

Cramer's Rule (H1)

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Solving a System of Linear Equations

3x3 A A^{-1} Inverse Matrix.

$$\begin{aligned} p_1 x + p_2 y + p_3 z &= b_1 \\ q_1 x + q_2 y + q_3 z &= b_2 \\ r_1 x + r_2 y + r_3 z &= b_3 \end{aligned}$$

$$\begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$A \quad x = b$

① $A^{-1} \quad x = A^{-1} \cdot m$

② Cramer's Rule $x = \frac{\begin{vmatrix} \color{green}{|} & \color{purple}{|} & \color{purple}{|} \\ \color{purple}{|} & \color{purple}{|} & \color{purple}{|} \end{vmatrix}}{\begin{vmatrix} \color{purple}{|} & \color{purple}{|} & \color{purple}{|} \\ \color{purple}{|} & \color{purple}{|} & \color{purple}{|} \\ \color{purple}{|} & \color{purple}{|} & \color{purple}{|} \end{vmatrix}} \quad y = \frac{\begin{vmatrix} \color{purple}{|} & \color{green}{|} & \color{purple}{|} \\ \color{purple}{|} & \color{purple}{|} & \color{purple}{|} \\ \color{purple}{|} & \color{purple}{|} & \color{purple}{|} \end{vmatrix}}{\begin{vmatrix} \color{purple}{|} & \color{purple}{|} & \color{purple}{|} \\ \color{purple}{|} & \color{purple}{|} & \color{purple}{|} \\ \color{purple}{|} & \color{purple}{|} & \color{purple}{|} \end{vmatrix}} \quad z = \frac{\begin{vmatrix} \color{purple}{|} & \color{purple}{|} & \color{green}{|} \\ \color{purple}{|} & \color{purple}{|} & \color{purple}{|} \\ \color{purple}{|} & \color{purple}{|} & \color{purple}{|} \end{vmatrix}}{\begin{vmatrix} \color{purple}{|} & \color{purple}{|} & \color{purple}{|} \\ \color{purple}{|} & \color{purple}{|} & \color{purple}{|} \\ \color{purple}{|} & \color{purple}{|} & \color{purple}{|} \end{vmatrix}}$

③ Gauss-Jordan Elimination RREF

$R_{ij} = \begin{bmatrix} \color{blue}{\rule{1cm}{0.4pt}} \\ \color{orange}{\rule{1cm}{0.4pt}} \end{bmatrix} \begin{matrix} i \\ j \end{matrix}$ $CR_i = c \times \begin{bmatrix} \color{blue}{\rule{1cm}{0.4pt}} \end{bmatrix} \begin{matrix} i \end{matrix}$ $CR_i + R_j = c \times \begin{bmatrix} \color{blue}{\rule{1cm}{0.4pt}} \\ \color{orange}{\rule{1cm}{0.4pt}} \end{bmatrix} \begin{matrix} i \end{matrix}$

Solving Linear Equations

A set of linear equations

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$



$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} e \\ f \end{bmatrix}$$

If the inverse matrix exists

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$$



$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix}$$

$$\begin{vmatrix} e & b \\ f & d \end{vmatrix} = de - bf$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} de - bf \\ -ce + af \end{bmatrix}$$

$$\begin{vmatrix} a & e \\ c & f \end{vmatrix} = af - ce$$

Cramer's Rule

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

ODE Background :
Complex Variables (4A)

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Cramer's Rule

Determinant of order 3

$$\begin{aligned} a_1x + a_2y + a_3z &= d \\ b_1x + b_2y + b_3z &= e \\ c_1x + c_2y + c_3z &= f \end{aligned}$$



$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} d & a_2 & a_3 \\ e & b_2 & b_3 \\ f & c_2 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} a_1 & d & a_3 \\ b_1 & e & b_3 \\ c_1 & f & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}$$

$$z = \frac{\begin{vmatrix} a_1 & a_2 & d \\ b_1 & b_2 & e \\ c_1 & c_2 & f \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}$$

