

Laurent Series and z-Transform

- Geometric Series

Permutations B

20240807 Wed

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a^n \times $R(n)$

a^n	a^{-n}

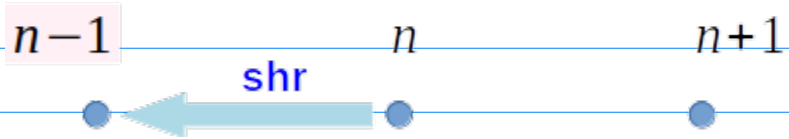
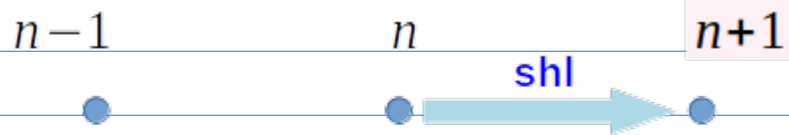
 \times

$u(n)$	$u(-n-1)$
$u(-n)$	$u(n-1)$

(1)	$a^n u(n)$	$a^{-n} u(n)$	(2)
(3)	$a^n u(-n)$	$a^{-n} u(-n)$	(4)
(5)	$a^n u(-n-1)$	$a^{-n} u(-n-1)$	(6)
(7)	$a^n u(n-1)$	$a^{-n} u(n-1)$	(8)

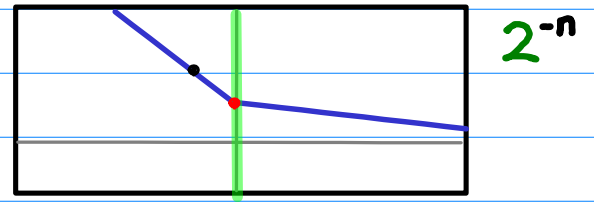
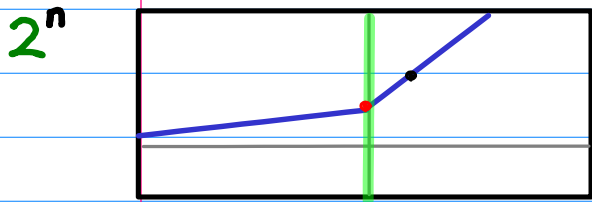
b^n	$shl(b^n)$	b^n	$shr(b^n)$
a^n	$a^{(n+1)}$	a^n	$a^{(n-1)}$
a^{-n}	$a^{-(n+1)}$	a^{-n}	$a^{-(n-1)}$

$R(n)$	$shl(R(n))$	$R(n)$	$shr(R(n))$
$u(n-1)$	$u(n)$	$u(n)$	$u(n-1)$
$u(-n)$	$u(-(n+1))$	$u(-(n+1))$	$u(-n)$



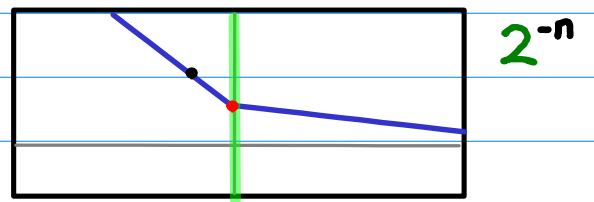
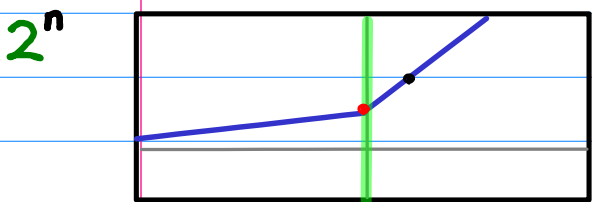
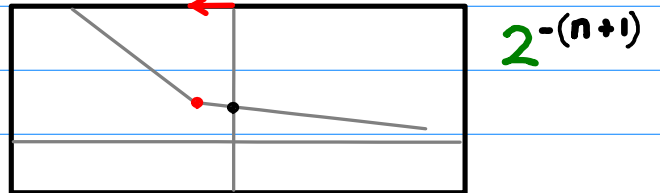
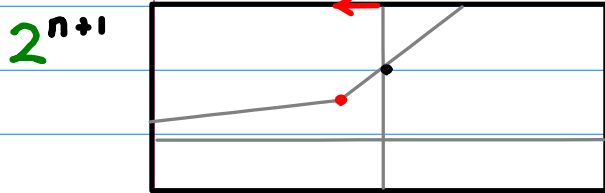
shl(b^n)
shr(b^n)

shl(n) = $n + 1$
shr(n) = $n - 1$



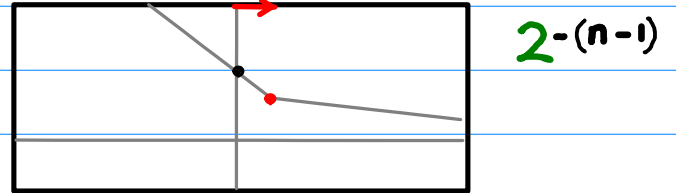
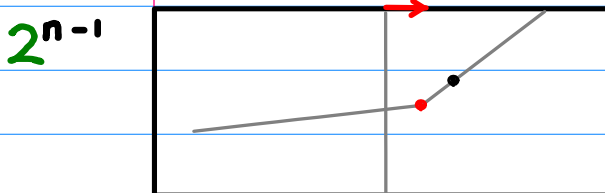
shift left
 $n \leftarrow n+1$

shift left
 $n \leftarrow n+1$



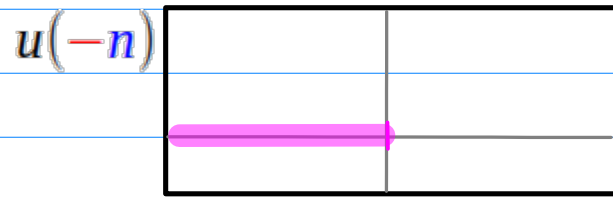
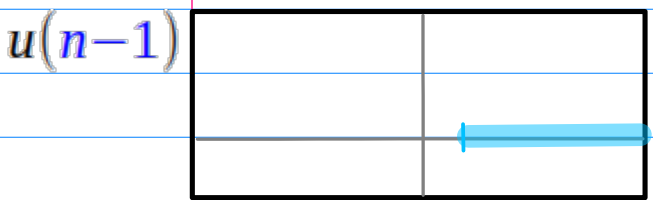
shift right
 $n \leftarrow n-1$

shift right
 $n \leftarrow n-1$



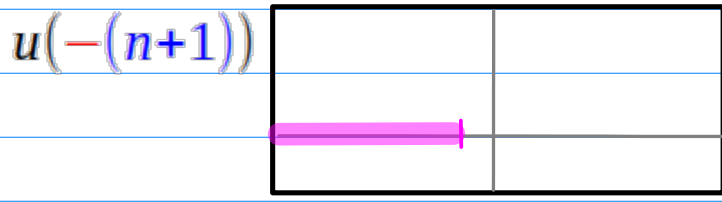
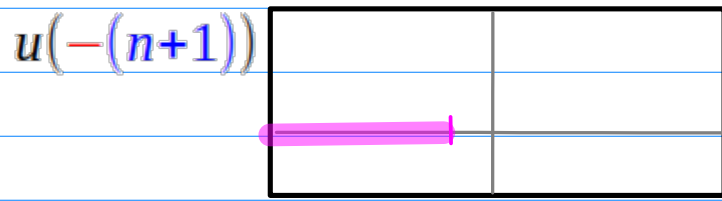
$\text{shl}(R(n))$
 $\text{shr}(R(n))$

$\text{shl}(n) = n + 1$
 $\text{shr}(n) = n - 1$



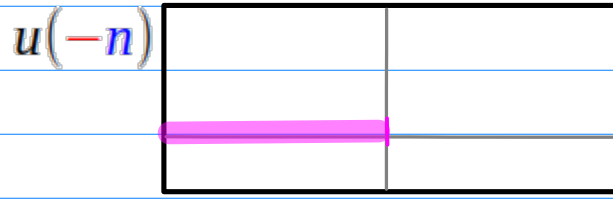
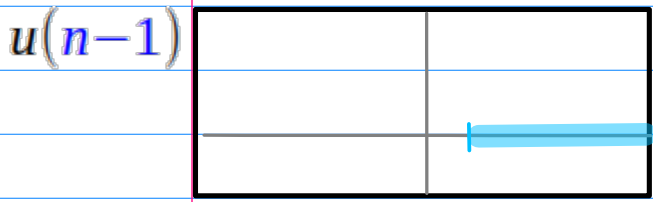
shift left
 $n \leftarrow n+1$

shift left
 $n \leftarrow n+1$



shift right
 $n \leftarrow n-1$

shift right
 $n \leftarrow n-1$



a^n \times $R(n)$

a^{n+1}	a^{-n-1}
a^{n-1}	a^{-n+1}

 \times

$u(n)$	$u(-n-1)$
$u(-n)$	$u(n-1)$

(1')	$a^{n+1}u(n)$	$a^{-n-1}u(n)$	(2')
(3')	$a^{n-1}u(-n)$	$a^{-n+1}u(-n)$	(4')
(5')	$a^{n+1}u(-n-1)$	$a^{-n-1}u(-n-1)$	(6')
(7')	$a^{n-1}u(n-1)$	$a^{-n+1}u(n-1)$	(8')

(1'')	$a^{n-1}u(n)$	$a^{-n+1}u(n)$	(2'')
(3'')	$a^{n+1}u(-n)$	$a^{-n-1}u(-n)$	(4'')
(5'')	$a^{n-1}u(-n-1)$	$a^{-n+1}u(-n-1)$	(6'')
(7'')	$a^{n+1}u(n-1)$	$a^{-n-1}u(n-1)$	(8'')

a^n \times $R(n)$

a^{n+1}	a^{-n-1}

 \times

$u(n)$	$u(-n-1)$
$u(-n)$	$u(n-1)$

$$(1') \quad a^{n+1} u(n) \quad a^{-n-1} u(n) \quad (2')$$

$$(3') \quad a^{n-1} u(-n) \quad a^{-n+1} u(-n) \quad (4')$$

$$(5') \quad a^{n+1} u(-n-1) \quad a^{-n-1} u(-n-1) \quad (6')$$

$$(7') \quad a^{n-1} u(n-1) \quad a^{-n+1} u(n-1) \quad (8')$$

a^n \times $R(n)$

a^{n+1}	a^{-n-1}
a^{n-1}	a^{-n+1}

 \times

$u(n)$	$u(-n-1)$
$u(-n)$	$u(n-1)$

(1'') $a^{n-1} u(n)$ $a^{-n+1} u(n)$ (2'')

(3'') $a^{n+1} u(-n)$ $a^{-n-1} u(-n)$ (4'')

(5'') $a^{n-1} u(-n-1)$ $a^{-n+1} u(-n-1)$ (6'')

(7'') $a^{n+1} u(n-1)$ $a^{-n-1} u(n-1)$ (8'')

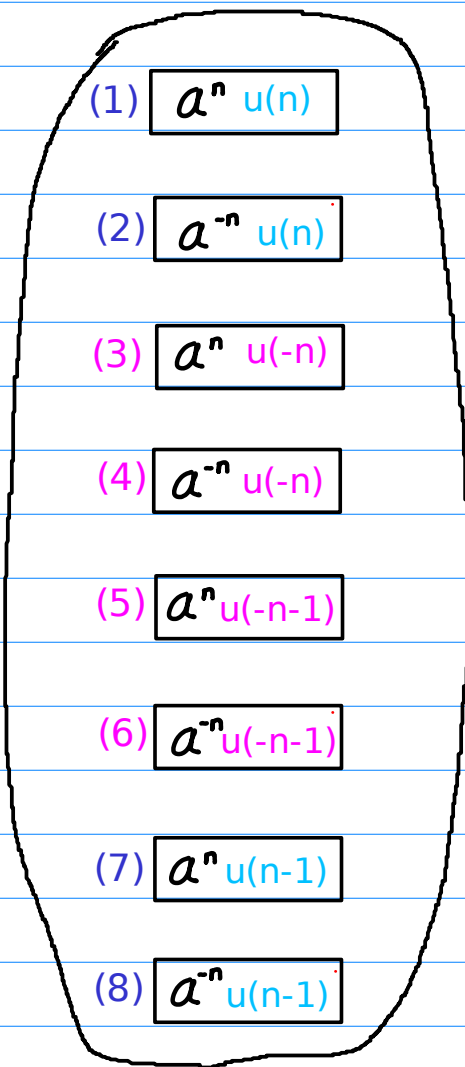
Unshifted Sequence x	Shifted Sequence 1 x'	Shifted Sequence 2 x''	
(1) $a^n u(n)$	$shl(b^n) a^{n+1} u(n) (1')$	$shl(b^{-n}) a^{-n-1} u(n) (1'')$	$n \leftarrow n+1$
(2) $a^{-n} u(n)$	$shl(b^n) a^{-n-1} u(n) (2')$	$shl(b^{-n}) a^{n+1} u(n) (2'')$	$n \leftarrow n+1$
(3) $a^n u(-n)$	$shr(b^n) a^{n-1} u(-n) (3')$	$shr(b^{-n}) a^{-n+1} u(-n) (3'')$	$n \leftarrow n-1$
(4) $a^{-n} u(-n)$	$shr(b^n) a^{-n+1} u(-n) (4')$	$shr(b^{-n}) a^{n-1} u(-n) (4'')$	$n \leftarrow n-1$
(5) $a^n u(-n-1)$	$shl(b^n) a^{n+1} u(-n-1) (5')$	$shl(b^{-n}) a^{-n-1} u(-n-1) (5'')$	$n \leftarrow n+1$
(6) $a^{-n} u(-n-1)$	$shl(b^n) a^{-n-1} u(-n-1) (6')$	$shl(b^{-n}) a^{n+1} u(-n-1) (6'')$	$n \leftarrow n+1$
(7) $a^n u(n-1)$	$shr(b^n) a^{n-1} u(n-1) (7')$	$shr(b^{-n}) a^{-n+1} u(n-1) (7'')$	$n \leftarrow n-1$
(8) $a^{-n} u(n-1)$	$shr(b^n) a^{-n+1} u(n-1) (8')$	$shr(b^{-n}) a^{n-1} u(n-1) (8'')$	$n \leftarrow n-1$

many possible permutations are possible
but consider these two

Unshifted Sequence x



Shifted Sequence $1 x'$



$shl(b^n)$

$a^{n+1} u(n)$ (1')

$shl(b^n)$

$a^{-n-1} u(n)$ (2')

$shr(b^n)$

$a^{n-1} u(-n)$ (3')

$shr(b^n)$

$a^{-n+1} u(-n)$ (4')

$shl(b^n)$

$a^{n+1} u(-n-1)$ (5')

$shl(b^n)$

$a^{-n-1} u(-n-1)$ (6')

$shr(b^n)$

$a^{n-1} u(n-1)$ (7')

$shr(b^n)$

$a^{-n+1} u(n-1)$ (8')

