region D between circles C_1 and C_2 centered at z_0 where f(z) is **analytic**

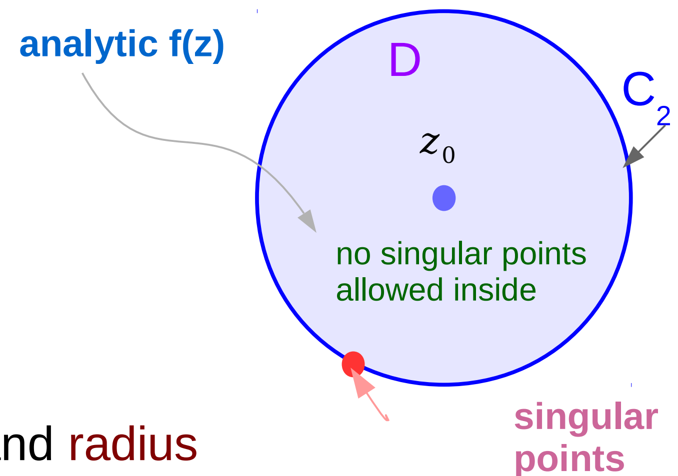


The **Taylor series** of a function $f(z)$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$

converges for all z in the open disk with center z_0 and radius generally equal to the distance from z_0 to the nearest singularity of $f(z)$

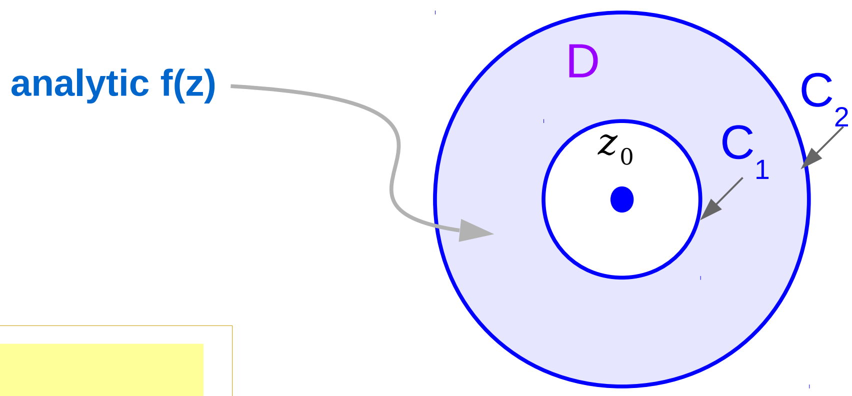


Laurent Series in different forms

$f(z)$: **analytic** in the annular domain D

between concentric circles C_1 and C_2

centered at z_0



convergent in the region D

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$
$$+ b_1(z-z_0)^{-1} + b_2(z-z_0)^{-2} + \cdots$$

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0) + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$
$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$
$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$$

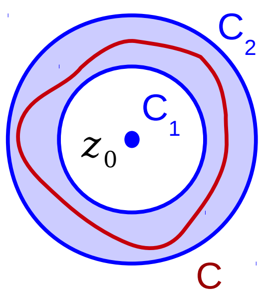
$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

$$f(z) = \cdots + a_n (z-z_0)^n + \cdots$$

$$\frac{f(z)}{(z-z_0)^{n+1}} = \cdots + \frac{a_n}{(z-z_0)} + \cdots$$

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \oint_C \frac{a_n}{(z-z_0)}\,dz$$

$$= 2\pi i \cdot a_n$$



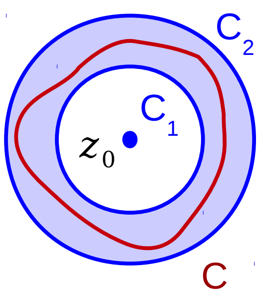
$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

$$f(z) = \cdots + \frac{b_n}{(z-z_0)^n} + \cdots$$

$$\frac{f(z)}{(z-z_0)^{-n+1}} = \cdots + \frac{b_n}{(z-z_0)} + \cdots$$

$$\oint_C \frac{f(z)}{(z-z_0)^{-n+1}}\,dz = \oint_C \frac{b_n}{(z-z_0)}\,dz$$

$$= 2\pi i \cdot b_n$$



$$b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{-n+1}}\,dz$$

Laurent's Series Coefficients

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + b_1(z-z_0)^{-1} + b_2(z-z_0)^{-2} + \cdots$$

$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} dw$$

$$b_n = \frac{1}{2\pi i} \oint_C f(w)(w-z)^{n-1} dw$$

$$(w-z)^{n-1} = \frac{1}{(w-z)^{-n+1}}, \quad n = 1, 2, 3, \cdots$$

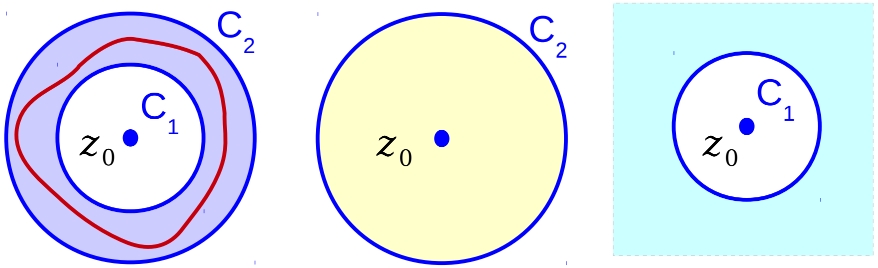
$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$$

