Cramer's Rule (H1)

20160106

Solving a System of Linear Equations

3x3 A A^{-1} Inverse Matrix.

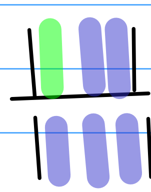
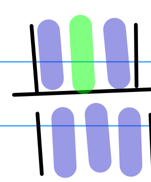
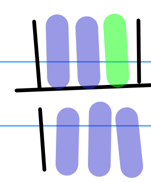
$$\begin{aligned} p_1 x + p_2 y + p_3 z &= b_1 \\ q_1 x + q_2 y + q_3 z &= b_2 \\ r_1 x + r_2 y + r_3 z &= b_3 \end{aligned}$$

$$\begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

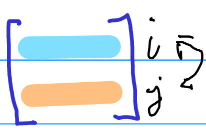
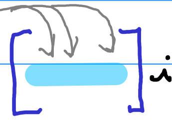
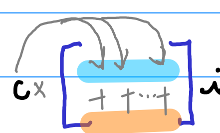
$$A \quad x = b$$

① A^{-1} $\qquad x = A^{-1} \cdot m$

② Cramer's Rule

$x =$  $y =$  $z =$ 

③ Gauss - Jordan Elimination RREF

$R_{ij} =$  $cR_i = c\times$  $cR_i + R_j = c\times$ 

Solving Linear Equations

A set of linear equations

If the inverse matrix exists

$$
\begin{aligned} a\,x + b\,y &= e \\ c\,x + d\,y &= f \end{aligned}
\quad \Longleftrightarrow \quad
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}
\quad \Rightarrow \quad
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} e \\ f \end{bmatrix}
$$

$$
\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0
\quad \Rightarrow \quad
\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix}
$$

$$
\begin{vmatrix} e & b \\ f & d \end{vmatrix} = de - bf
\qquad
\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} de - bf \\ -ce + af \end{bmatrix}
$$

$$
\begin{vmatrix} a & e \\ c & f \end{vmatrix} = af - ce
$$

Cramer's Rule

$$
x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}
\qquad
y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}
$$

Cramer's Rule

Determinant of order 3

$$
\begin{aligned} a_1 x + a_2 y + a_3 z &= d \\ b_1 x + b_2 y + b_3 z &= e \\ c_1 x + c_2 y + c_3 z &= f \end{aligned}
\quad \Longleftrightarrow \quad
\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}
$$

$$
x = \frac{\begin{vmatrix} d & a_2 & a_3 \\ e & b_2 & b_3 \\ f & c_2 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}
\qquad
y = \frac{\begin{vmatrix} a_1 & d & a_3 \\ b_1 & e & b_3 \\ c_1 & f & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}
\qquad
z = \frac{\begin{vmatrix} a_1 & a_2 & d \\ b_1 & b_2 & e \\ c_1 & c_2 & f \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}
$$

Cramer's Rule (1) – solutions

$$\overset{A}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}} \overset{x}{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}} \begin{matrix} = \\ = \\ \\ = \end{matrix} \overset{b}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}$$

$$A_1 = \begin{pmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad A_2 = \begin{pmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{pmatrix} \quad A_n = \begin{pmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{pmatrix}$$

$$x_1 = \frac{det(A_1)}{det(A)} \qquad x_2 = \frac{det(A_2)}{det(A)} \qquad x_n = \frac{det(A_n)}{det(A)}$$