Inter-permutations over unshifted sequence and shifted sequence

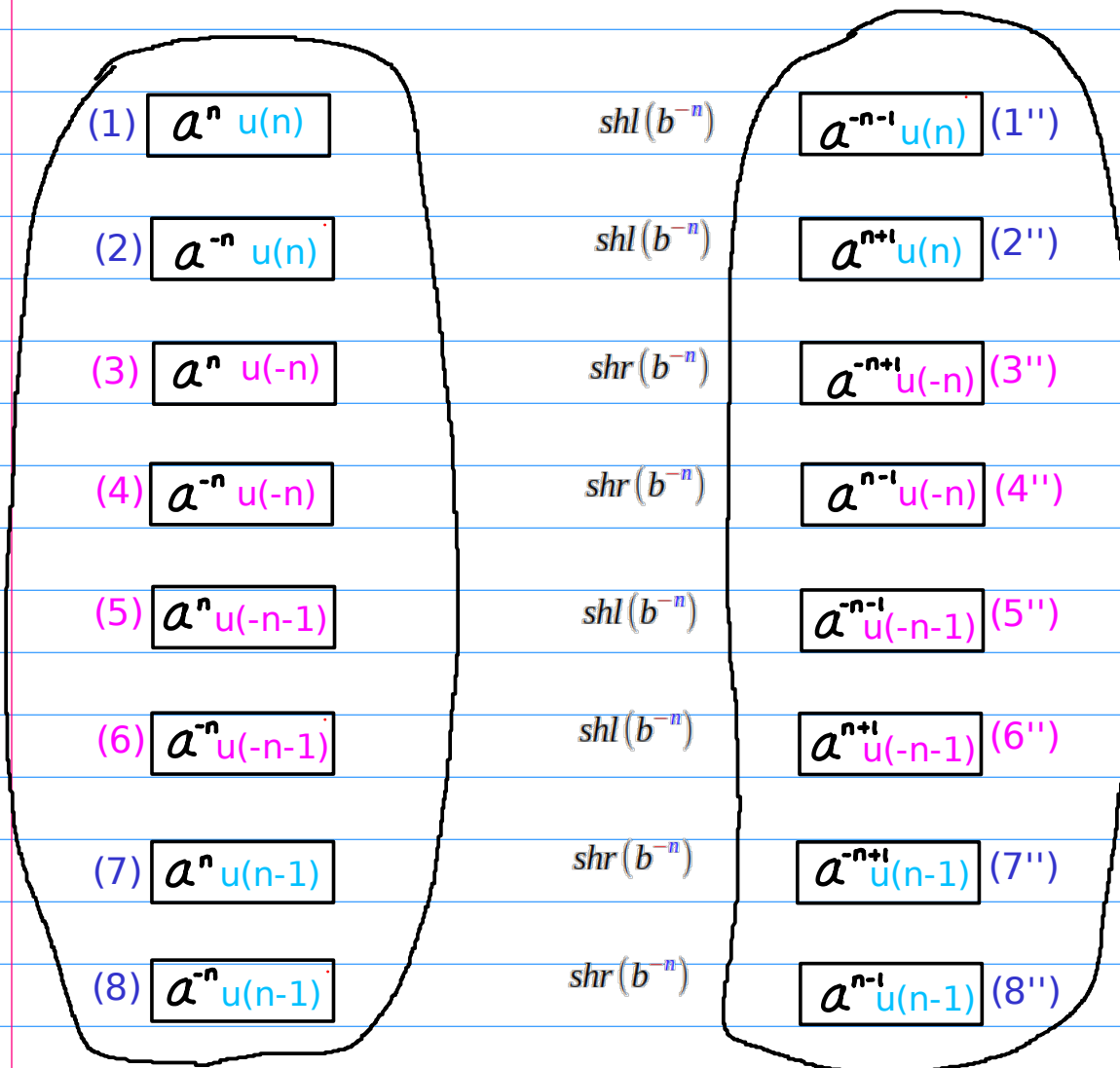
Intra-permutations over unshifted sequence

Intra-permutations over shifted sequence

Unshifted Sequence x



Shifted Sequence 2 x''



Inter-permutations over unshifted sequence and shifted sequence

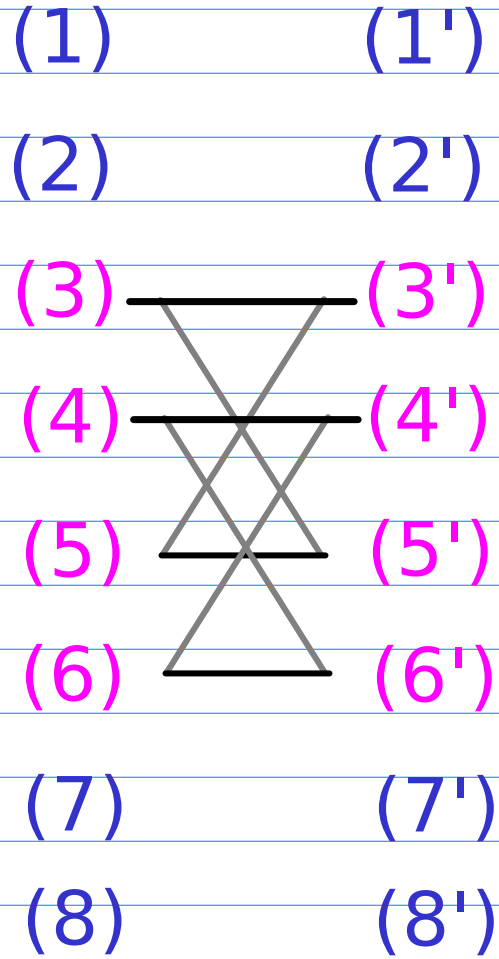
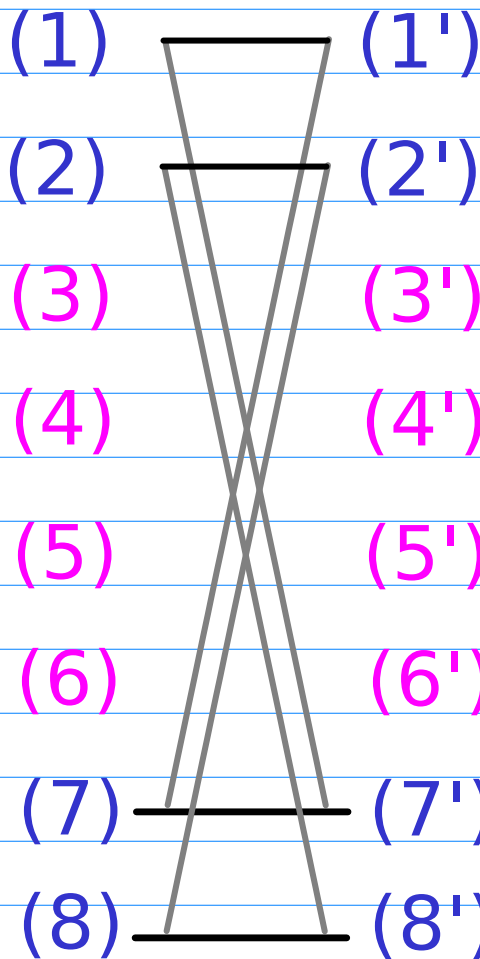
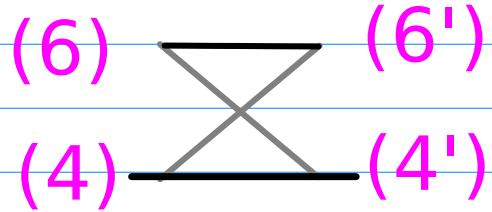
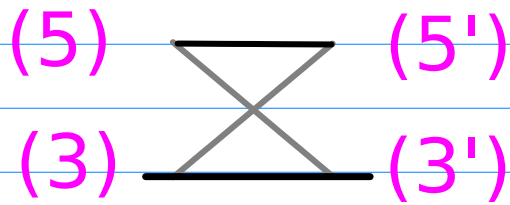
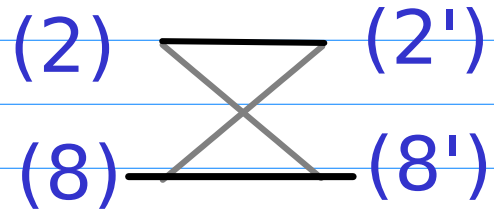
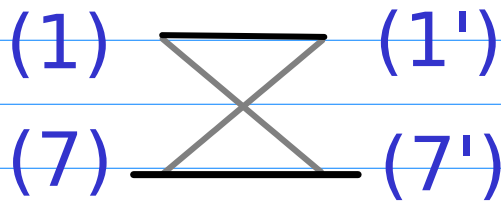
Intra-permutations over unshifted sequence

Intra-permutations over shifted sequence

(1) - (1')
(7) - (7')
(5) - (5')
(3) - (3')

(2) - (2')
(8) - (8')
(6) - (6')
(4) - (4')

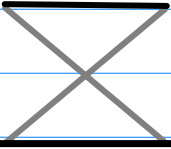

(x)  **(x')**
(1)~(8) (1')~(8')

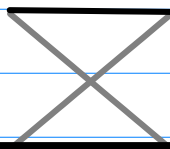



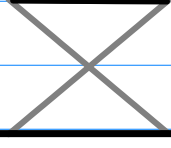

(1) - (1')
(7) - (7')
(5) - (5')
(3) - (3')

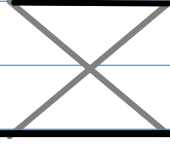

(2) - (2')
(8) - (8')
(6) - (6')
(4) - (4')


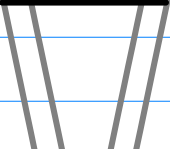
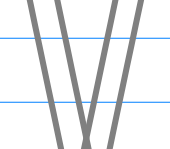
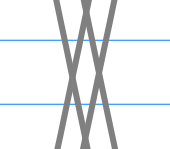
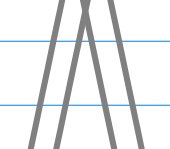
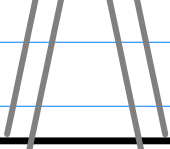


(x)  **(x'')**
(1)~(8) (1')~(8')





(1)  (1'')
(7)  (7'')

(2)  (2'')
(8)  (8'')

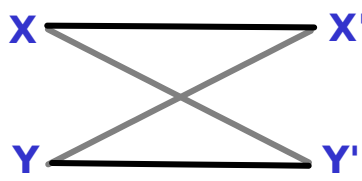
(5)  (5'')
(3)  (3'')

(6)  (6'')
(4)  (4'')

(1)  (1'')
(2)  (2'')
(3)  (3'')
(4)  (4'')
(5)  (5'')
(6)  (6'')
(7)  (7'')
(8)  (8'')

(1) (1'')
(2) (2'')
(3)  (3'')
(4)  (4'')
(5)  (5'')
(6)  (6'')
(7) (7'')
(8) (8'')

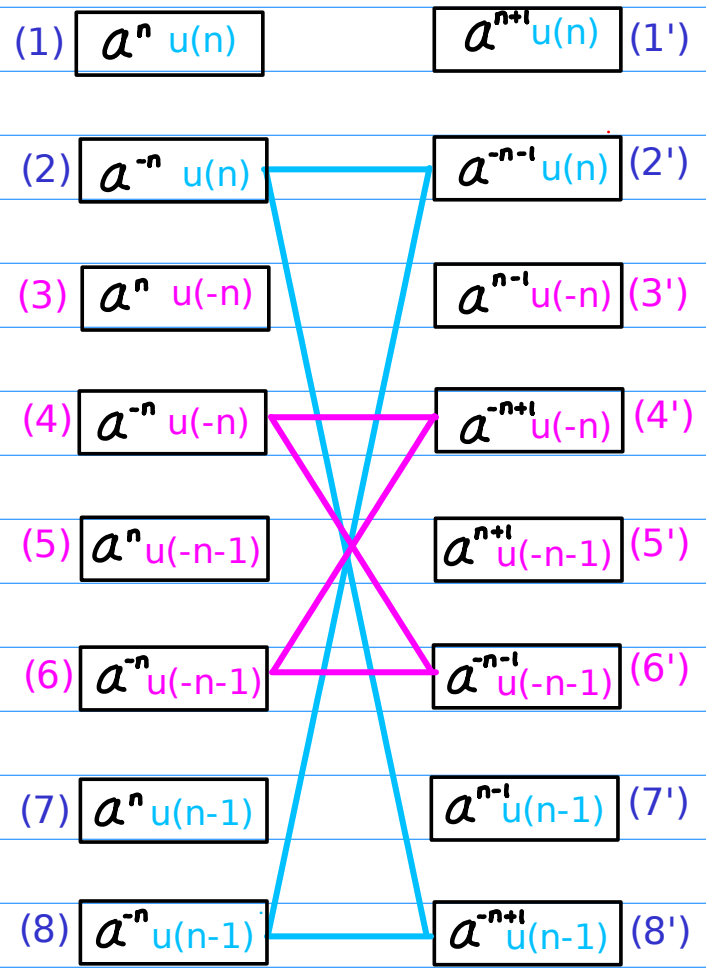
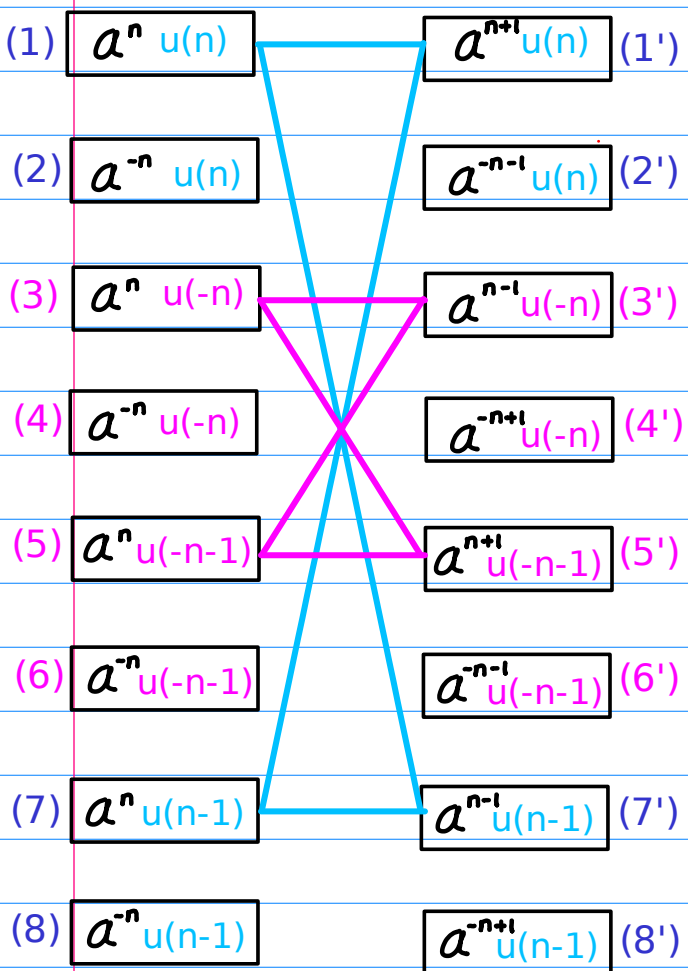
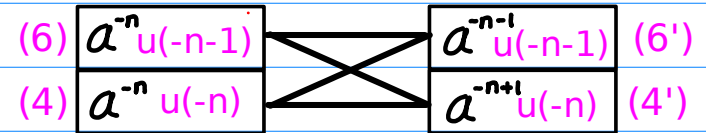
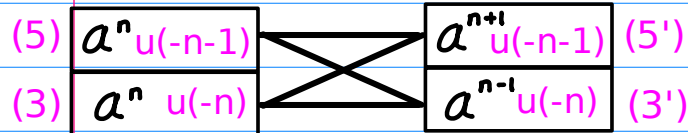
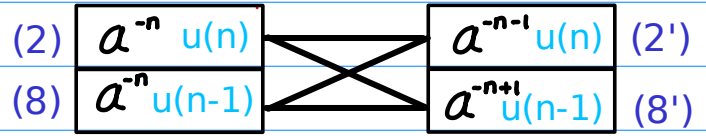
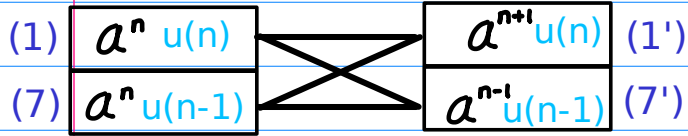
Unshifted Sequence