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{k+1}} dw$$

$$k = \cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$$

Laurent's Theorem and Coefficients

$f(z)$: **analytic** in the annular domain D
between concentric circles C_1 and C_2
centered at z_0 $\qquad r < |z - z_0| < R$



$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$
$$+ b_1(z-z_0)^{-1} + b_2(z-z_0)^{-2} + \cdots$$

convergent in the domain D

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

$$b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{-n+1}}\,dz$$

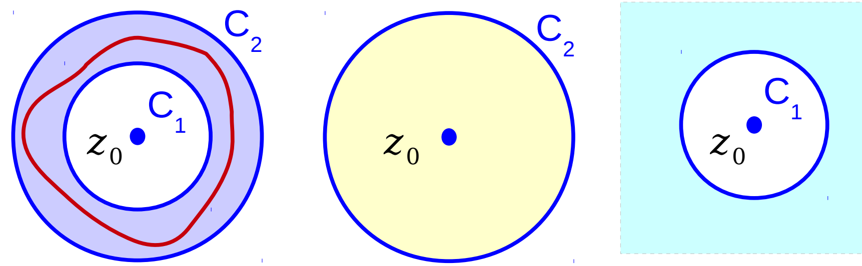
any simple closed path C in D

$$f(z) = \sum_{n=-\infty}^{\infty} a_k(z-z_0)^k$$

$$a_k = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{k+1}}\,dz$$

Residue

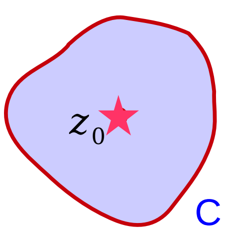
$f(z)$: **analytic** in the annular domain D
between concentric circles C_1 and C_2
centered at z_0 $\quad r < |z - z_0| < R$



$$f(z) = \sum_{n=-\infty}^{\infty} a_k (z - z_0)^k \qquad a_k = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{k+1}} \, dz \qquad k = -1$$

z_0 can be an isolated singularity

$$f(z) = \cdots + \frac{a_{-1}}{(z - z_0)} + \cdots$$



$$a_{-1} = Res(f(z), z_0)$$

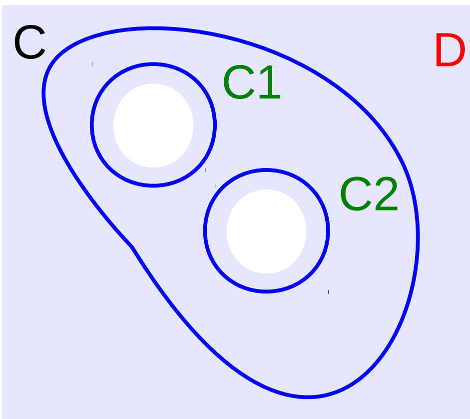
: **residue** of the function $f(z)$
at the isolated singularity z_0

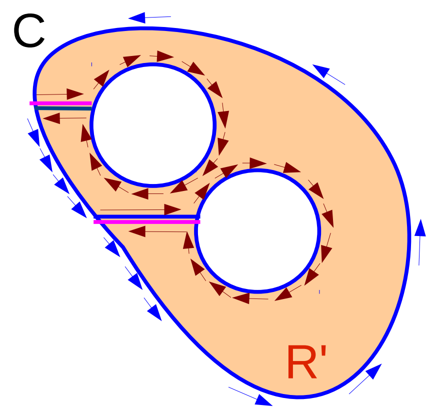
$$\oint_C f(z) \, dz = 2\pi i \, Res(f(z), z_0)$$

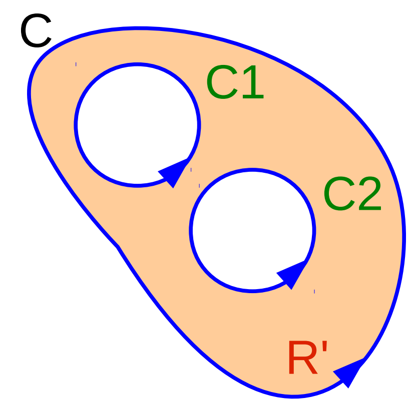
$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z) \, dz$$

Cauchy-Goursat Theorem

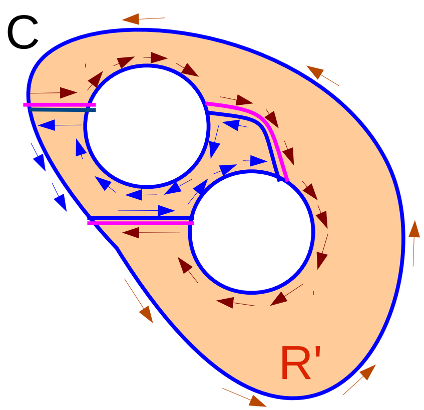
triply connected domain D ➡ **simply connected region** R' **contour C, C1, C2**