Cramer's Rule (2) – determinants

$$A_1 = \begin{pmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$det(A_1) = b_1 C_{11} + b_2 C_{21} + \cdots + b_n C_{n1}$$

cofactor expansion along the first column

$$x_1 = \frac{det(A_1)}{det(A)}$$

$$A_2 = \begin{pmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{pmatrix}$$

$$det(A_2) = b_1 C_{12} + b_2 C_{22} + \cdots + b_n C_{n2}$$

cofactor expansion along the second column

$$x_2 = \frac{det(A_2)}{det(A)}$$

$$A_n = \begin{pmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{pmatrix}$$

$$det(A_n) = b_1 C_{1n} + b_2 C_{2n} + \cdots + b_n C_{nn}$$

cofactor expansion along the last column

$$x_n = \frac{det(A_n)}{det(A)}$$

Cramer's Rule (3) – inverse matrix

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \qquad \boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b} = \frac{adj(\boldsymbol{A})}{det(\boldsymbol{A})}\boldsymbol{b}$$

$$\boldsymbol{x} = \frac{1}{det(\boldsymbol{A})}\begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

the transposed matrix
note reverse order index

$$\boldsymbol{x} = \frac{1}{det(\boldsymbol{A})}\begin{pmatrix} b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1} \\ b_1C_{12} + b_2C_{22} + \cdots + b_nC_{n2} \\ \vdots \\ b_1C_{1n} + b_2C_{2n} + \cdots + b_nC_{nn} \end{pmatrix} = \frac{1}{det(\boldsymbol{A})}\begin{pmatrix} det(\boldsymbol{A}_1) \\ det(\boldsymbol{A}_2) \\ \vdots \\ det(\boldsymbol{A}_n) \end{pmatrix}$$

Solving Linear Equations

A set of linear equations

If the inverse matrix exists

$$a\,x + b\,y = e \qquad c\,x + d\,y = f \quad \Longleftrightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\begin{bmatrix} e \\ f \end{bmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0 \quad \Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} e \\ f \end{bmatrix}$$

$$\begin{vmatrix} e & b \\ f & d \end{vmatrix} = de - bf \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} de - bf \\ -ce + af \end{bmatrix}$$

$$\begin{vmatrix} a & e \\ c & f \end{vmatrix} = af - ce$$

Cramer's Rule

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

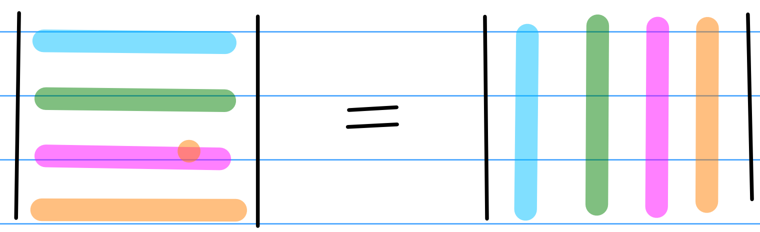
Cramer's Rule

Determinant of order 3

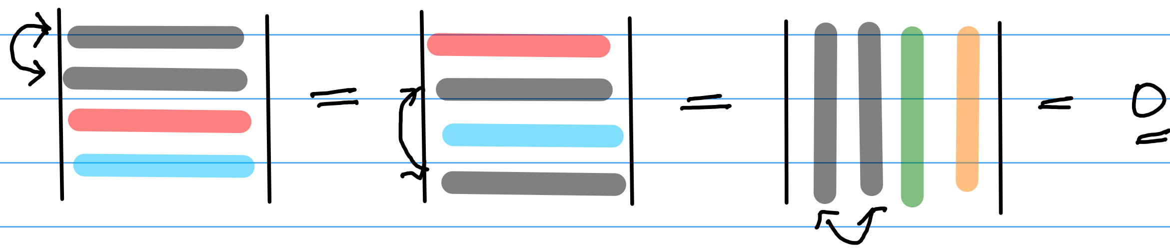
$$\begin{aligned} a_1 x + a_2 y + a_3 z &= d \\ b_1 x + b_2 y + b_3 z &= e \\ c_1 x + c_2 y + c_3 z &= f \end{aligned} \quad \Longleftrightarrow \quad \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} d & a_2 & a_3 \\ e & b_2 & b_3 \\ f & c_2 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & d & a_3 \\ b_1 & e & b_3 \\ c_1 & f & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_1 & a_2 & d \\ b_1 & b_2 & e \\ c_1 & c_2 & f \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}$$