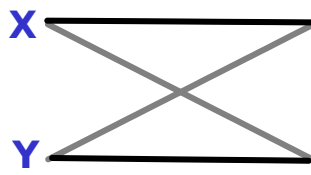
SL

Shifted Sequence

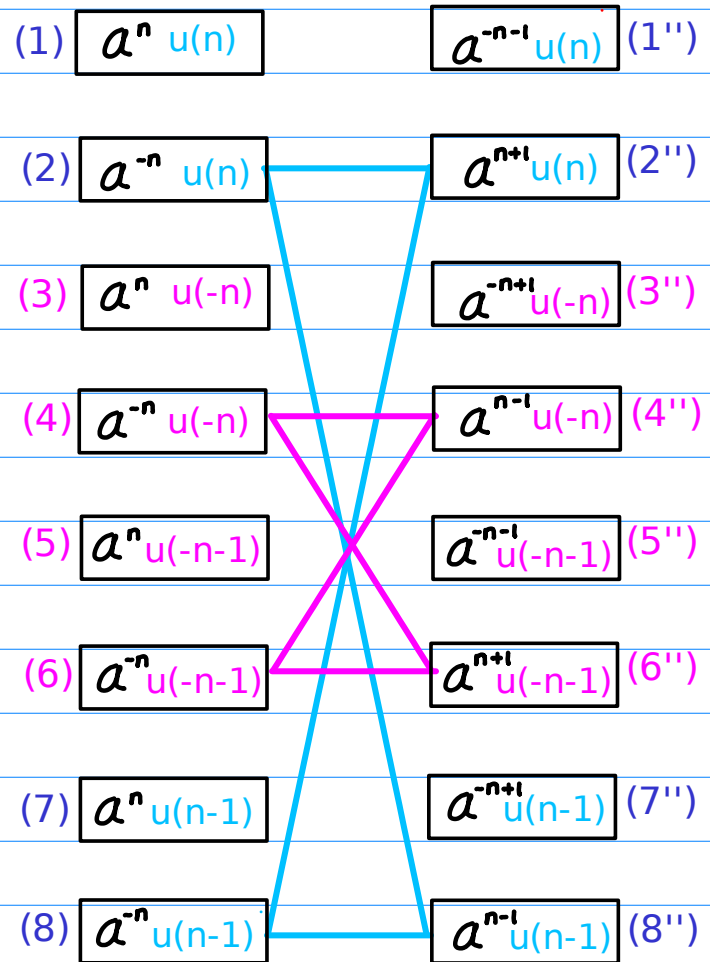
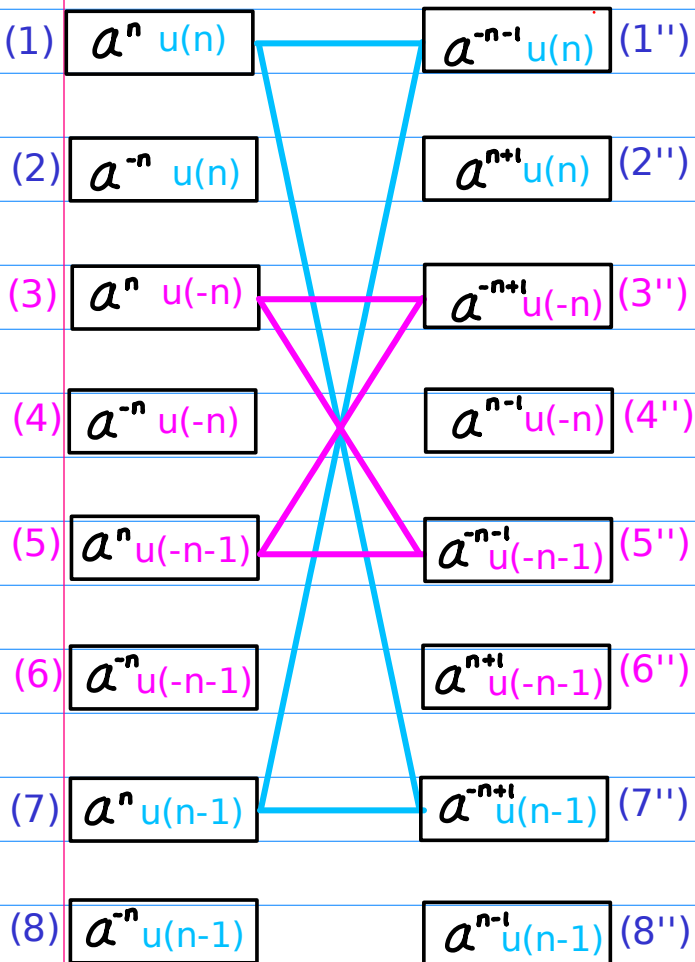
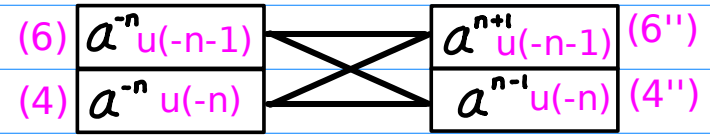
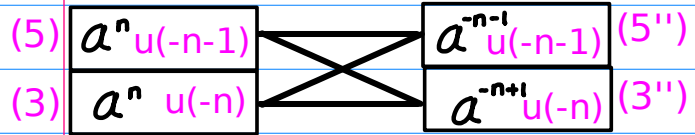
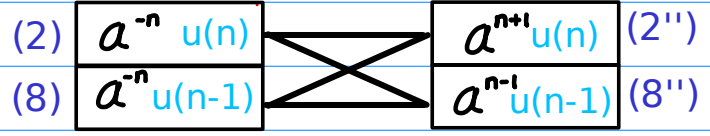
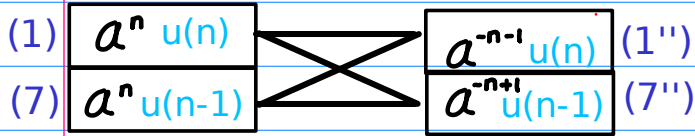
SR



Unshifted Sequence



Shifted Sequence

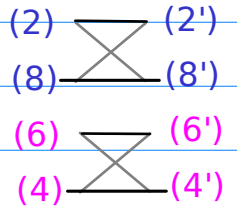
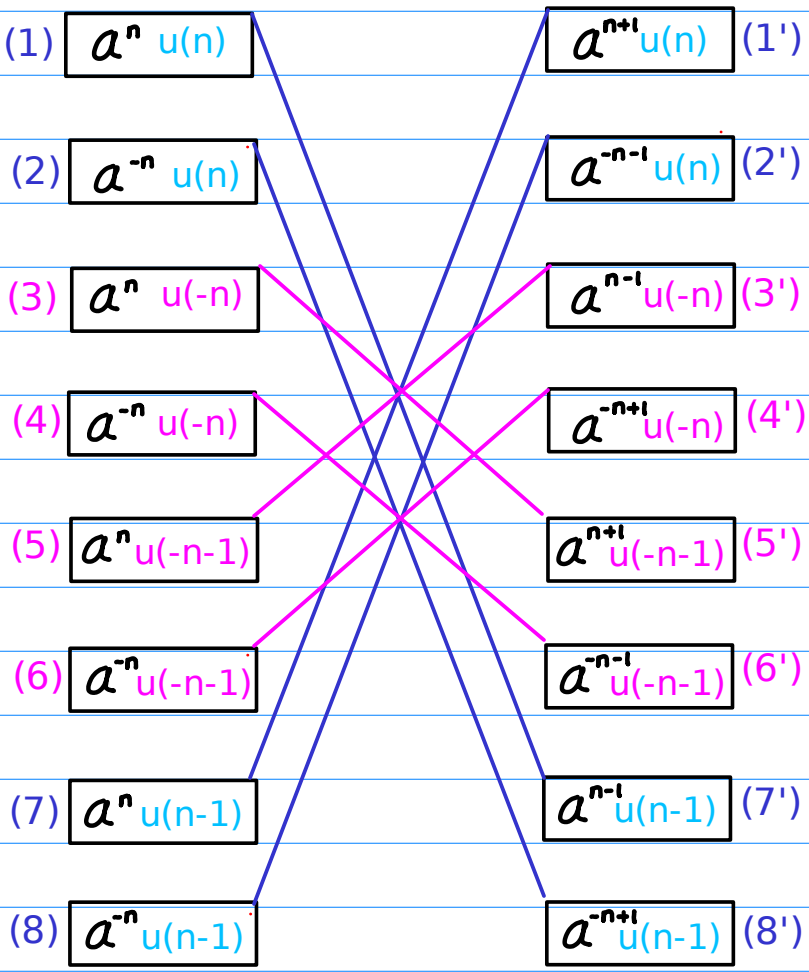
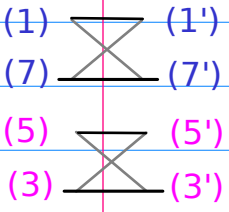


Unshifted Sequence x

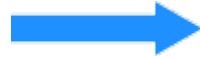


Shifted Sequence 1 x'

(1)	$\frac{1}{1-2z^{-1}} 2^n u(n)$	(2)	$\frac{1}{1-0.5z^{-1}} 2^{-n} u(n)$	(1')	$\frac{2}{1-2z^{-1}} 2^{n+1} u(n)$	(2')	$\frac{0.5}{1-0.5z^{-1}} 2^{-n-1} u(n)$
(3)	$\frac{1}{1-0.5z^{-1}} 2^n u(-n)$	(4)	$\frac{1}{1-2z^{-1}} 2^{-n} u(-n)$	(3')	$\frac{0.5}{1-0.5z^{-1}} 2^{n-1} u(-n)$	(4')	$\frac{2}{1-2z^{-1}} 2^{-n+1} u(-n)$
(5)	$\frac{0.5z^{-1}}{1-0.5z^{-1}} 2^n u(-n-1)$	(6)	$\frac{2z^{-1}}{1-2z^{-1}} 2^{-n} u(-n-1)$	(5')	$\frac{z^{-1}}{-0.5z^{-1}} 2^{n+1} u(-n-1)$	(6')	$\frac{z^{-1}}{1-2z^{-1}} 2^{-n-1} u(-n-1)$
(7)	$\frac{2z}{1-2z} 2^n u(n-1)$	(8)	$\frac{0.5z}{1-0.5z} 2^{-n} u(n-1)$	(7')	$\frac{z}{1-2z} 2^{n-1} u(n-1)$	(8')	$\frac{z}{1-0.5z} 2^{-n+1} u(n-1)$

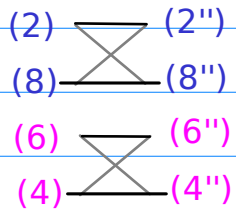
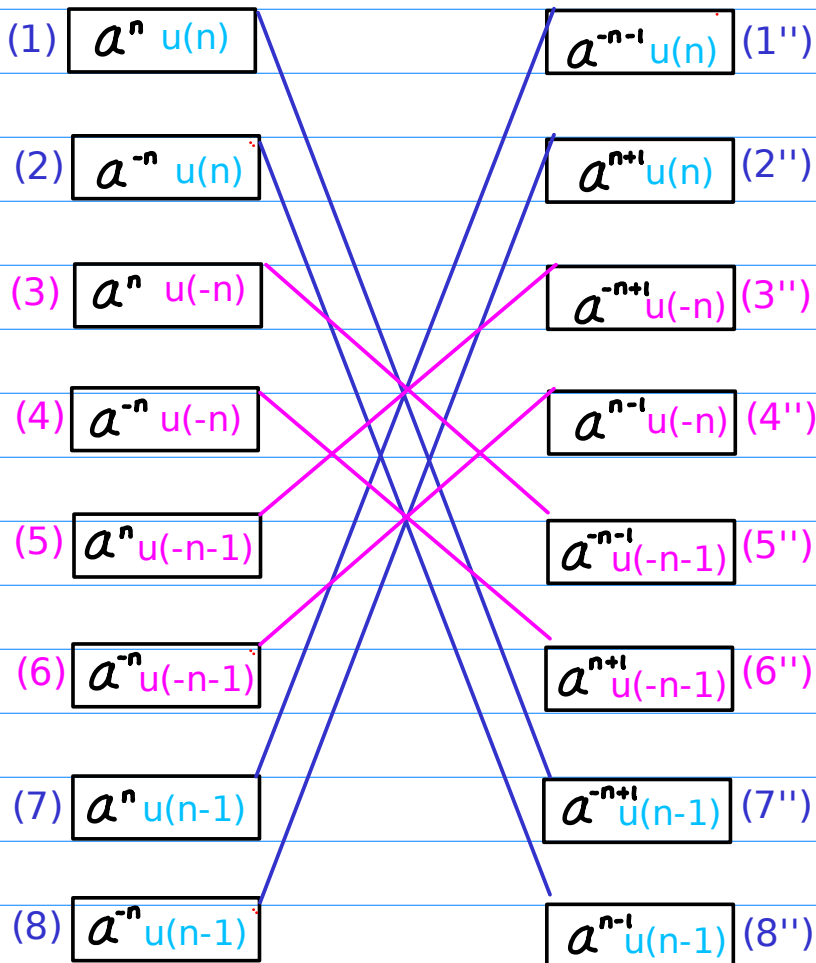
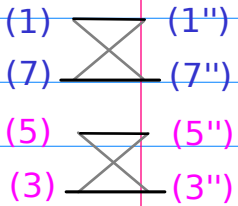


Unshifted Sequence x



Shifted Sequence $2x''$

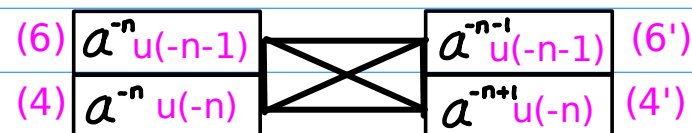
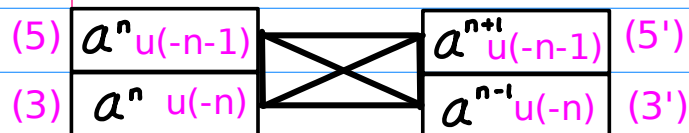
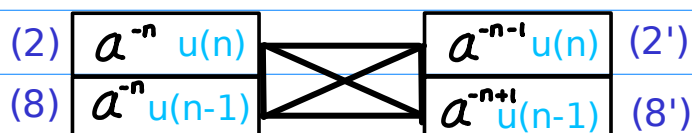
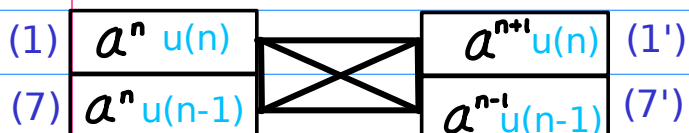
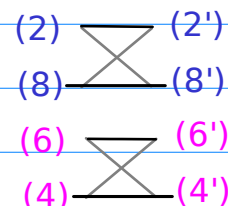
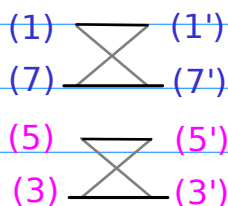
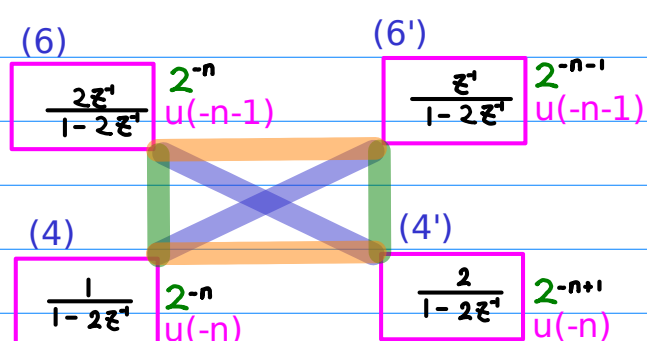
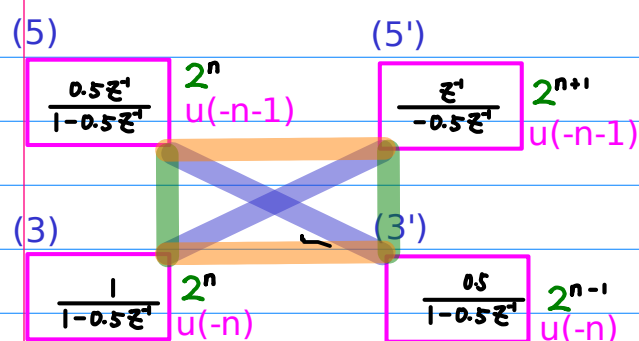
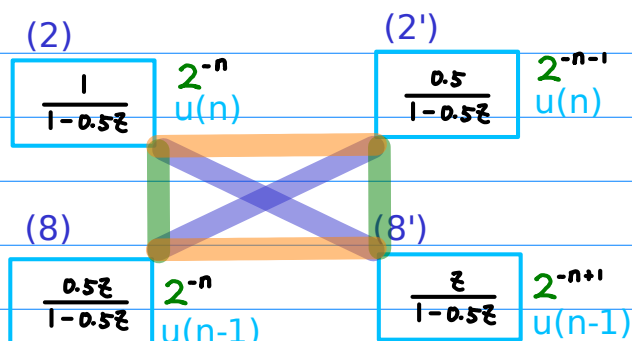
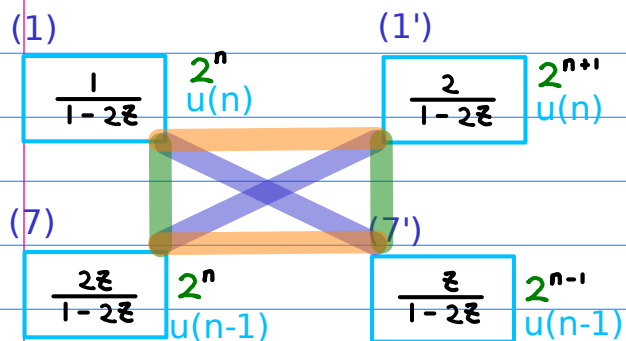
(1)	$\frac{1}{1-2z} 2^n u(n)$	(2)	$\frac{1}{1-0.5z} 2^{-n} u(n)$	(1'')	$\frac{0.5}{1-0.5z} 2^{-n-1} u(n)$	(2'')	$\frac{2}{1-2z} 2^{n+1} u(n)$
(3)	$\frac{1}{1-0.5z^{-1}} 2^n u(-n)$	(4)	$\frac{1}{1-2z^{-1}} 2^{-n} u(-n)$	(3'')	$\frac{2}{1-2z^{-1}} 2^{-n+1} u(-n)$	(4'')	$\frac{0.5}{1-0.5z^{-1}} 2^{n-1} u(-n)$
(5)	$\frac{0.5z^{-1}}{1-0.5z^{-1}} 2^n u(-n-1)$	(6)	$\frac{2z^{-1}}{1-2z^{-1}} 2^{-n} u(-n-1)$	(5'')	$\frac{z^{-1}}{1-2z^{-1}} 2^{-n-1} u(-n-1)$	(6'')	$\frac{z^{-1}}{-0.5z^{-1}} 2^{n+1} u(-n-1)$
(7)	$\frac{2z}{1-2z} 2^n u(n-1)$	(8)	$\frac{0.5z}{1-0.5z} 2^{-n} u(n-1)$	(7'')	$\frac{z}{1-0.5z} 2^{-n+1} u(n-1)$	(8'')	$\frac{z}{1-2z} 2^{n-1} u(n-1)$