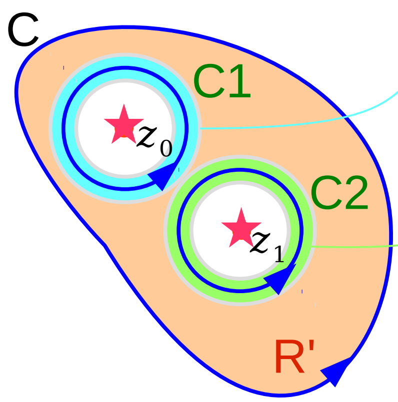
cuts



$$\oint_{ccw\ C} f(z)\, dz + \oint_{cw\ C1} f(z)\, dz + \oint_{cw\ C2} f(z)\, dz = 0$$

$$\oint_{C} f(z)\, dz = \oint_{C1} f(z)\, dz + \oint_{C2} f(z)\, dz$$

Residue Theorem



z_0, z_1 : isolated singularities

Laurent series expansion around z_0

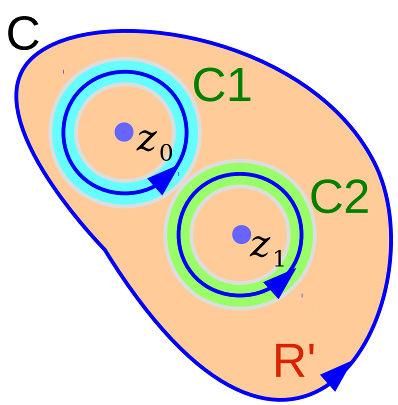
$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n \quad \Rightarrow \quad \oint_{C1} f(z)\,dz = 2\pi i \cdot a_{-1}$$

Laurent series expansion around z_1

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z - z_1)^n \quad \Rightarrow \quad \oint_{C2} f(z)\,dz = 2\pi i \cdot c_{-1}$$

$$\oint_C f(z)\,dz = \oint_{C1} f(z)\,dz + \oint_{C2} f(z)\,dz = 2\pi i \cdot a_{-1} + 2\pi i \cdot c_{-1}$$