ODE Background :
Complex Variables (4A)

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Cramer's Rule (1) - solutions

$$\mathbf{A} \mathbf{x} = \mathbf{b}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\mathbf{A}_1 \quad \mathbf{A}_2 \quad \mathbf{A}_n$$

$$\begin{pmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad \begin{pmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{pmatrix}$$

$$x_1 = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})} \quad x_2 = \frac{\det(\mathbf{A}_2)}{\det(\mathbf{A})} \quad x_n = \frac{\det(\mathbf{A}_n)}{\det(\mathbf{A})}$$

Determinant (3A)

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Cramer's Rule (2) - determinants

$$\mathbf{A}_1 \quad \det(\mathbf{A}_1) = b_1 C_{11} + b_2 C_{21} + \cdots + b_n C_{n1}$$

cofactor expansion along the first column

$$x_1 = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})}$$

$$\mathbf{A}_2 \quad \det(\mathbf{A}_2) = b_1 C_{12} + b_2 C_{22} + \cdots + b_n C_{n2}$$

cofactor expansion along the second column

$$x_2 = \frac{\det(\mathbf{A}_2)}{\det(\mathbf{A})}$$

$$\mathbf{A}_n \quad \det(\mathbf{A}_n) = b_1 C_{1n} + b_2 C_{2n} + \cdots + b_n C_{nn}$$

cofactor expansion along the last column

$$x_n = \frac{\det(\mathbf{A}_n)}{\det(\mathbf{A})}$$

Determinant (3A)

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Cramer's Rule (3) - inverse matrix

$$\mathbf{Ax} = \mathbf{b} \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{\text{adj}(\mathbf{A})}{\det(\mathbf{A})}\mathbf{b}$$

$$\mathbf{x} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

the transposed matrix
note reverse order index

$$\mathbf{x} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} b_1 C_{11} + b_2 C_{21} + \cdots + b_n C_{n1} \\ b_1 C_{12} + b_2 C_{22} + \cdots + b_n C_{n2} \\ \vdots \\ b_1 C_{1n} + b_2 C_{2n} + \cdots + b_n C_{nn} \end{pmatrix} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} \det(\mathbf{A}_1) \\ \det(\mathbf{A}_2) \\ \vdots \\ \det(\mathbf{A}_n) \end{pmatrix}$$



Solving Linear Equations

A set of linear equations

if the inverse matrix exists

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned} \iff \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} e \\ f \end{bmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0 \implies \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix}$$

$$\begin{vmatrix} e & b \\ f & d \end{vmatrix} = de - bf \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} de - bf \\ -ce + af \end{bmatrix}$$

$$\begin{vmatrix} a & e \\ c & f \end{vmatrix} = af - ce$$

Cramer's Rule

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

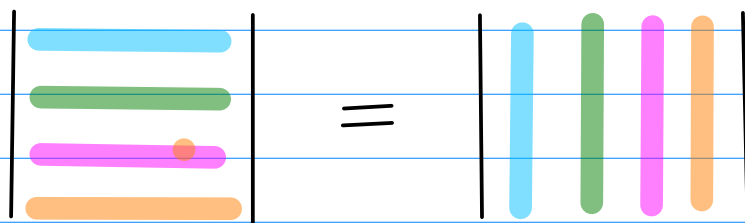
Cramer's Rule

Determinant of order 3

$$\begin{aligned} a_1x + a_2y + a_3z &= d \\ b_1x + b_2y + b_3z &= e \\ c_1x + c_2y + c_3z &= f \end{aligned} \iff \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