Inter-permutation

(x) \longrightarrow (x')

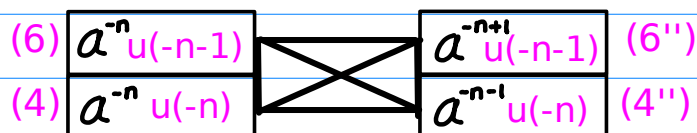
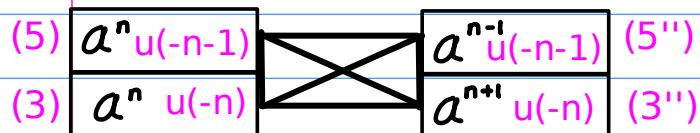
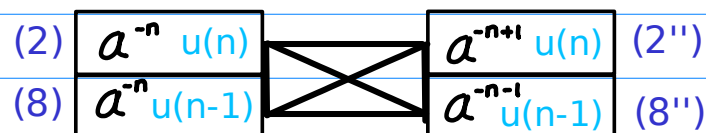
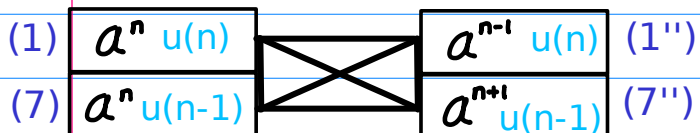
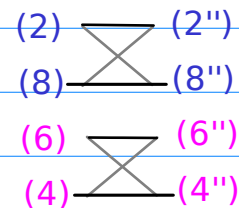
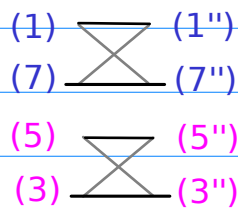
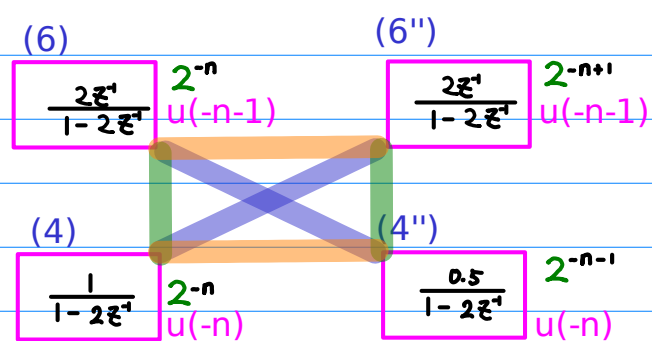
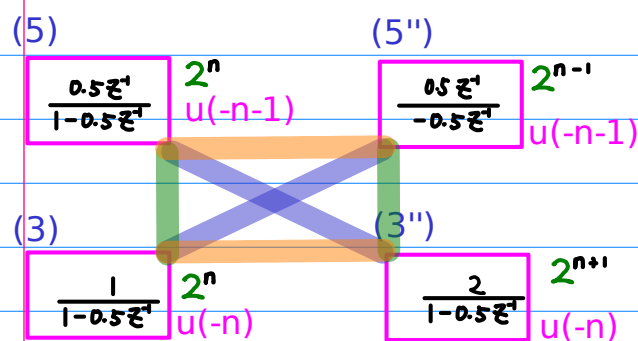
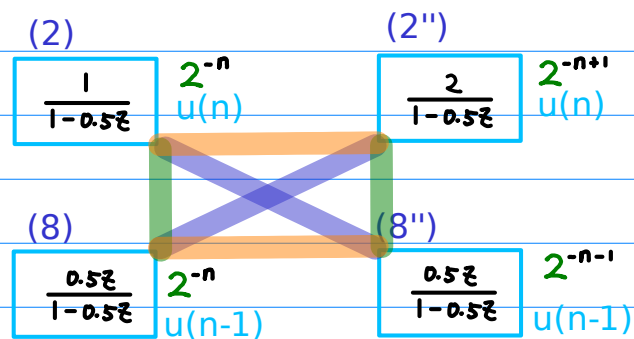
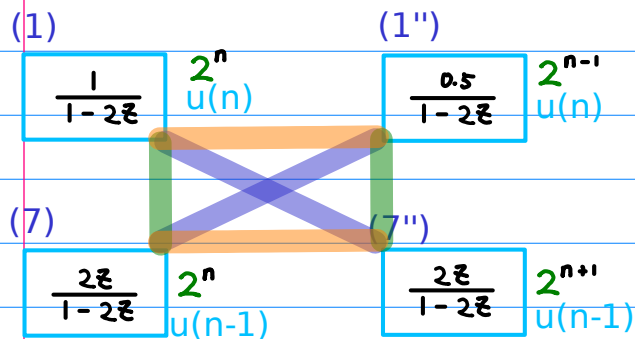
(1)~(8) \longrightarrow (1')~(8')



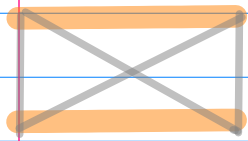
Inter-permutation

(x) \longrightarrow (x'')

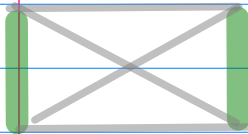
(1)~(8) (1'')~(8'')



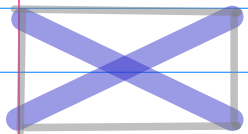
Decomposing Shift Operations



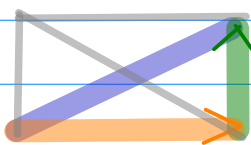
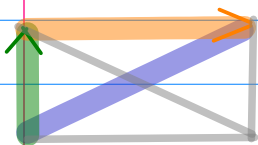
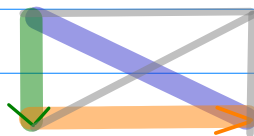
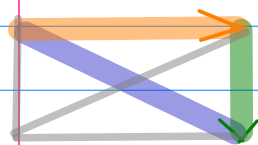
(exponent shift, identity)



(identity, range shift)



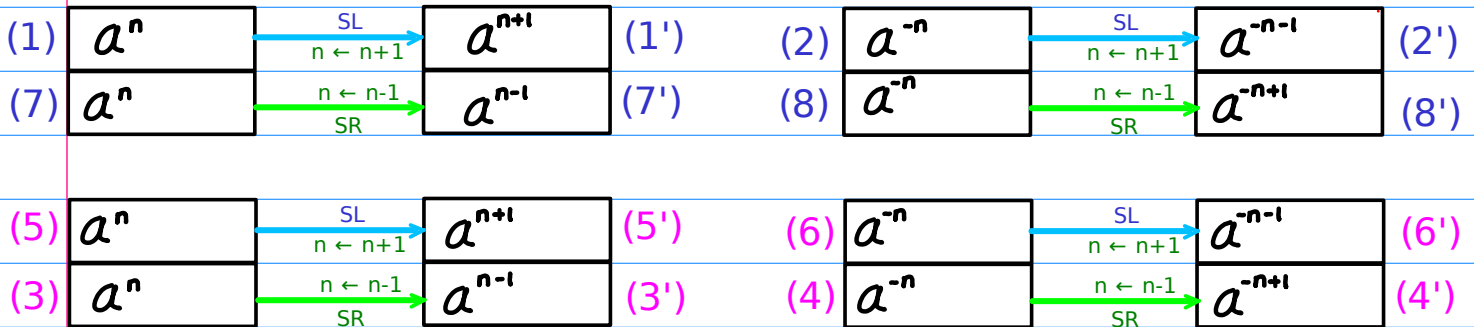
(exponent shift, range shift)
= (exponent shift, identity)
+ (identity, range shift)



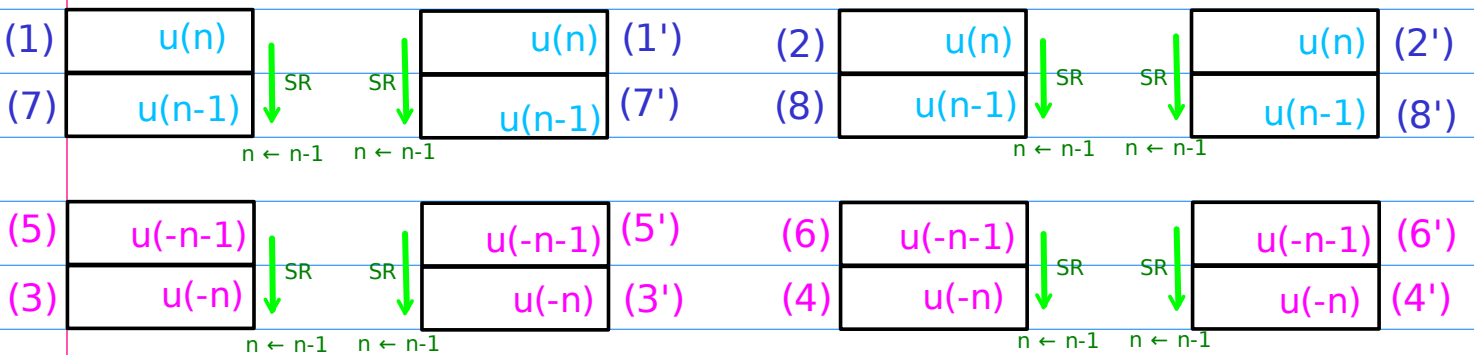
(id, SR) + (SR, id) = (SR, SR)
 (id, SL) + (SL, id) = (SL, SL)

(x) \longrightarrow **(x')**
 (1)~(8) (1')~(8')

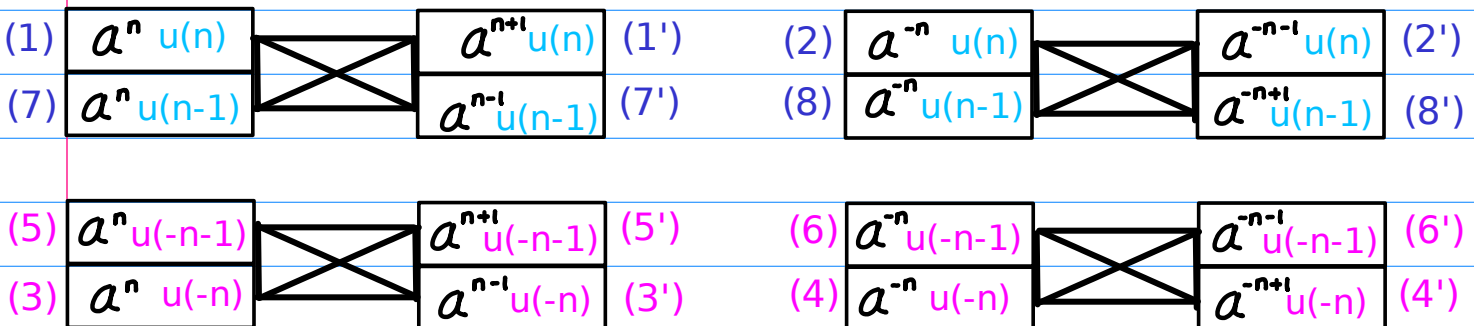
Exponent Shifts : (SR, id) or (SL, id)



Range Shifts : (id, SR) or (id, SL)



Exponent & Range Permutations



Decomposition

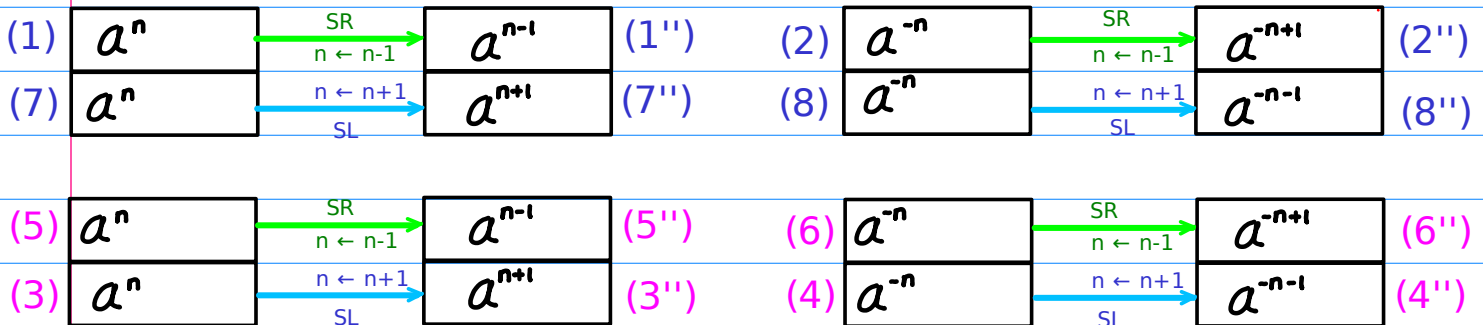
(EP, RP) = (EP, id) + (id, RP)