$$\oint_C f(z)\,dz = 2\pi i \{Res(f(z), z_0) + Res(f(z), z_1)\}$$

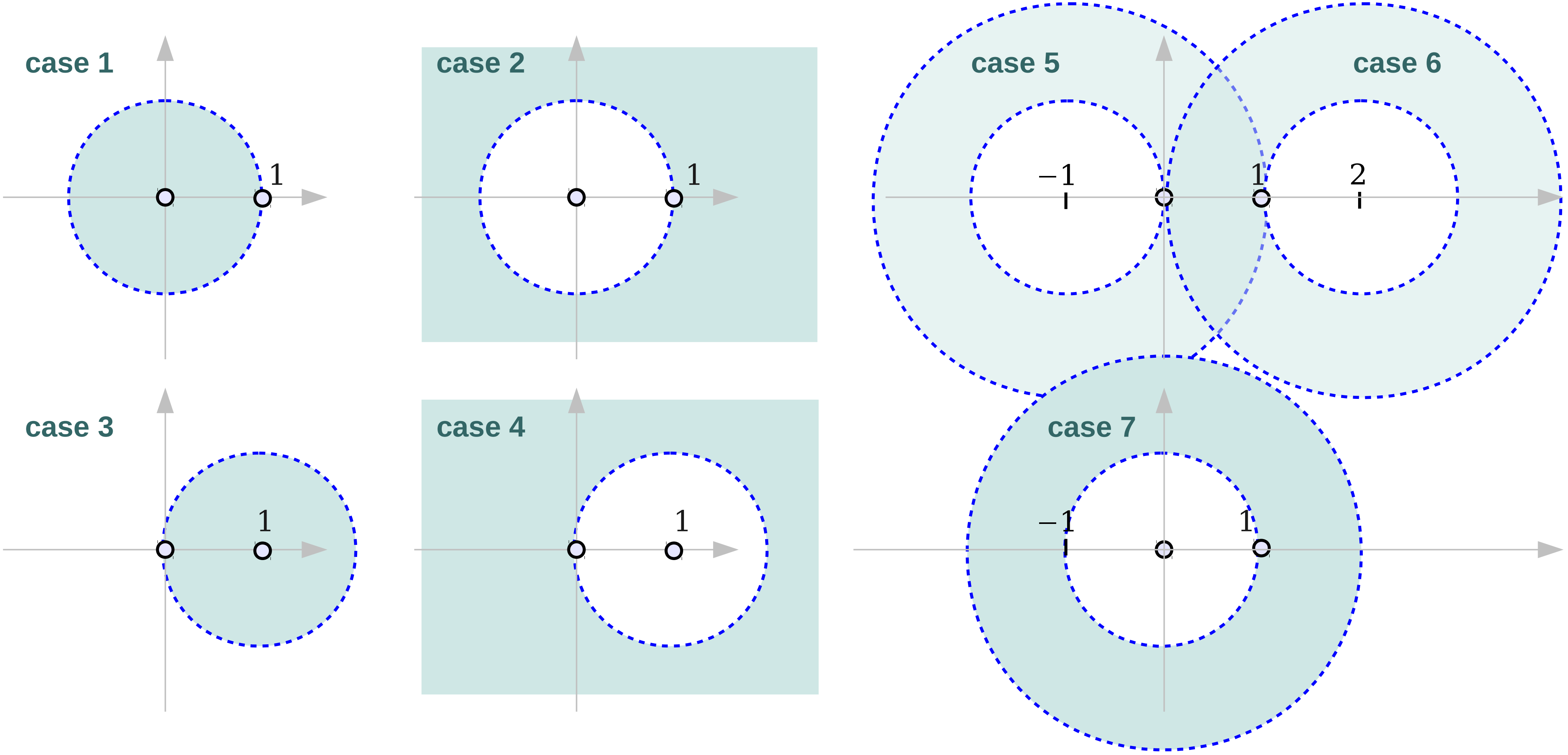


z_0, z_1 : regular points

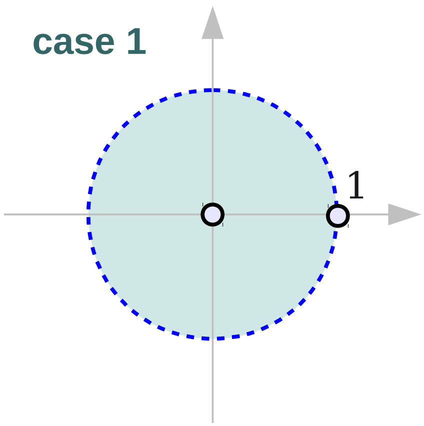
$$\oint_C f(z)\,dz = \oint_{C1} f(z)\,dz = \oint_{C2} f(z)\,dz = 0$$

Laurent Expansion Example (1)

$$f(z) = \frac{1}{z(z-1)}$$

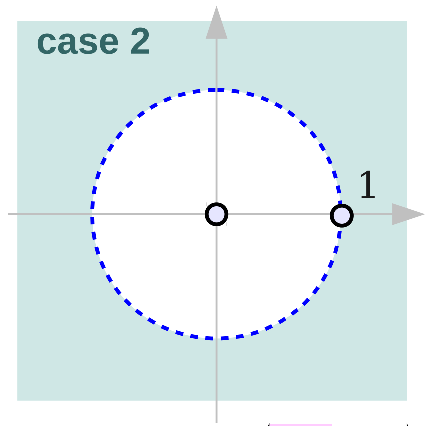


Laurent Expansion Example (2)



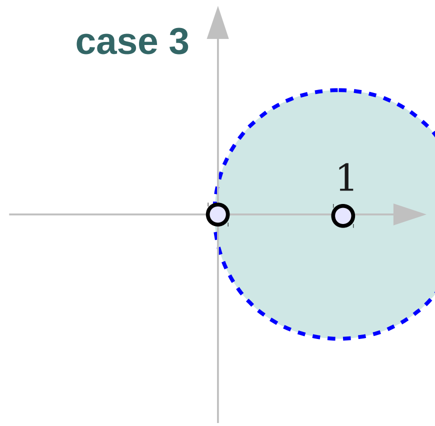
$0 < |z| < 1$

$$\frac{1}{1 - z} = 1 + z + z^2 + \cdots$$



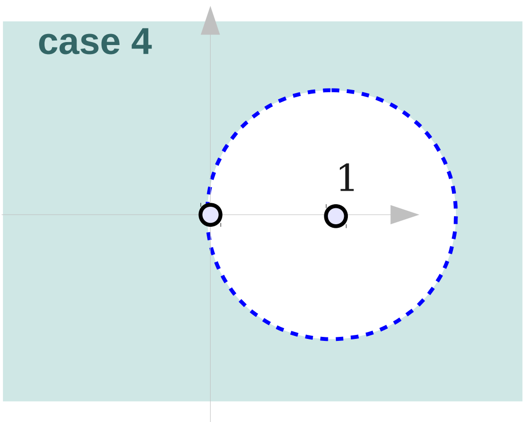
$|z| > 1 \quad \left(\frac{1}{|z|} < 1 \right)$

$$\frac{1}{1 - \frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots$$



$0 < |z - 1| < 1$

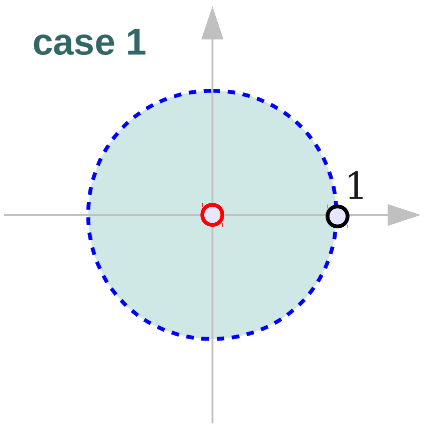
$$\frac{1}{1 - (z-1)} = 1 + (z-1) + (z-1)^2 + \cdots$$



$|z - 1| > 1 \quad \left(\frac{1}{|z - 1|} < 1 \right)$

$$\frac{1}{1 - \frac{1}{(z-1)}} = 1 + \frac{1}{(z-1)} + \frac{1}{(z-1)^2} + \cdots$$