$$\det(A^T) = \det(A)$$



$\left(\begin{array}{l}\text{any 2 rows}\\ \text{any 2 columns}\end{array}\right)$ are the same $\quad \det(A) = 0$



$$\begin{vmatrix} 0 & 0 & 0 & \cdots & 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 4 & 2 & 1 \\ 5 & -2 & 1 \\ 7 & 4 & -2 \end{vmatrix} = \begin{vmatrix} 4 & 2\cdot 1 & 1 \\ 5 & 2\cdot -1 & 1 \\ 7 & 2\cdot 2 & 2 \end{vmatrix} = 2 \begin{vmatrix} 4 & 1 & 1 \\ 5 & -1 & 1 \\ 7 & 2 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 5 & 8 \\ 20 & 16 \end{vmatrix} = \begin{vmatrix} 5 & 8 \\ 4\cdot 5 & 4\cdot 4 \end{vmatrix} = 4 \begin{vmatrix} 5 & 8 \\ 5 & 4 \end{vmatrix} = 4\cdot 5\cdot \begin{vmatrix} 1 & 8 \\ 1 & 4 \end{vmatrix}$$

$$4\cdot 5\cdot \begin{vmatrix} 1 & 4\cdot 2 \\ 1 & 4\cdot 1 \end{vmatrix} = 4\cdot 5\, 4\cdot \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$$

$$\det(A \cdot B) = \det(A) \cdot \det(B)$$

Product of Elementary Matrices

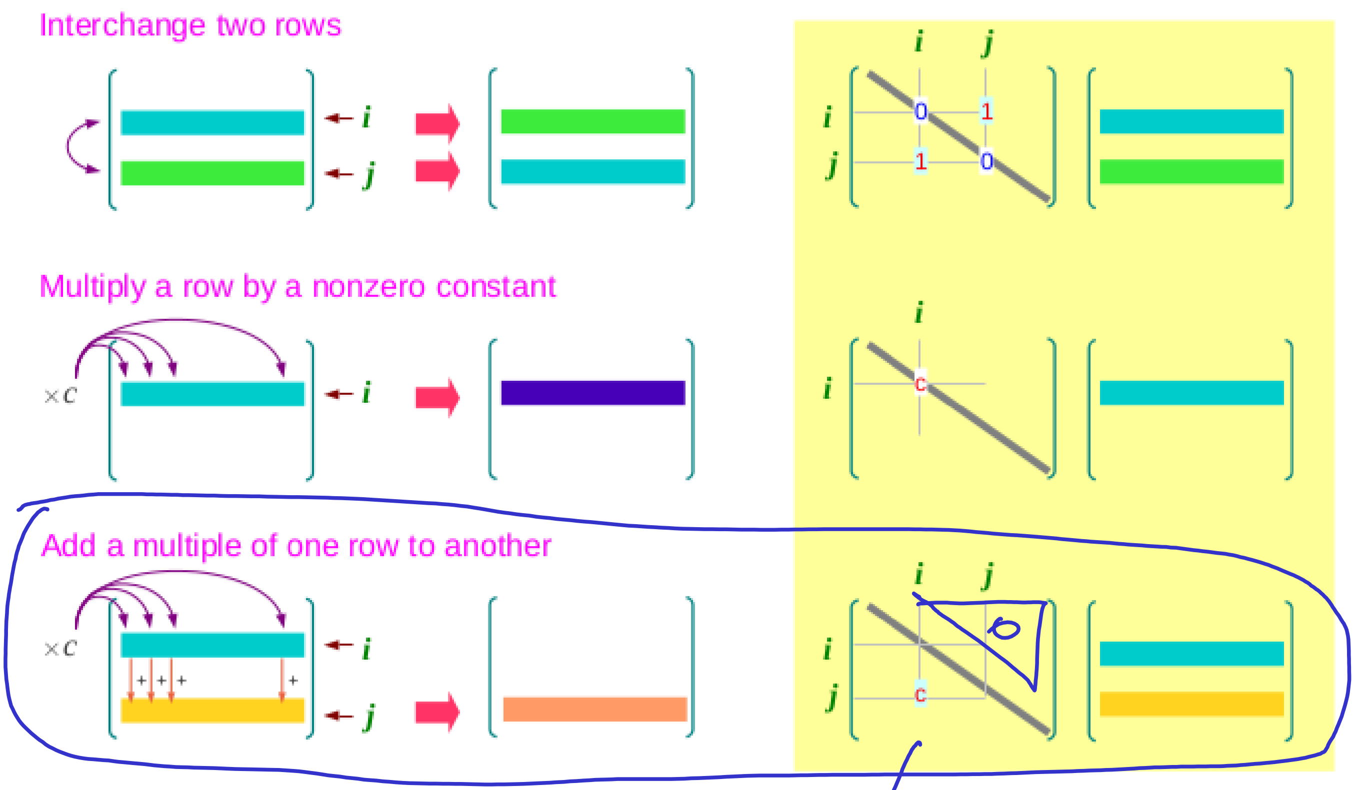
$$\underset{E_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}} \cdot \underset{E_2}{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{E_1}{\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \left[\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right]$$