EP : Exponent Permutations
RP : Range Permutations

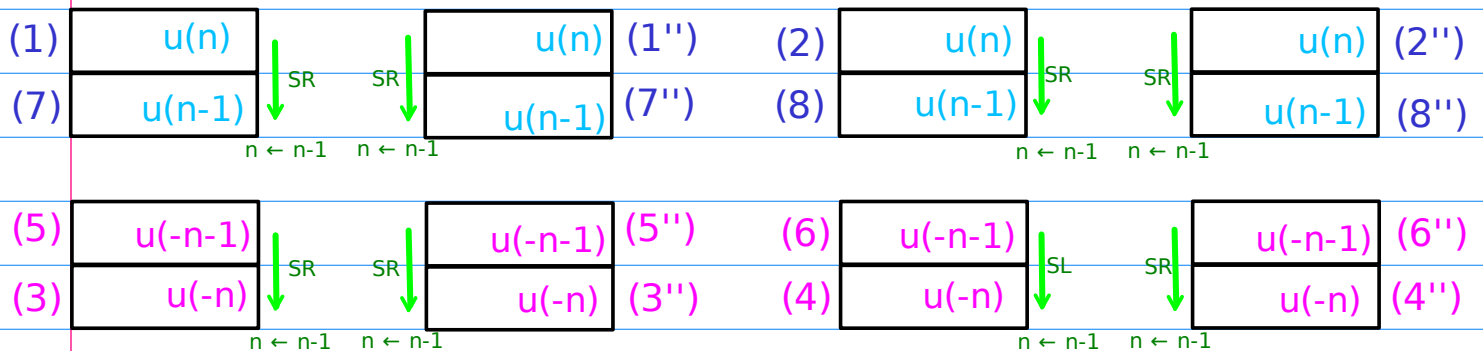
$$\begin{aligned}
 (\text{id}, \text{SR}) + (\text{SL}, \text{id}) &= (\text{SL}, \text{SR}) \\
 (\text{id}, \text{SL}) + (\text{SR}, \text{id}) &= (\text{SR}, \text{SL})
 \end{aligned}$$

$$\begin{aligned}
 (\mathbf{x}) &\longrightarrow (\mathbf{x}'') \\
 (1) \sim (8) &\qquad\qquad (1'') \sim (8'')
 \end{aligned}$$

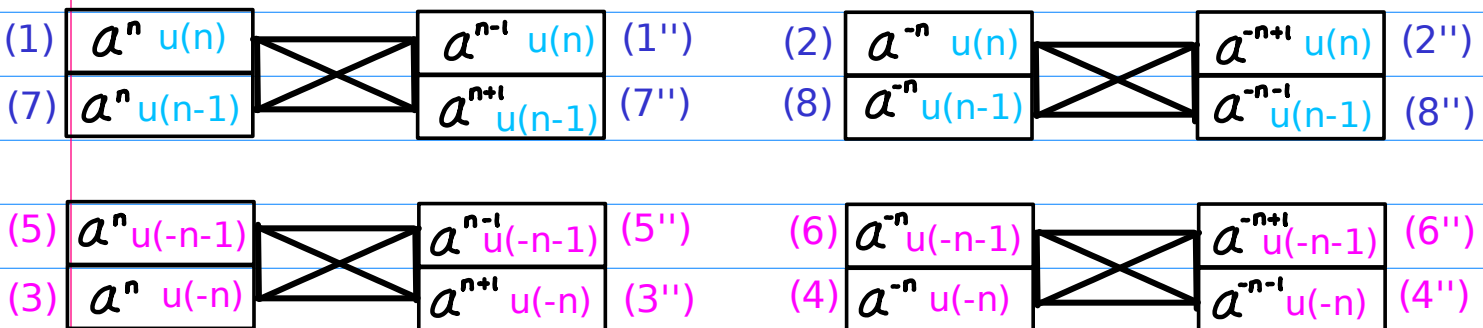
Exponent Shifts : (SR, id) or (SL, id)



Range Shifts : (id, SR) or (id, SL)



Exponent & Range Permutations



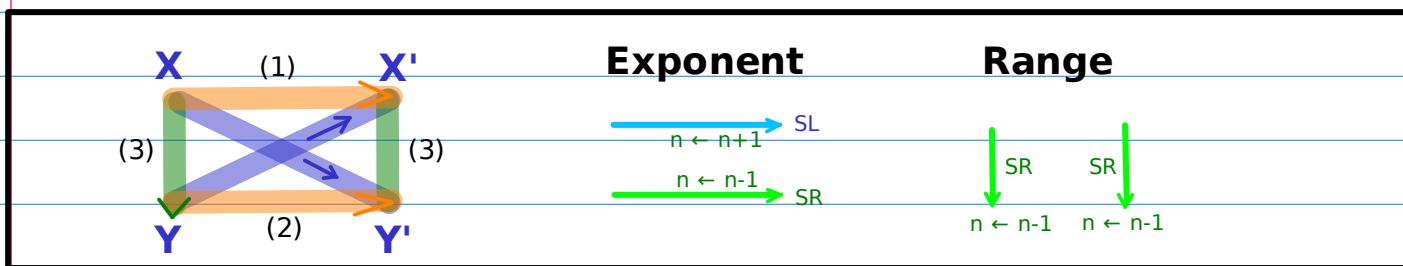
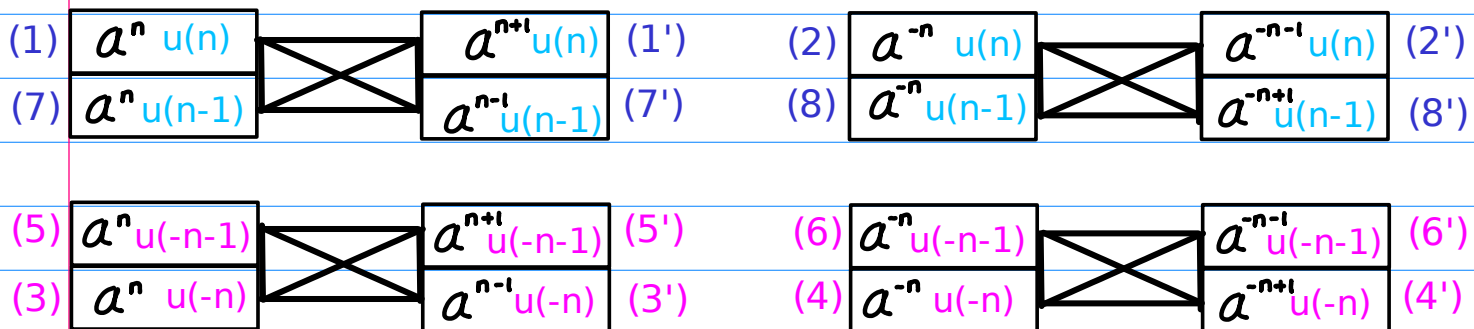
Decomposition

$$(\text{EP}, \text{RP}) = (\text{EP}, \text{id}) + (\text{id}, \text{RP})$$

EP : Exponent Permutations
RP : Range Permutations

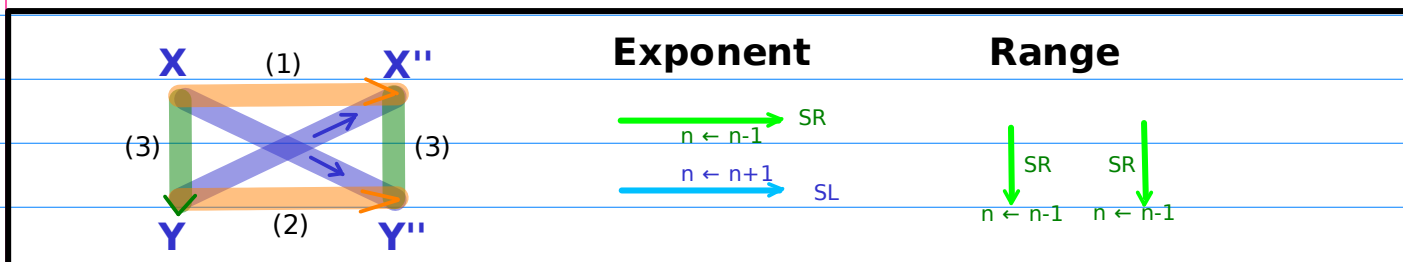
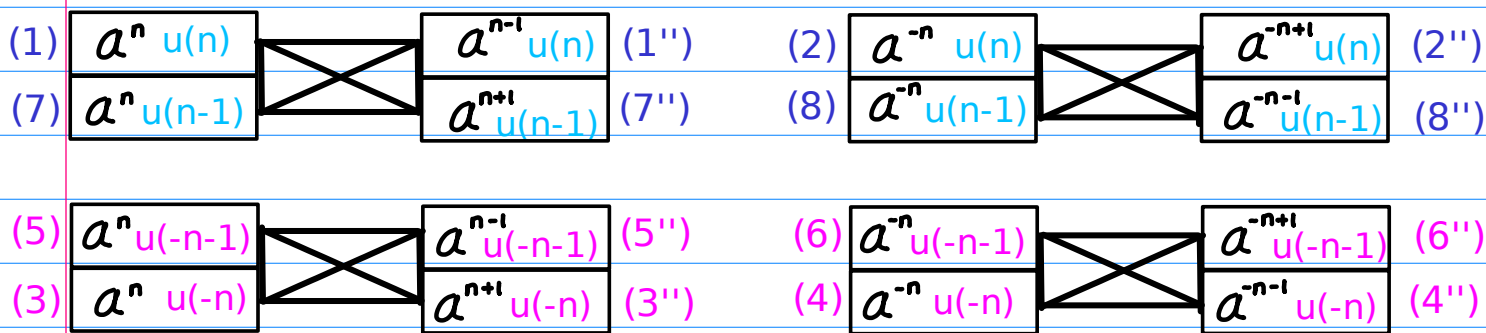
(id, SR) + (SR, id) = (SR, SR)
 (id, SL) + (SL, id) = (SL, SL)

(x) → (x')
 (1)~(8) (1')~(8')



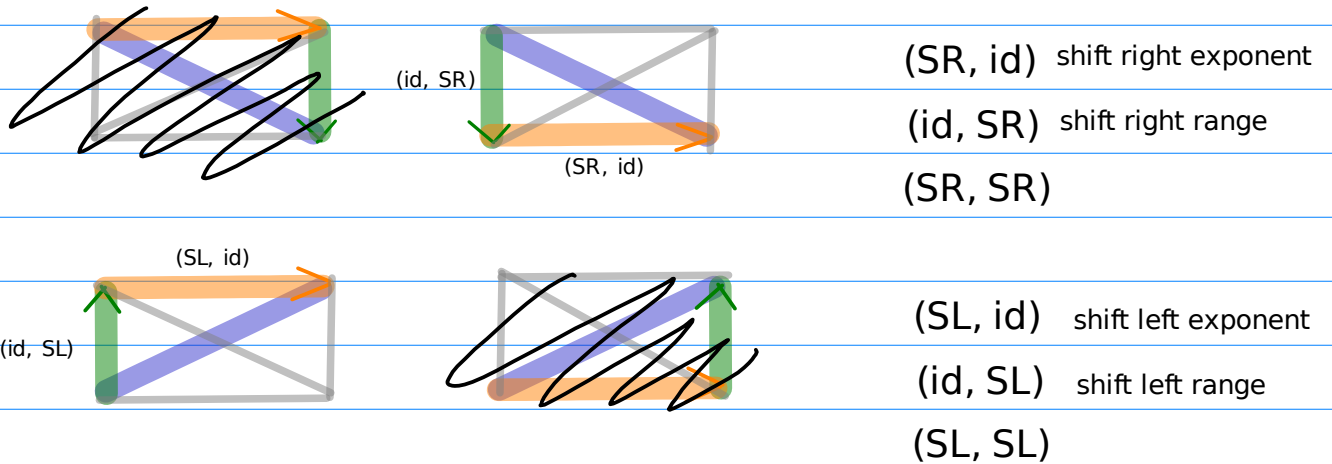
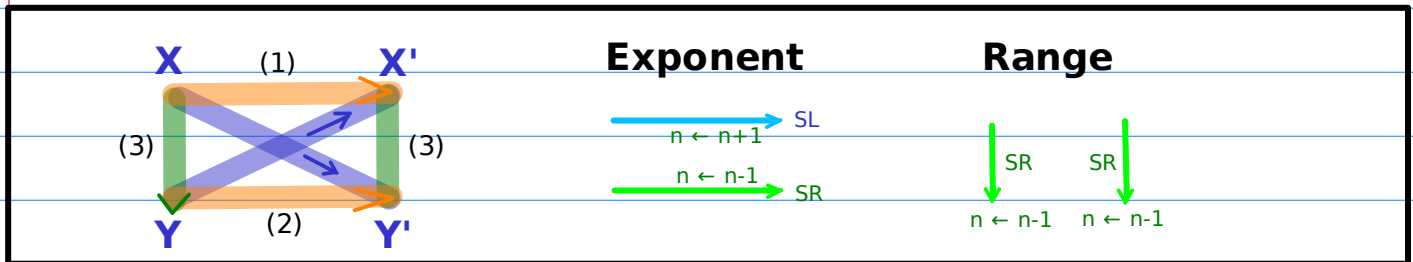
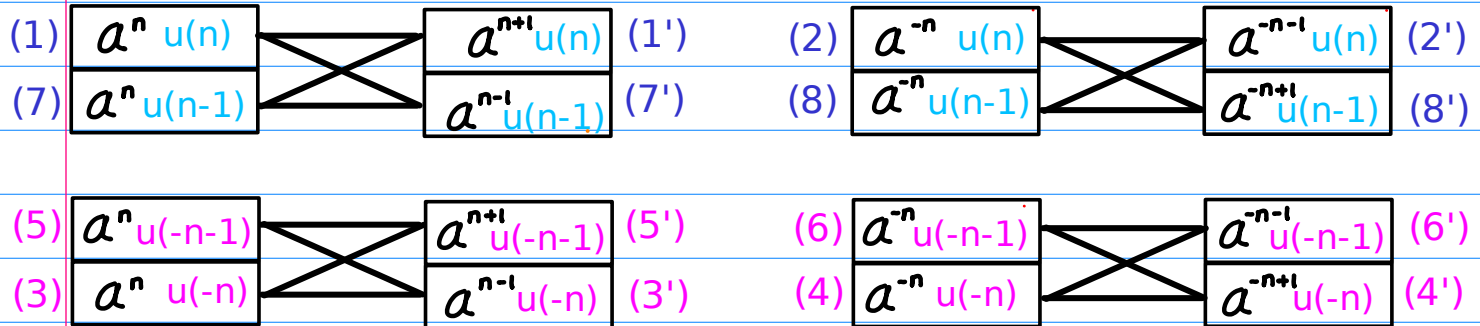
(id, SR) + (SL, id) = (SL, SR)
 (id, SL) + (SR, id) = (SR, SL)

(x) → (x'')
 (1)~(8) (1'')~(8'')



Inter-permutation (x) \longrightarrow (x')