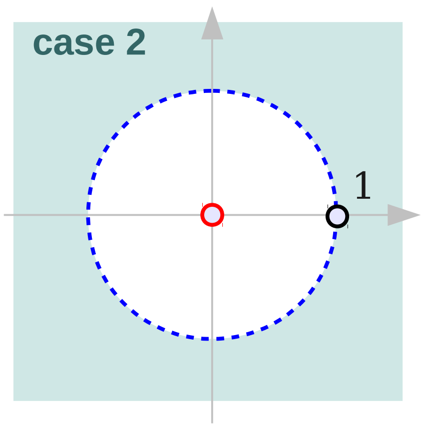
Laurent Expansion Example (3)



$|z| < 1$

$$\frac{1}{1-z}$$

$$= 1 + z + z^2 + \cdots$$



$|z| > 1 \qquad \left(\frac{1}{|z|} < 1\right)$

$$\frac{1}{1-\frac{1}{z}}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots$$

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z}\frac{1}{(1-z)}$$

$$= -\frac{1}{z}\left[1 + z + z^2 + z^3 + \cdots\right]$$

$$= \boxed{-\frac{1}{z}} - 1 + z + z^2 + \cdots$$

A **simple pole** $z = 0$

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{z^2\left(1-\frac{1}{z}\right)}$$

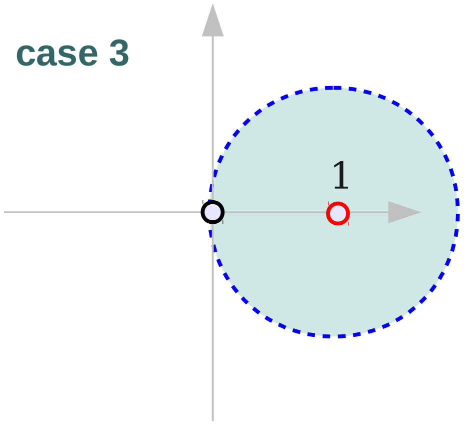
$$= \frac{1}{z^2}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right]$$

$$= \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots$$

Not annular domain

Laurent Expansion Example (4)

case 3

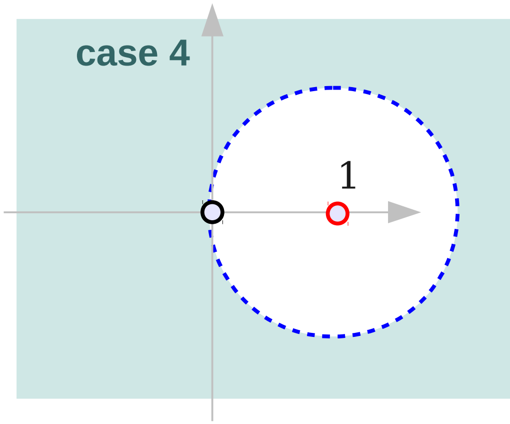


$$0 < |z-1| < 1$$

$$\frac{1}{1-(z-1)}$$

$$1 + (z-1) + (z-1)^2 + \cdots$$

case 4



$$|z-1| > 1 \quad \left(\frac{1}{|z-1|} < 1 \right)$$

$$\frac{1}{1-\frac{1}{(z-1)}}$$

$$1 + \frac{1}{(z-1)} + \frac{1}{(z-1)^2} + \cdots$$

Case 3:

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{(z-1)} \frac{1}{(1+(z-1))}$$

$$= \frac{1}{(z-1)} \left[1 - (z-1) + (z-1)^2 - \cdots \right]$$

$$= \frac{1}{(z-1)} - 1 + (z-1) - (z-1)^2 + \cdots$$

A **simple pole** $z = 1$

Case 4:

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{(z-1)^2 \left(1 + \frac{1}{(z-1)} \right)}$$

$$= \frac{1}{(z-1)^2} \left[1 - \frac{1}{(z-1)} + \frac{1}{(z-1)^2} - \cdots \right]$$

$$= \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + \frac{1}{(z-1)^4} - \cdots$$

Not annular domain

Laurent Expansion Example (5)

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$

$z = -2$ Not an isolated singular point

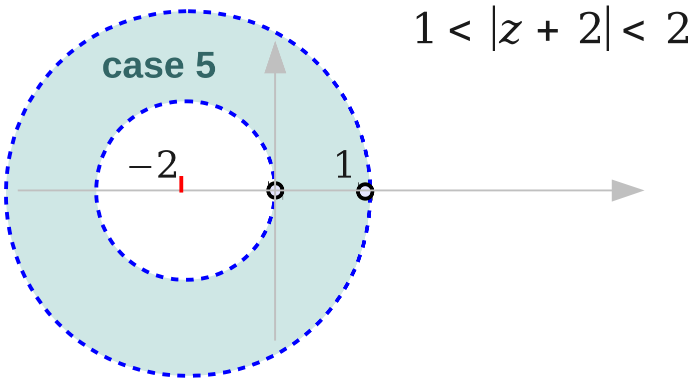
$$\frac{1}{z-1} = \frac{1}{z+2-3} = \frac{-1}{3\left(1 - \frac{1}{z+2}\right)}$$