$$\underset{E_6}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}} \cdot \underset{E_5}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}} \cdot \underset{E_4}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\underset{E_9}{\begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{E_8}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{E_7}{\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\det(E_i) = 1$$

Elementary Matrix



det = 1

$E\,A = B$

$\det(E) \cdot \det(A) = \det(B)$

$\det(E) = 1$

$\det(A) = \det(B)$

Gauss-Jordan Elimination – Step 1

$$+2x_1 + x_2 - x_3 = 8 \qquad (L_1)$$
$$-3x_1 - x_2 + 2x_3 = -11 \qquad (L_2)$$
$$-2x_1 + x_2 + 2x_3 = -3 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right)$$

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = 4 \qquad (\tfrac{1}{2} \times L_1)$$

+2/2 +1/2 −1/2 +8/2

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = 4 \qquad (\tfrac{1}{2} \times L_1)$$
$$-3x_1 - x_2 + 2x_3 = -11 \qquad (L_2)$$
$$-2x_1 + x_2 + 2x_3 = -3 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right)$$

Gauss-Jordan Elimination – Step 2

out

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$-3x_1 - x_2 + 2x_3 = -11 \qquad (L_2)$$
$$-2x_1 + x_2 + 2x_3 = -3 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ \boxed{-3} & -1 & +2 & -11 \\ \boxed{-2} & +1 & +2 & -3 \end{array}\right)$$

$$+3x_1 + \tfrac{3}{2}x_2 - \tfrac{3}{2}x_3 = +12 \qquad (\boxed{3 \times L_1})$$
$$-3x_1 - x_2 + 2x_3 = -11 \qquad (L_2)$$

+3 +3/2 −3/2 +12
−3 −1 +2 −11

$$+2x_1 + \tfrac{2}{2}x_2 - \tfrac{2}{2}x_3 = +8 \qquad (\boxed{2 \times L_1})$$
$$-2x_1 + x_2 + 2x_3 = -3 \qquad (L_3)$$

+2 +2/2 −2/2 +8
−2 +1 +2 −3

out

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 = +1 \qquad (\boxed{3 \times L_1} + L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \qquad (\boxed{2 \times L_1} + L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ \boxed{0} & +1/2 & +1/2 & +1 \\ \boxed{0} & +2 & +1 & +5 \end{array}\right)$$

$$\begin{vmatrix} a & & * & \\ 0 & b & & \\ 0 & 0 & c & \\ 0 & 0 & 0 & d \end{vmatrix} = a\cdot b\cdot c\cdot d = \begin{vmatrix} a & 0 & 0 & 0 \\ & b & 0 & 0 \\ * & & c & 0 \\ & & & d \end{vmatrix}$$