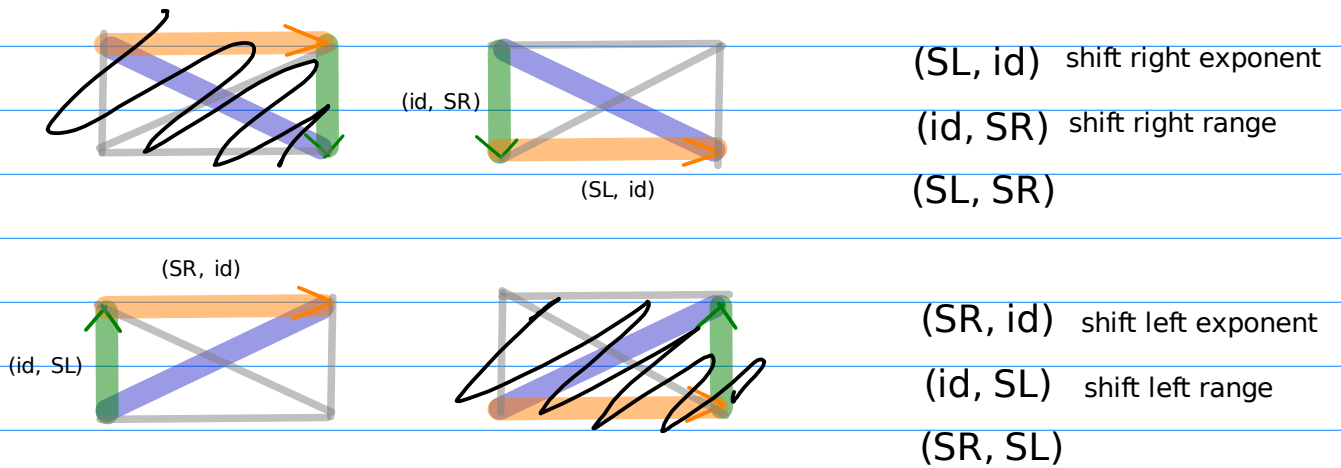
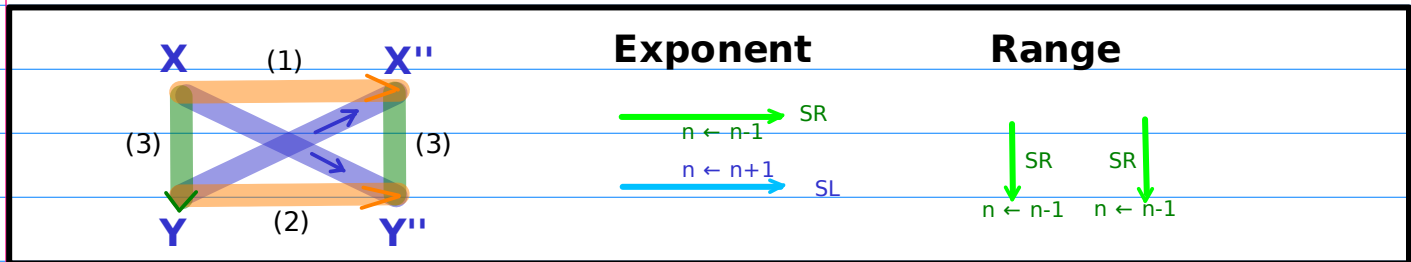
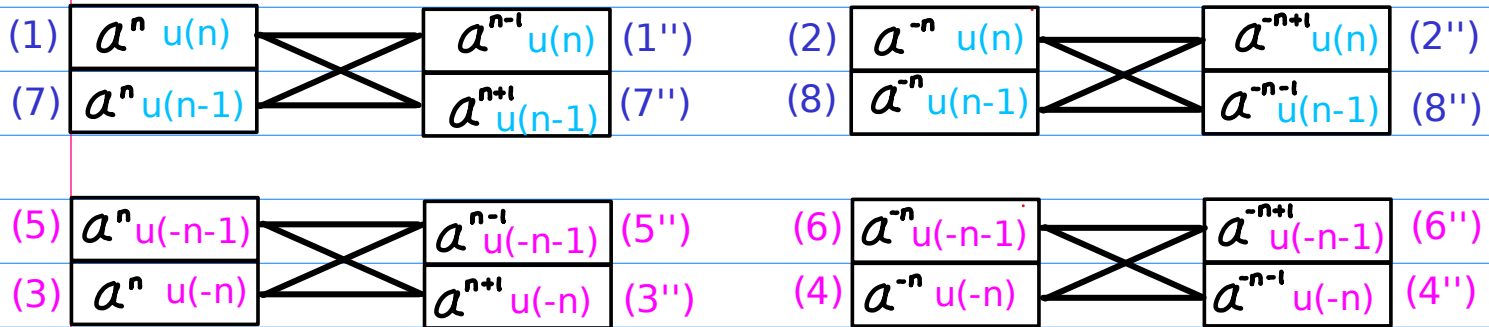
(1)~(8) (1')~(8')



(SR, id) + (id, SR) = (SR, SR)
(SL, id) + (id, SL) = (SL, SL)

Inter-permutation (x) \longrightarrow (x'')

Summary (1)~(8) (1'')~(8'')

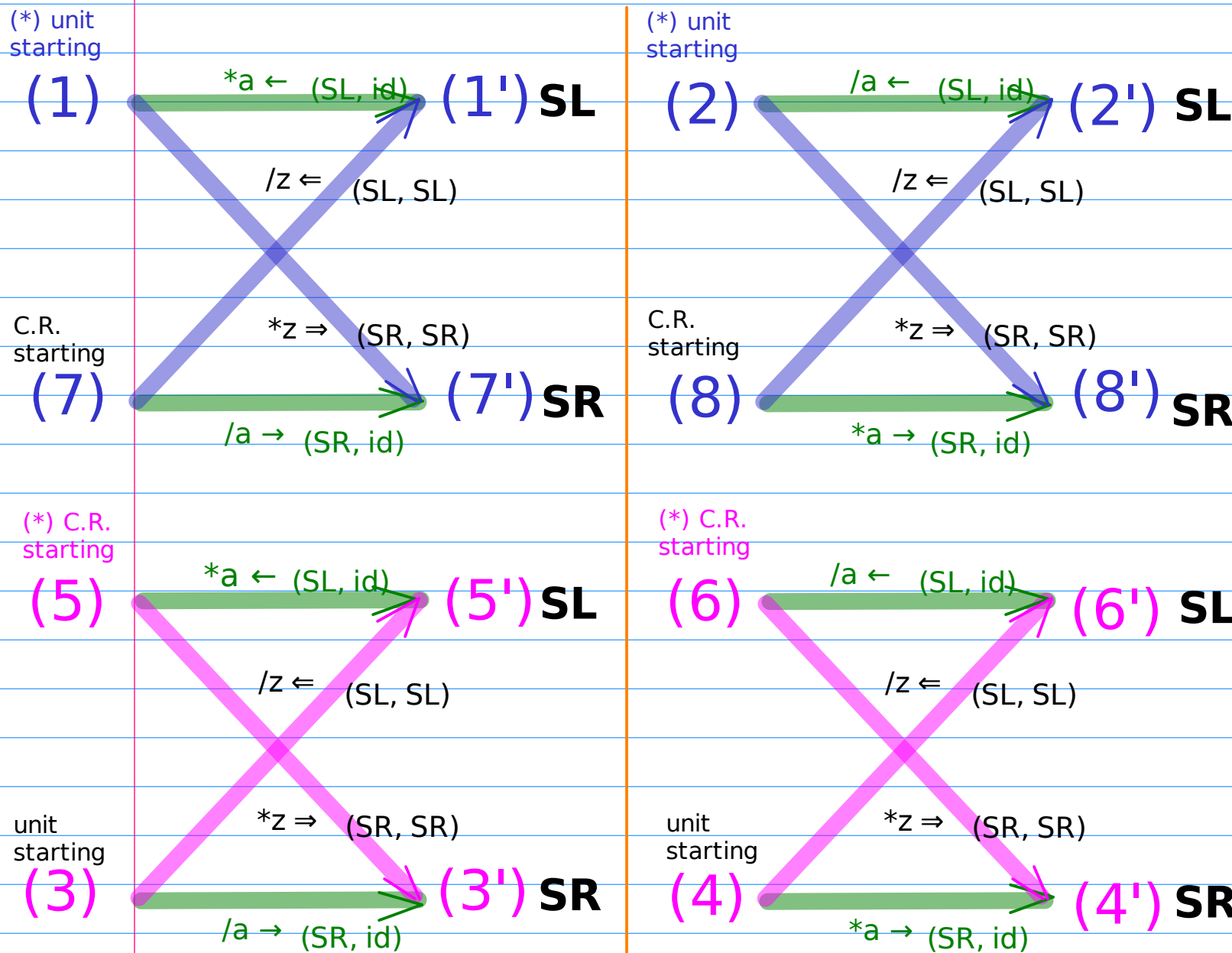
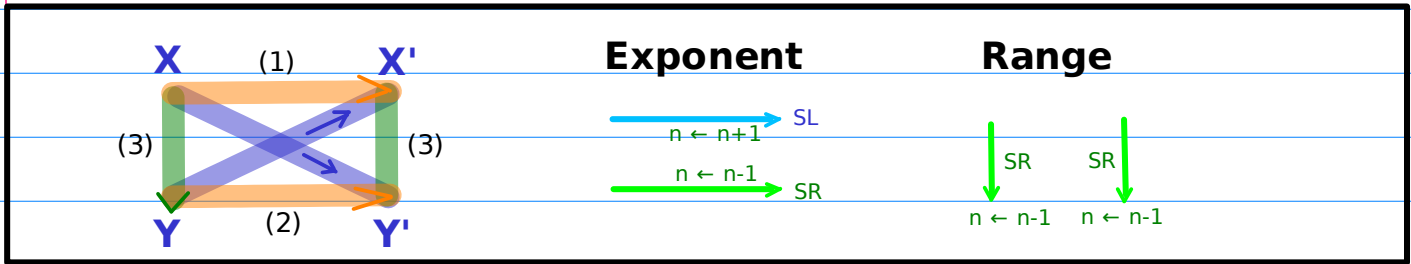


(SL, id) + (id, SR) = (SL, SR)
(SR, id) + (id, SL) = (SR, SL)



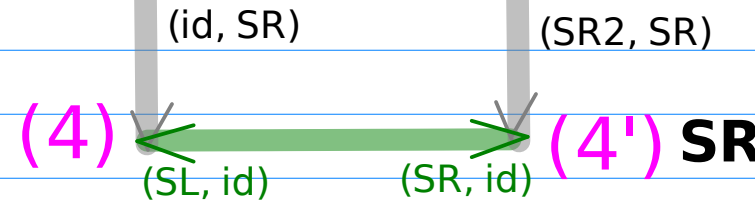
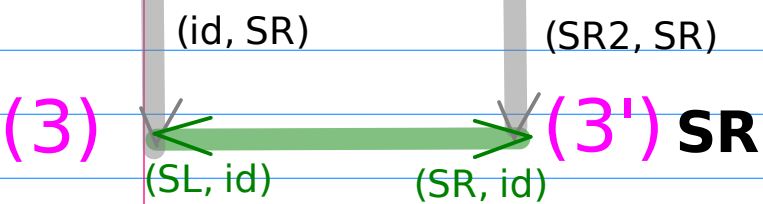
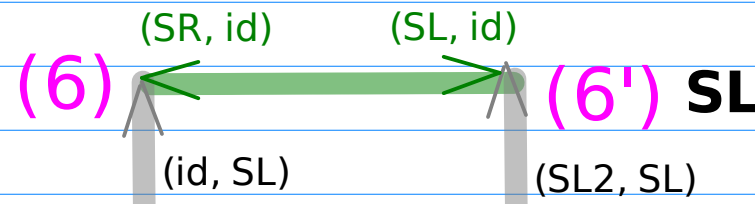
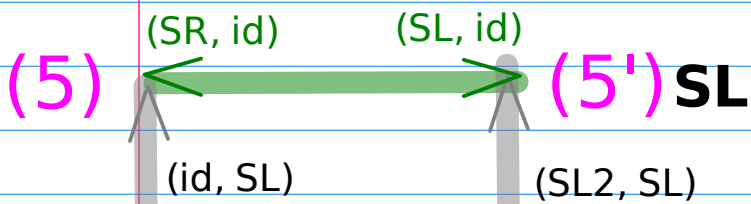
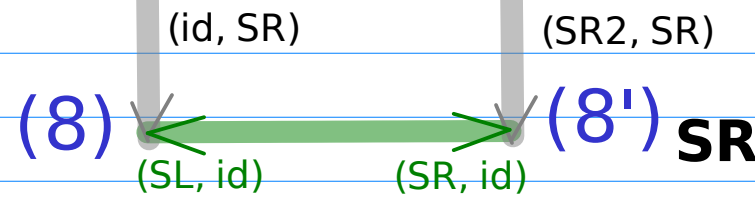
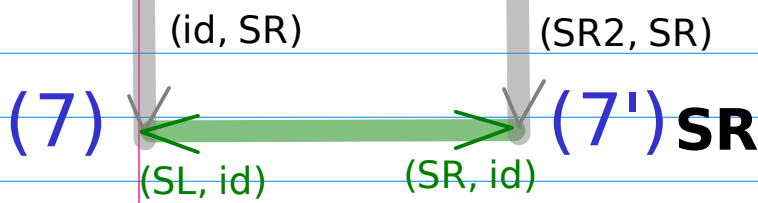
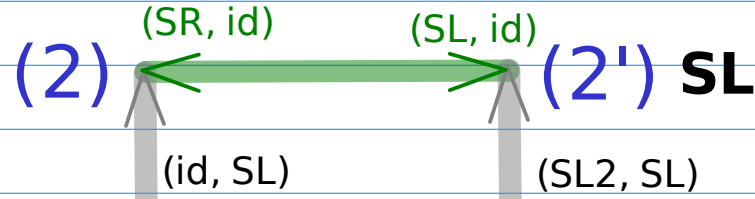
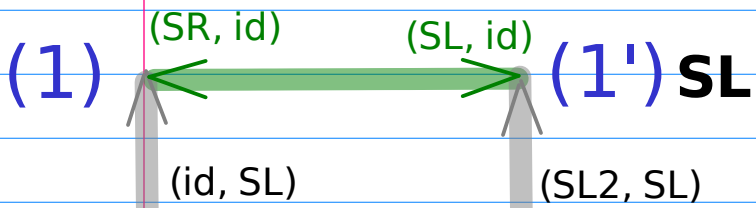
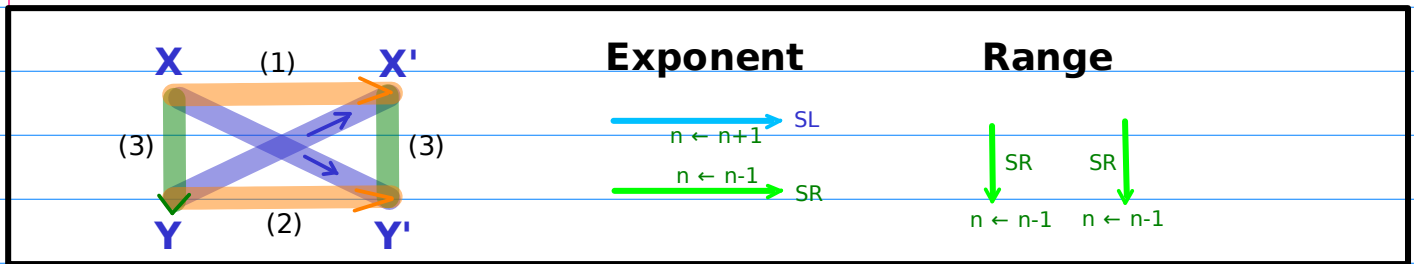
Butterfly Relations