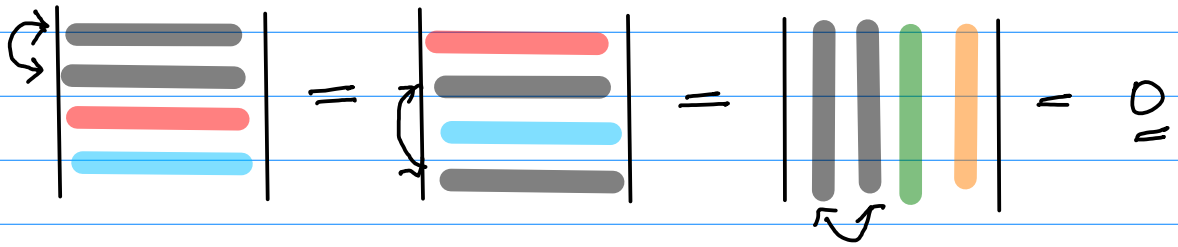
$$x = \frac{\begin{vmatrix} d & a_2 & a_3 \\ e & b_2 & b_3 \\ f & c_2 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}} \quad y = \frac{\begin{vmatrix} a_1 & d & a_3 \\ b_1 & e & b_3 \\ c_1 & f & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}} \quad z = \frac{\begin{vmatrix} a_1 & a_2 & d \\ b_1 & b_2 & e \\ c_1 & c_2 & f \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}$$

$$\det(A^T) = \det(A)$$



The diagram illustrates the equality of the determinant of a matrix and its transpose. On the left, a 4x4 matrix is shown with horizontal rows of different colors: blue, green, pink, and orange. On the right, the same 4x4 matrix is shown with vertical columns of the same colors: blue, green, pink, and orange. An equals sign is placed between the two matrices, indicating that $\det(A^T) = \det(A)$.

(any 2 rows
any 2 columns) are the same $\det(A) = 0$



$$\begin{vmatrix} 0 & 0 & \dots & 0 \end{vmatrix} = \begin{vmatrix} 0 & \dots & 0 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 4 & 2 & 1 \\ 5 & -2 & 1 \\ 7 & 4 & -2 \end{vmatrix} = \begin{vmatrix} 4 & 2 \cdot 1 & 1 \\ 5 & 2 \cdot -1 & 1 \\ 7 & 2 \cdot 2 & 2 \end{vmatrix} = 2 \begin{vmatrix} 4 & 1 & 1 \\ 5 & -1 & 1 \\ 7 & 2 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 5 & 8 \\ 20 & 16 \end{vmatrix} = \begin{vmatrix} 5 & 8 \\ 4 \cdot 5 & 4 \cdot 4 \end{vmatrix} = 4 \begin{vmatrix} 5 & 8 \\ 5 & 4 \end{vmatrix} = 4 \cdot 5 \begin{vmatrix} 1 & 8 \\ 1 & 4 \end{vmatrix}$$

$$4 \cdot 5 \begin{vmatrix} 1 & 4 \cdot 2 \\ 1 & 4 \cdot 1 \end{vmatrix} = 4 \cdot 5 \cdot 4 \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$$

$$\det(A \cdot B) = \det(A) \cdot \det(B)$$

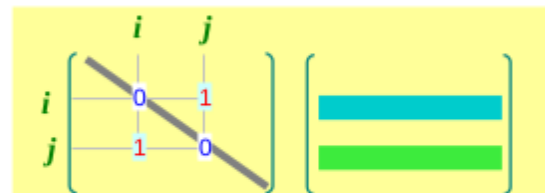
Product of Elementary Matrices

$$\begin{array}{c}
 E_3 \\
 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \cdot \\
 E_2 \\
 \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \\
 E_1 \\
 \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \\
 \left(\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array} \right) \\
 \\
 E_6 \\
 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \\
 E_5 \\
 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \cdot \\
 E_4 \\
 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 \\
 E_9 \\
 \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \\
 E_8 \\
 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \\
 E_7 \\
 \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
 \end{array}$$