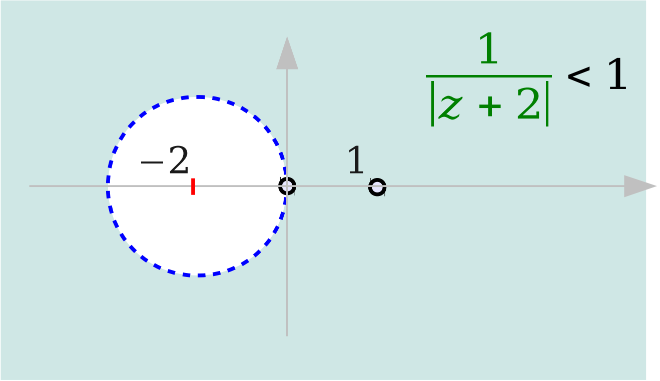
$$= -\frac{1}{3}\left[1 + \frac{1}{(z+2)} + \frac{1}{(z+2)^2} + \frac{1}{(z+2)^3} \cdots \right]$$

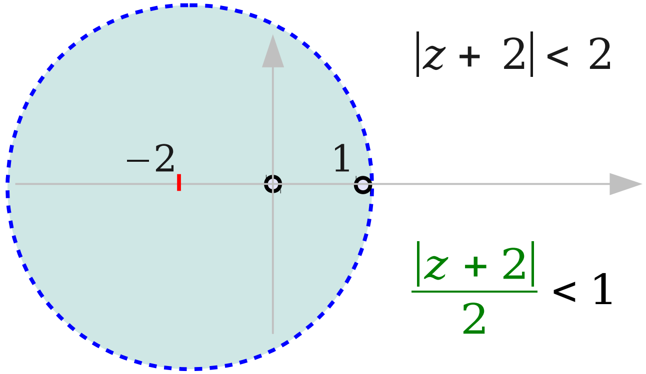
$$-\frac{1}{z} = -\frac{1}{z+2-2} = \frac{-1}{2\left(1 - \frac{z+2}{2}\right)}$$

$$= -\frac{1}{2}\left[1 + \frac{(z+2)}{2} + \frac{(z+2)^2}{2^2} + \frac{(z+2)^3}{2^3} \cdots \right]$$

~~essential singularity~~







Laurent Expansion Example (6)

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$

$z = +2$ Not an isolated singular point

$$\frac{1}{z-1} = \frac{1}{1+z-2} = \frac{1}{(z-2)\left(1+\frac{1}{z-2}\right)}$$

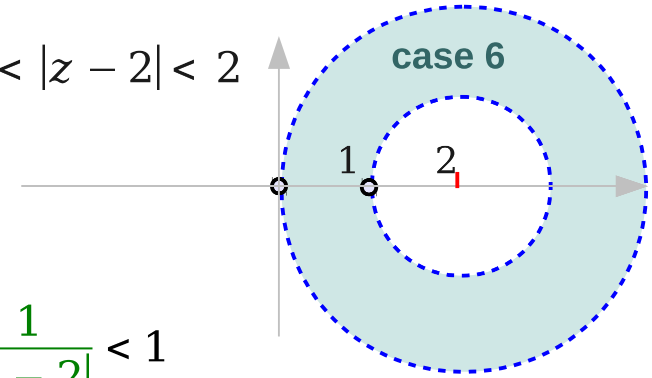
$$= \frac{1}{z-2}\left[1 - \frac{1}{(z-2)} + \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} \cdots\right]$$

$$-\frac{1}{z} = -\frac{1}{2+z-2} = -\frac{1}{2\left(1+\frac{z-2}{2}\right)}$$

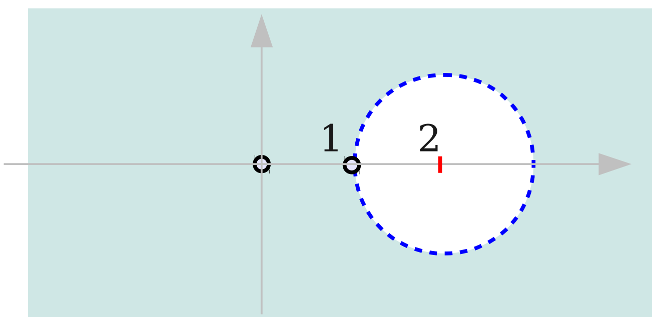
$$= -\frac{1}{2}\left[1 - \frac{(z-2)}{2} + \frac{(z-2)^2}{2^2} - \frac{(z-2)^3}{2^3} \cdots\right]$$

~~*essential singularity*~~

$1 < |z-2| < 2$

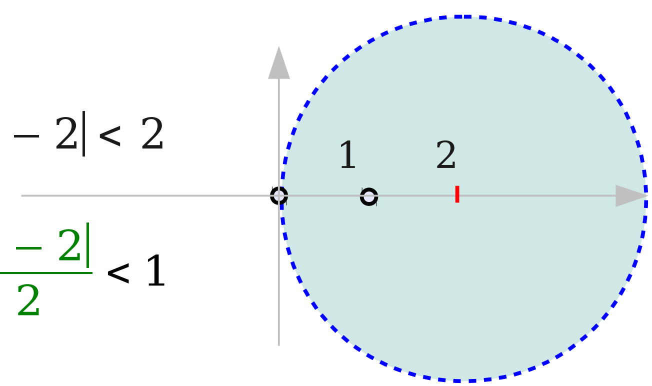


$$\frac{1}{|z-2|} < 1$$



$|z-2| < 2$

$$\frac{|z-2|}{2} < 1$$



Laurent Expansion Example (7)