(x) \longrightarrow (x')
 $(1) \sim (8)$ \longrightarrow $(1') \sim (8')$



Butterfly Relations

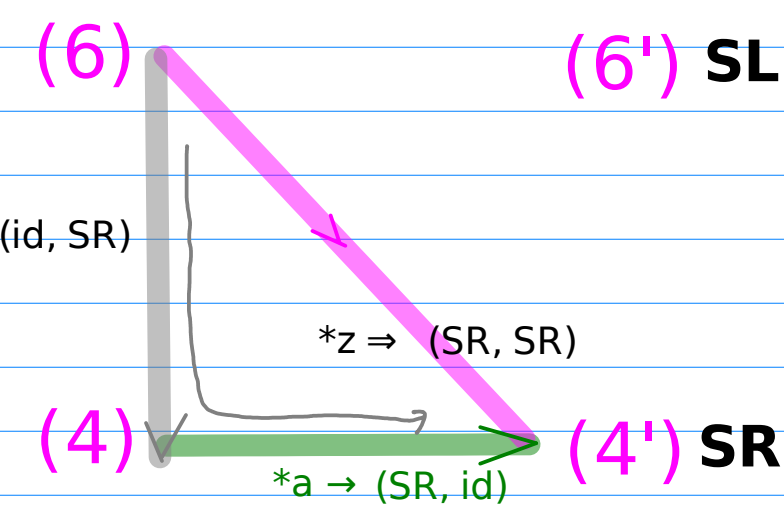
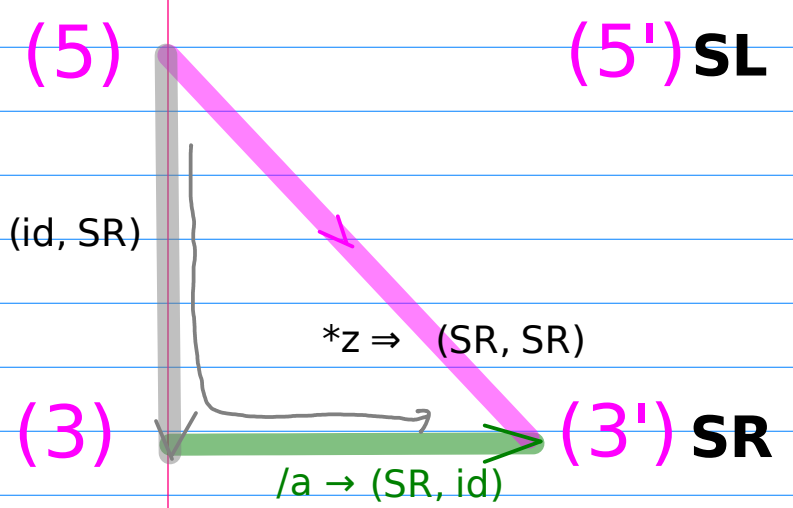
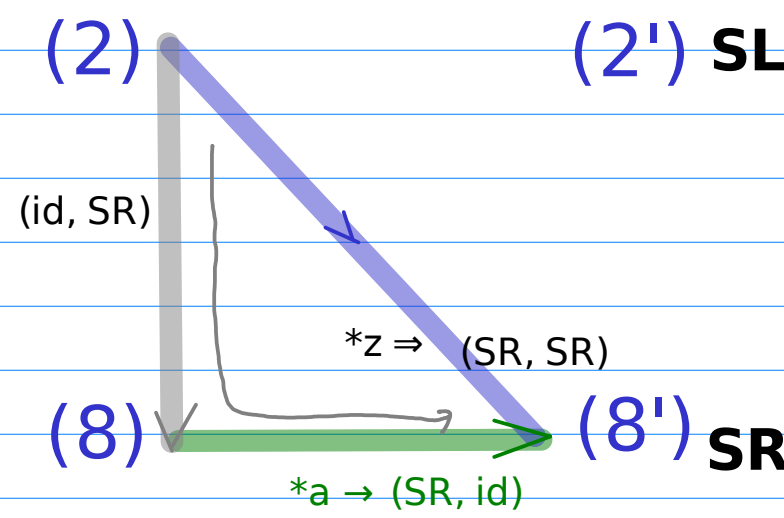
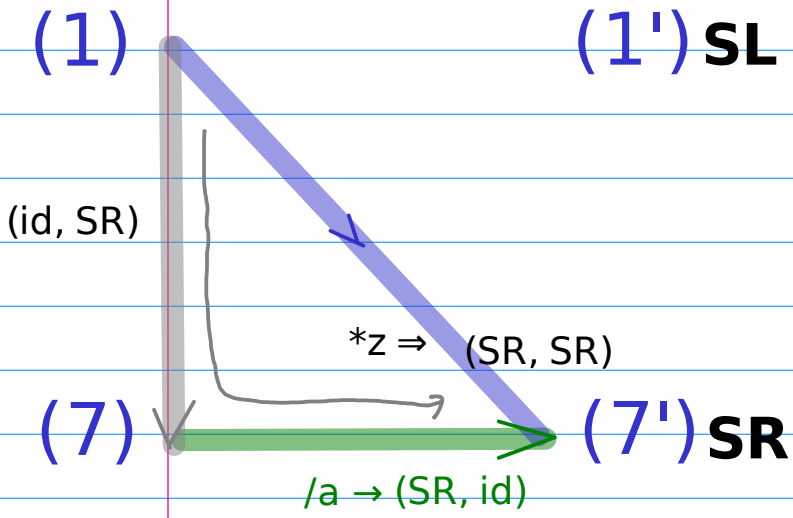
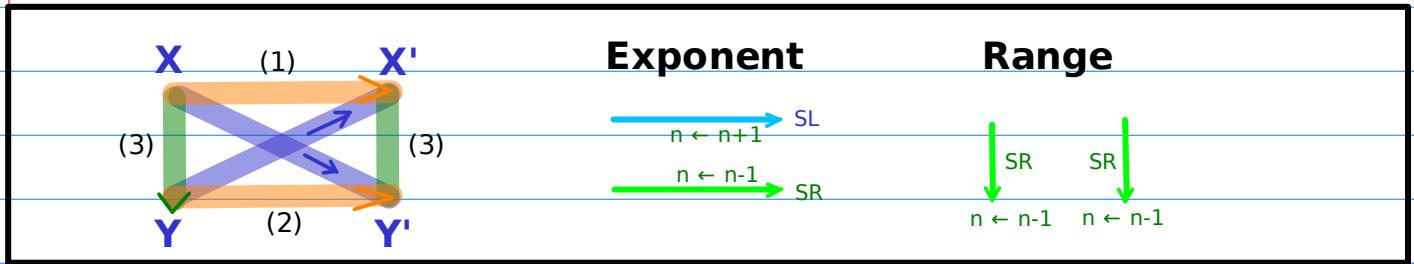
(x) \longrightarrow (x')
 $(1) \sim (8)$ \longrightarrow $(1') \sim (8')$



Decomposition of Exp and Rng Shifts (1)

$(x) \xrightarrow{\quad} (x')$
 $(1) \sim (8) \quad (1') \sim (8')$

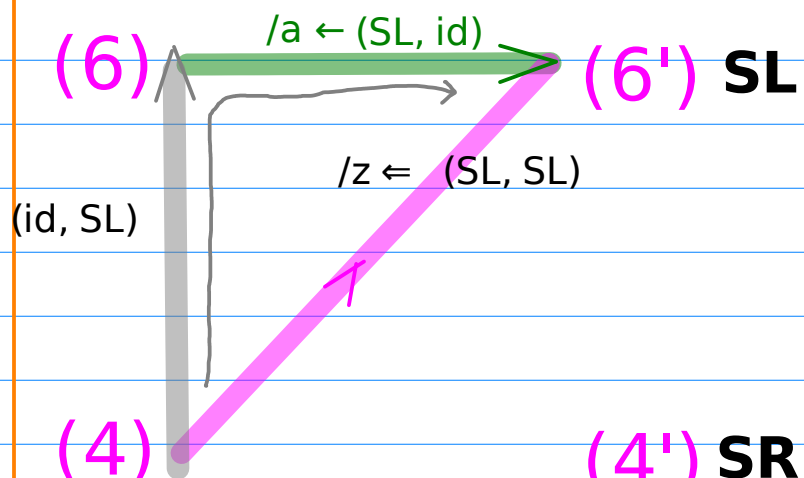
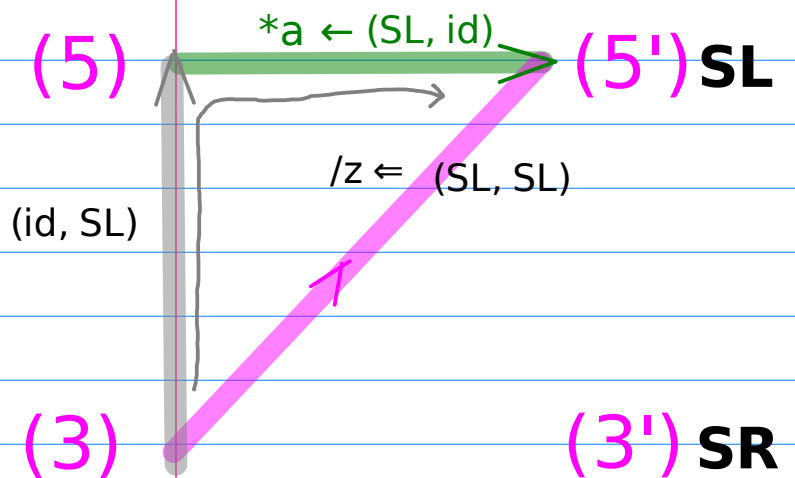
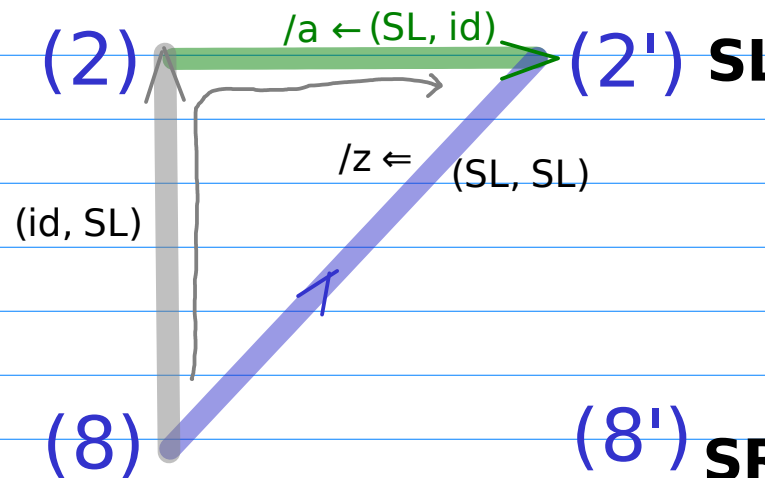
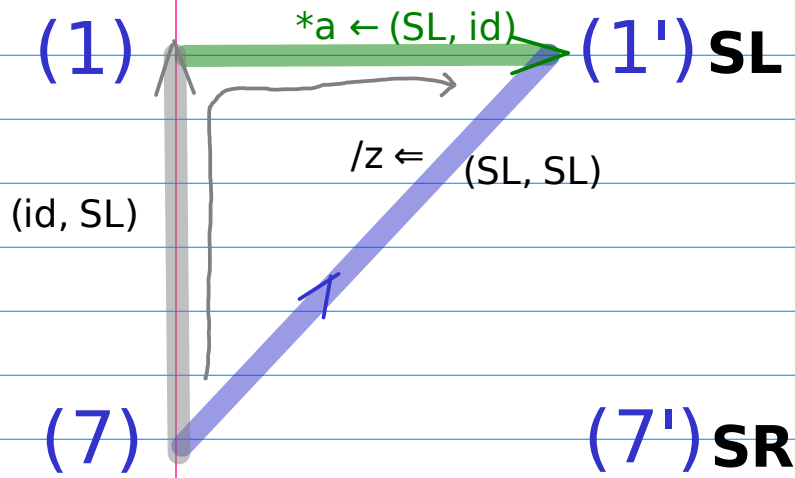
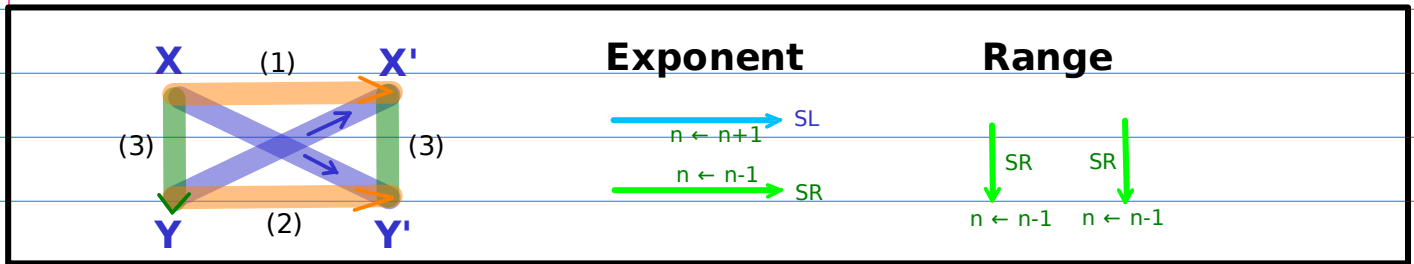
$(id, SR) + (SR, id) = (SR, SR)$



Decomposition of Exp and Rng Shifts (2)

$(x) \xrightarrow{\quad} (x')$
 (1)~(8) (1')~(8')

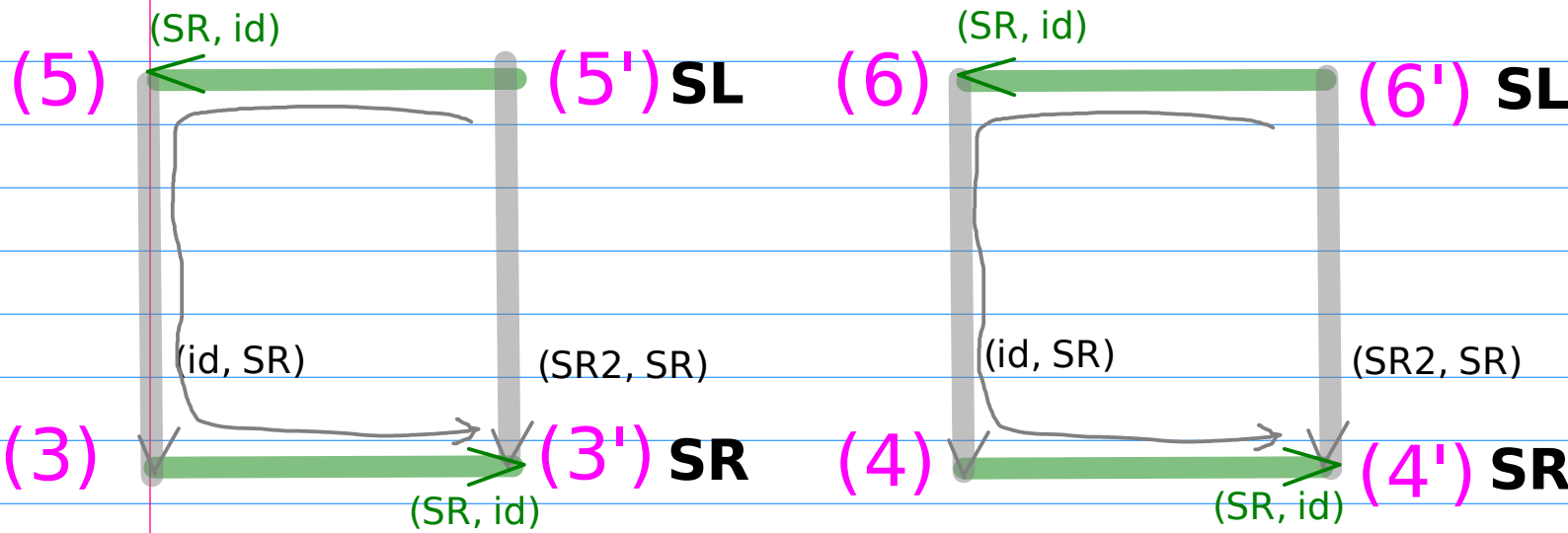
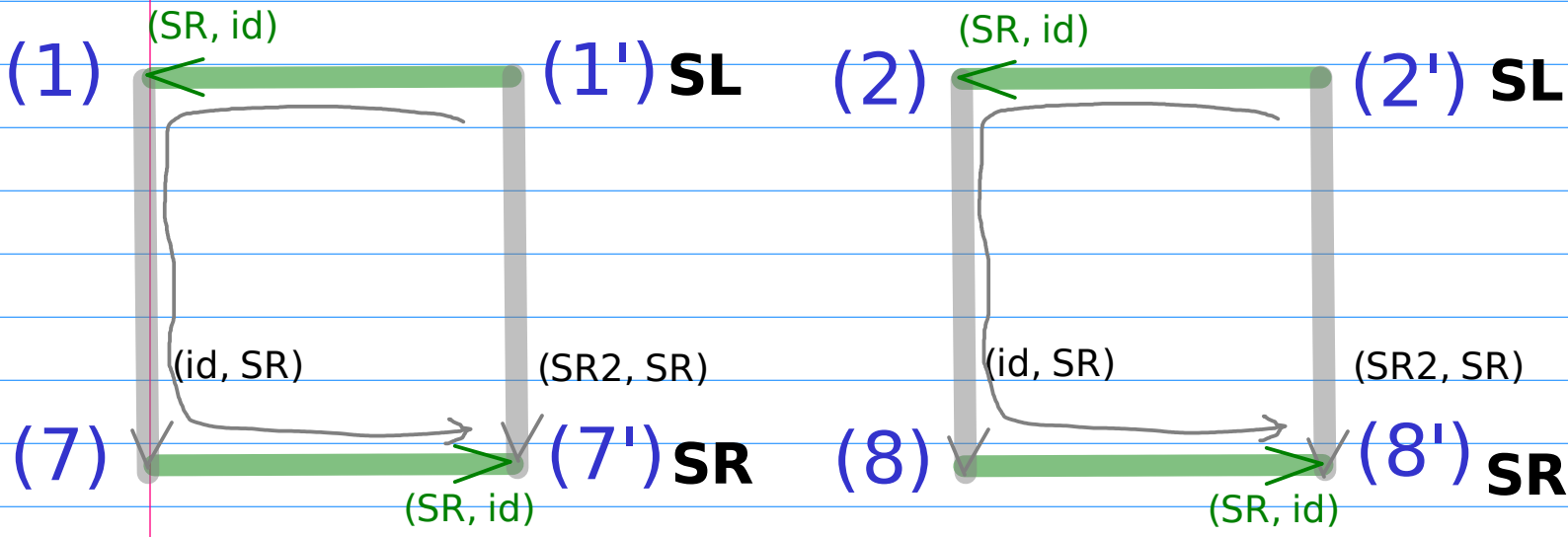
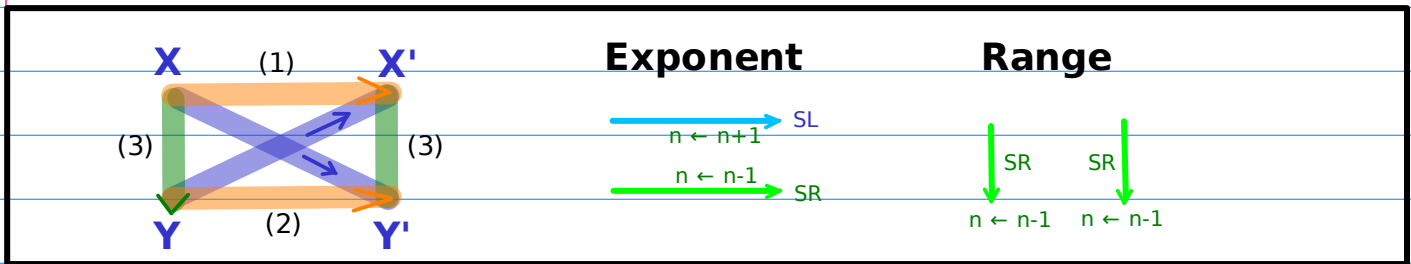
$(id, SL) + (SL, id) = (SL, SL)$