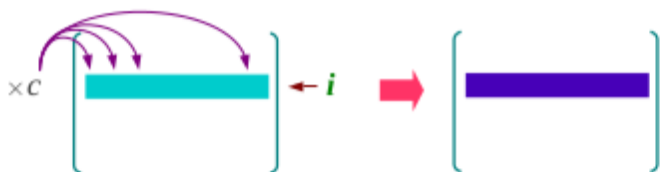
$$\det(E_i) = 1$$

Elementary Matrix

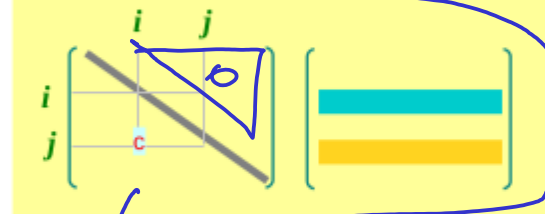
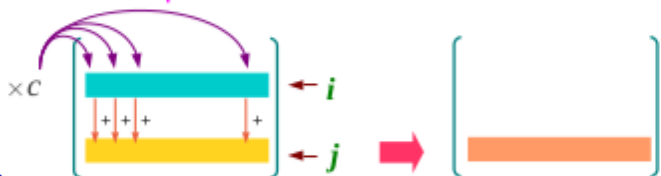
Interchange two rows



Multiply a row by a nonzero constant



Add a multiple of one row to another



Elementary Matrix (2A)

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$\det = 1$

$$E A = B$$

$$\det(E) \cdot \det(A) = \det(B)$$

$$\stackrel{1}{\det(E)}$$

$$\det(A) = \det(B)$$

Gauss-Jordan Elimination - Step 1

$$\begin{array}{rcl}
 +2x_1 + x_2 - x_3 = 8 & (L_1) & \left(\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array} \right) \\
 -3x_1 - x_2 + 2x_3 = -11 & (L_2) & \\
 -2x_1 + x_2 + 2x_3 = -3 & (L_3) &
 \end{array}$$

$$+1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = 4 \quad \left(\frac{1}{2} \times L_1\right) \quad +2/2 \quad +1/2 \quad -1/2 \quad +8/2$$

$$\begin{array}{rcl}
 +1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = 4 & \left(\frac{1}{2} \times L_1\right) & \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array} \right) \\
 -3x_1 - x_2 + 2x_3 = -11 & (L_2) & \\
 -2x_1 + x_2 + 2x_3 = -3 & (L_3) &
 \end{array}$$

Row Reduciton (1A)

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Gauss-Jordan Elimination - Step 2

$$\begin{array}{rcl}
 +1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 & (L_1) & \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array} \right) \\
 -3x_1 - x_2 + 2x_3 = -11 & (L_2) & \\
 -2x_1 + x_2 + 2x_3 = -3 & (L_3) &
 \end{array}$$

$$\begin{array}{rcl}
 +3x_1 + \frac{3}{2}x_2 - \frac{3}{2}x_3 = +12 & \boxed{3 \times L_1} & \begin{array}{ccc|c} +3 & +3/2 & -3/2 & +12 \\ -3 & -1 & +2 & -11 \end{array} \\
 -3x_1 - x_2 + 2x_3 = -11 & (L_2) &
 \end{array}$$

$$\begin{array}{rcl}
 +2x_1 + \frac{2}{2}x_2 - \frac{2}{2}x_3 = +8 & \boxed{2 \times L_1} & \begin{array}{ccc|c} +2 & +2/2 & -2/2 & +8 \\ -2 & +1 & +2 & -3 \end{array} \\
 -2x_1 + x_2 + 2x_3 = -3 & (L_3) &
 \end{array}$$

$$\begin{array}{rcl}
 +1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 & (L_1) & \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array} \right) \\
 0x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 = +1 & \boxed{3 \times L_1} + L_2 & \\
 0x_1 + 2x_2 + 1x_3 = +5 & \boxed{2 \times L_1} + L_3 &
 \end{array}$$

Row Reduciton (1A)

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$$\begin{vmatrix} a & & & \\ 0 & b & & \\ 0 & 0 & c & \\ 0 & 0 & 0 & d \end{vmatrix} = a \cdot b \cdot c \cdot d = \begin{vmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{vmatrix}$$