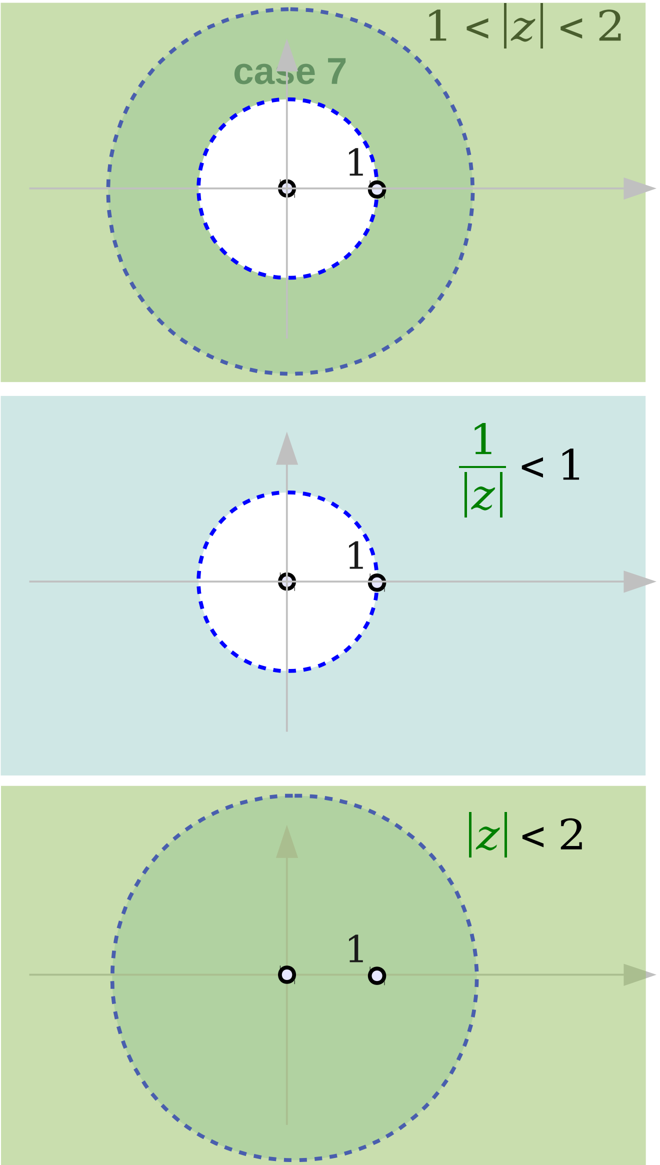
$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$

$$\frac{1}{z-1} = \frac{1}{z\left(1 - \frac{1}{z}\right)} = \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} \cdots \right]$$

$$\frac{1}{z(z-1)} = \left[\frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} \cdots \right]$$

$$\frac{1}{z-1} - \frac{1}{z} = \left[\frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} \cdots \right]$$

essential singularity



Young Won Lim
2/21/14

Singular Point

Regular point of $f(z)$ $\triangleq$ a point at which $f(z)$ is analytic

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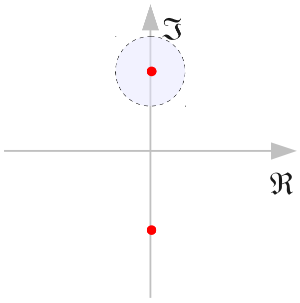
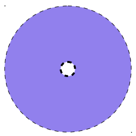
Singular point of $f(z)$ $\triangleq$ a point at which $f(z)$ is not analytic

Isolated Singular point of $f(z)$ $\triangleq$ a point at which $f(z)$ is analytic everywhere else inside some small circle about the singular point



Isolated Singularity of $f(z)$ at z_0 $\triangleq$ If $z=z_0$ has a neighborhood *without further singularities* of *f*(*z*)

$\triangleq$ There exists some *deleted neighborhood* or *punctured open disk* of z_0 throughout which *f*(*z*) is analytic



$$0 < |z - z_0| < R$$

Non-isolated Singularity

Cluster points: limit points of isolated singularities. If they are all poles, despite admitting Laurent series expansions on each of them, no such expansion is possible at its limit

$$f(z) = \tan(1/z)$$

simple poles $\quad z_n = \dfrac{1}{(\pi/2 + n\pi)}$

$$\lim_{n \to 0} z_n = 0$$

Every punctured disk centered at 0 has an infinite number of singularities. No Laurent expansion

---

Natural boundaries: non-isolated set (e.g. a curve) which functions can not be analytically continued around (or outside them if they are closed curves in the Riemann sphere).

$$f(z) = Ln\ z$$

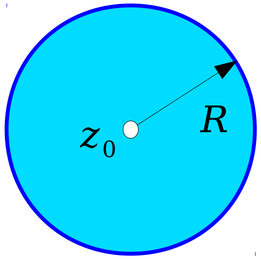
the branch point 0

and the negative axis

Every neighborhood of z0 contains at least one singularity of f(z) other than z0

Isolated Singularity Classification (1)

When **Laurent expansion** is valid
for the punctured open disk $0 < |z - z_0| < R$
around z_0 : **isolated singularity** of f



Depending on the number of terms of the principal part
An **isolated singular point** z_0 is called

A **removable singularity**	no principal part
A **simple pole**	one term in the principal part
A **pole of order n**	*n* terms in the principal part
An **essential singularity**	infinite terms in the principal part

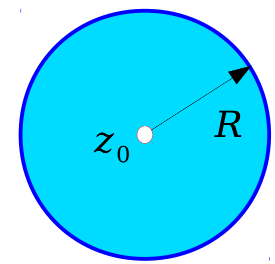
$$f(z) = \sum_{n=-\infty}^{\infty} a_k (z - z_0)^k = \underbrace{\sum_{n=1}^{\infty} a_{-k} (z - z_0)^{-k}}_{\text{principal part}} + \sum_{n=0}^{\infty} a_k (z - z_0)^k$$

Isolated Singularity Classification (2)

When **Laurent expansion** is valid

for the punctured open disk $0 < |z - z_0| < R$

around z_0 : **isolated singularity** of f



$b_k = 0$ $\quad f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$ $\quad$ z_0 ***removable singularity***

$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$ $\quad$ z_0 ***simple pole***

$b_1 = a_{-1}$ $\quad + \dfrac{b_1}{(z-z_0)}$ ⬅ one term

$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$ $\quad$ z_0 ***pole of order n***

$b_1 = a_{-1}$ $\quad + \dfrac{b_1}{(z-z_0)} + \dfrac{b_2}{(z-z_0)^2} + \cdots + \dfrac{b_n}{(z-z_0)^n}$ ⬅ n terms

$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$ $\quad$ z_0 ***essential singularity***

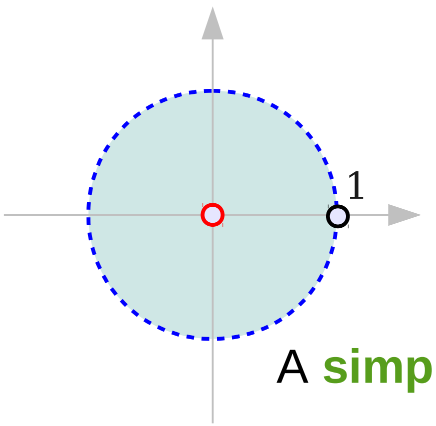
$b_1 = a_{-1}$ $\quad + \dfrac{b_1}{(z-z_0)} + \dfrac{b_2}{(z-z_0)^2} + \cdots + \dfrac{b_n}{(z-z_0)^n} + \cdots$ ⬅ infinite terms

Isolated Singularity Examples (1)

$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$	$z = 0$	*regular point*
$\frac{\sin(z)}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots$	$z = 0$	***removable singularity***
$\frac{\sin(z)}{z^2} = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} + \cdots$	$z = 0$	***simple pole***
$\frac{\sin(z)}{z^3} = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} + \cdots$	$z = 0$	***pole of order 2***

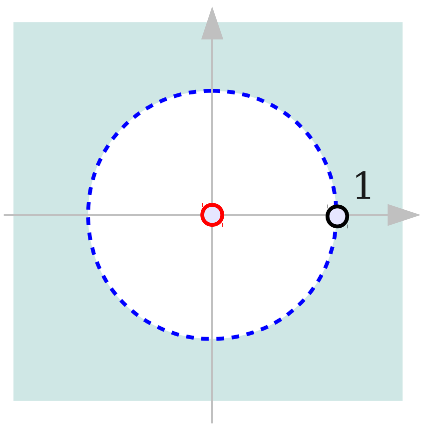
$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$	$z = 0$	*regular point*
$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$	$z = 0$	***essential singularity***

Isolated Singularity Examples (2)



$|z| < 1$

A **simple pole** $z = 0$



$|z| > 1$ $\quad \left(\frac{1}{|z|} < 1 \right)$

Not annular domain

~~$0 < |z - z_0| < R$~~

➡ ~~*essential singularity*~~

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z}\frac{1}{(1-z)}$$

$$= -\frac{1}{z}\left[1 + z + z^2 + z^3 + \cdots\right]$$

$$= -\frac{1}{z} - 1 + z + z^2 + \cdots$$

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{z^2\left(1 - \frac{1}{z}\right)}$$

$$= \frac{1}{z^2}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right]$$

$$= \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots$$

Singularities of Order n

z_0 A **zero** of a function f

f : **analytic** at $z = z_0$

$\triangleq \quad f(z_0) = 0$

z_0 A **zero** of order n of a function f

f : **analytic** at $z = z_0$

$\triangleq \quad f(z_0) = 0, \; f'(z_0) = 0, \; f''(z_0) = 0, \; \cdots, \; f^{(n)}(z_0) = 0$

z_0 A **pole** of order n of a function $F(z) = g(z) / f(z)$

f, g : **analytic** at $z = z_0$

$$\triangleq \begin{cases} g(z_0) \neq 0 \\ f(z_0) = 0, \; f'(z_0) = 0, \; f''(z_0) = 0, \; \cdots, \; f^{(n)}(z_0) = 0 \end{cases}$$

References

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