Decomposition of Exp and Rng Shifts (3)

$(X) \rightarrow (X')$
 $(1) \sim (8) \rightarrow (1') \sim (8')$

$$(SR, id) + (id, SR) + (SR, id) = (SR2, SR)$$





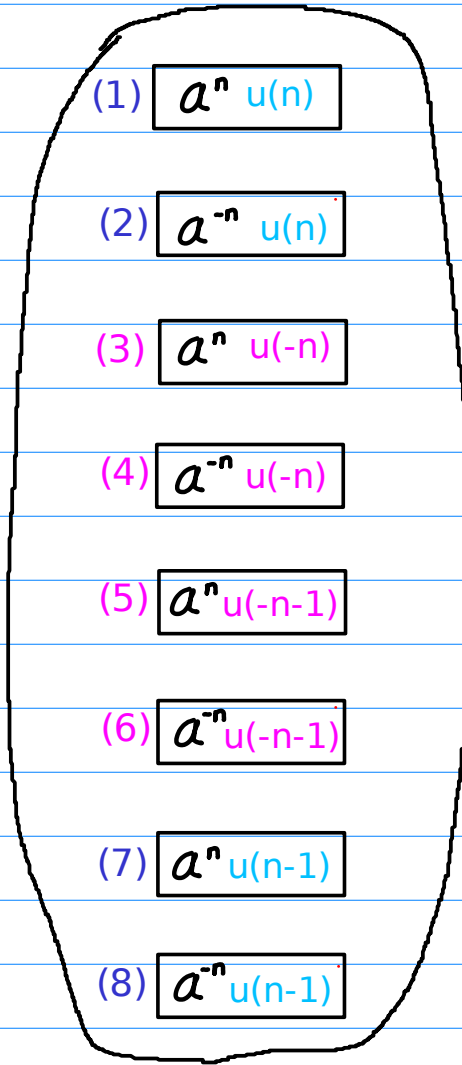
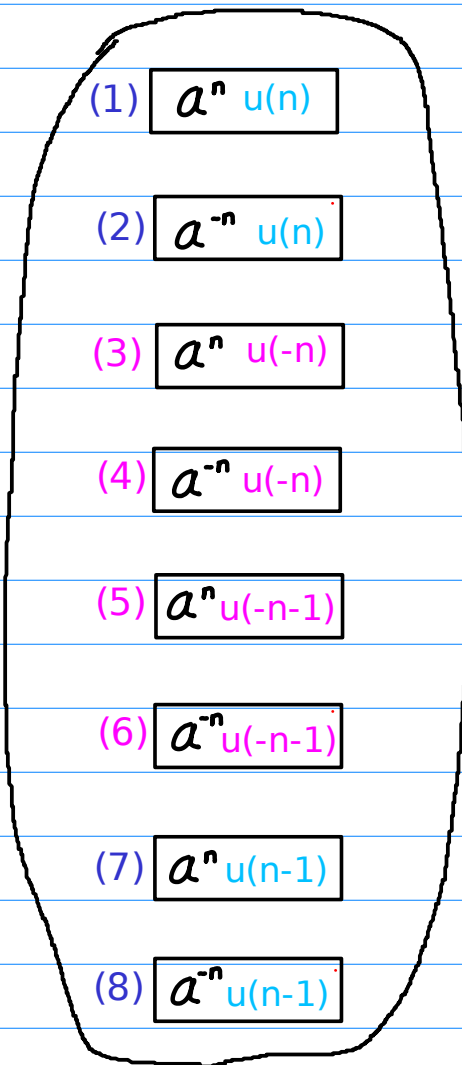


Intra-Permutations over (1) ~ (8)

Unshifted Sequence x



Unshifted Sequence x

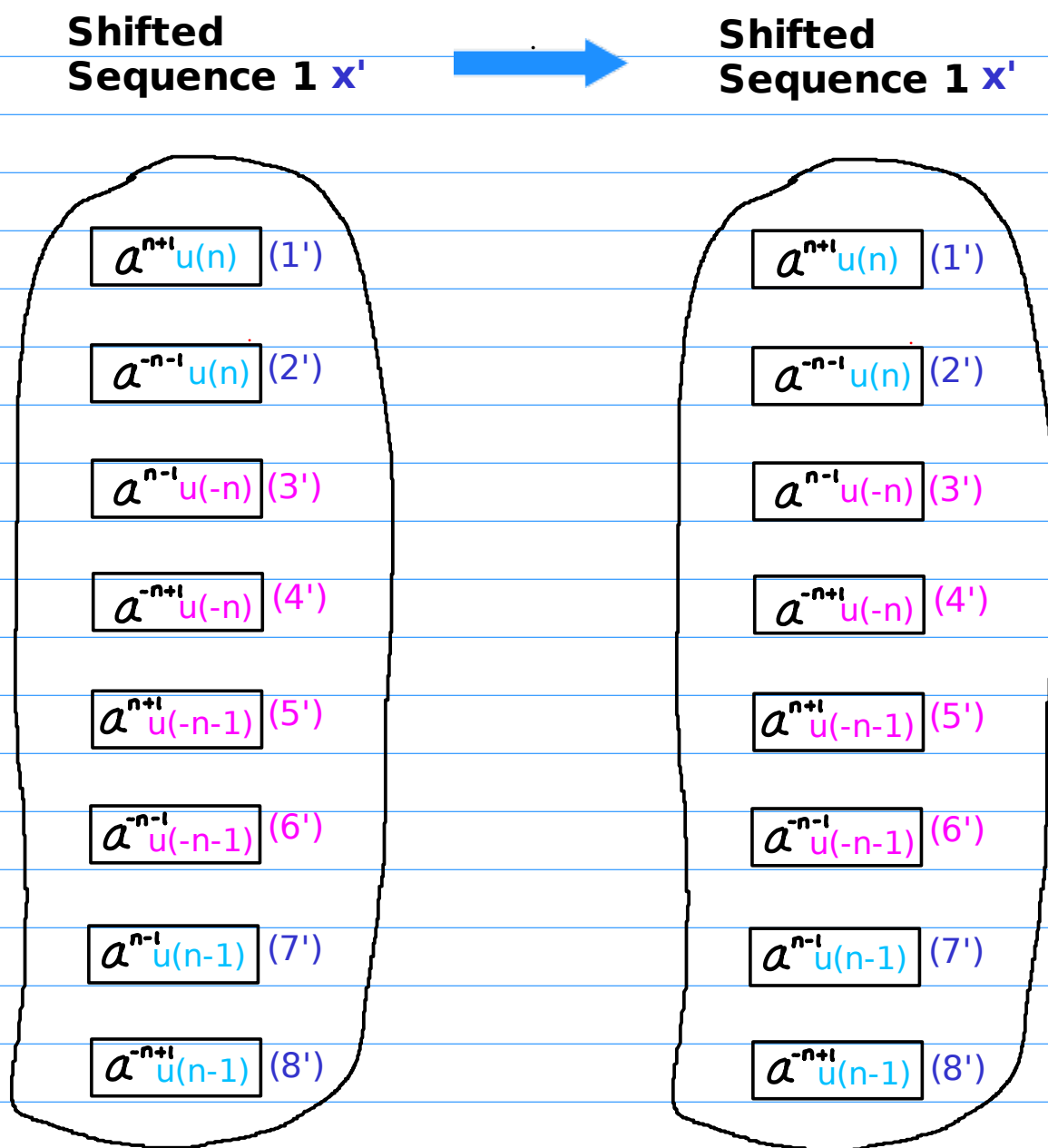


Inter-permutations over unshifted sequence and shifted sequence

Intra-permutations over unshifted sequence

Intra-permutations over shifted sequence

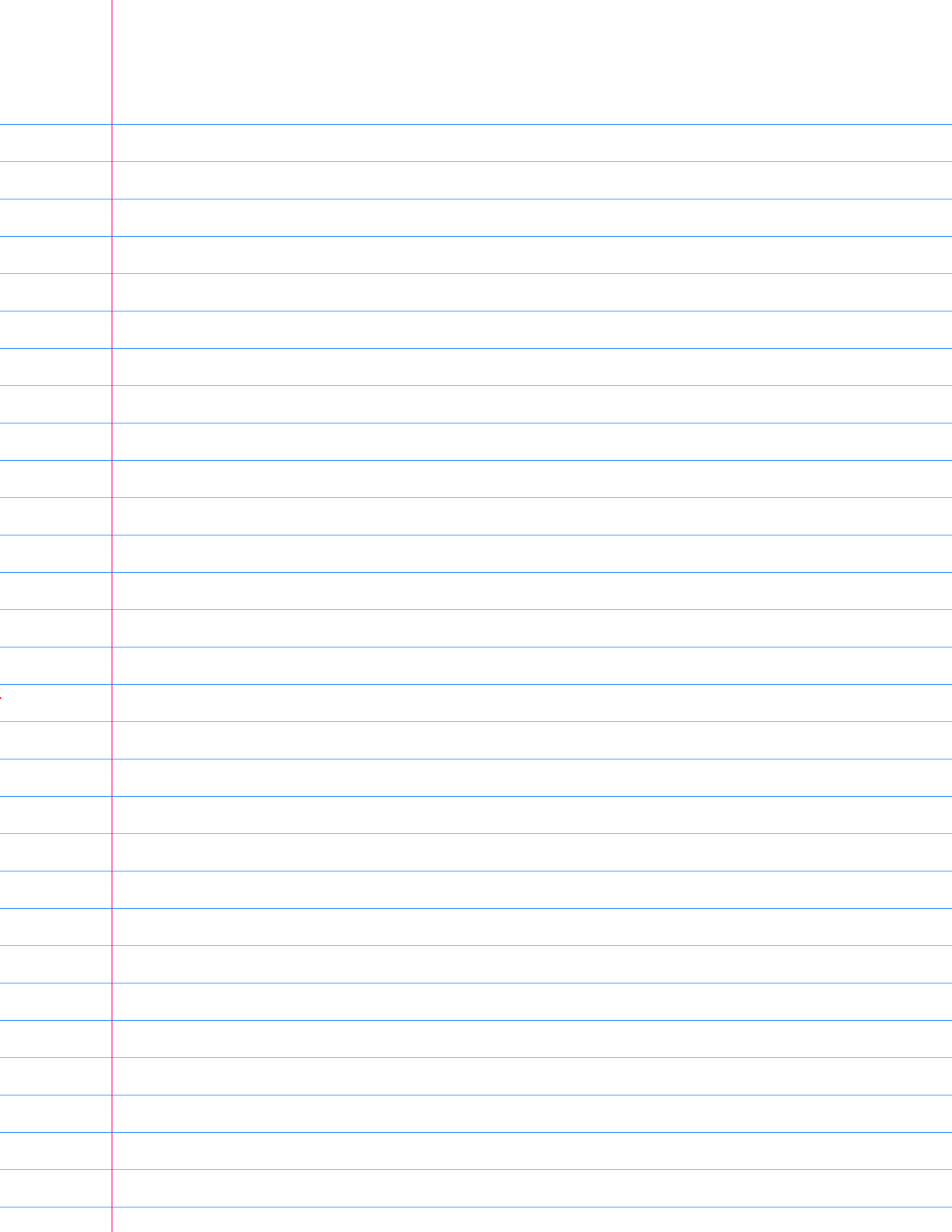
Intra-Permutations over (1') ~ (8')



Inter-permutations over unshifted sequence and shifted sequence

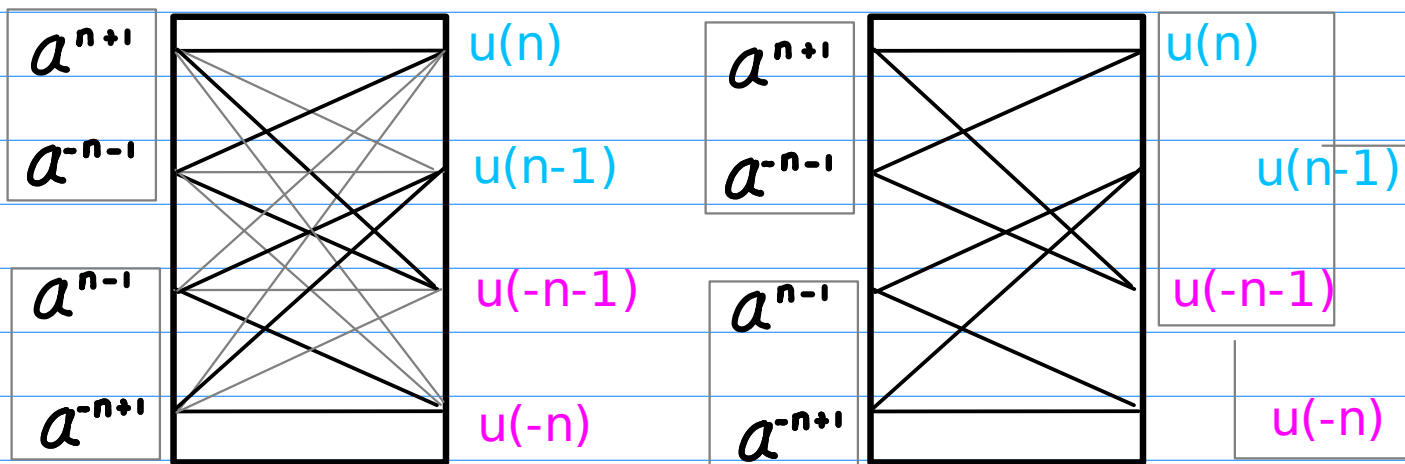
Intra-permutations over unshifted sequence

Intra-permutations over shifted sequence



$$a^n \times R(n)$$

a^{n+1}	a^{-n-1}	\times	$u(n)$	$u(-n-1)$
a^{n-1}	a^{-n+1}	\times	$u(n-1)$	$u(-n)$



$n \leftarrow n+2$ or $n \leftarrow n-2$

$(1')$	$a^{n+1}u(n)$	$a^{-n-1}u(n)$	$(2')$
$(3')$	$a^{n-1}u(-n)$	$a^{-n+1}u(-n)$	$(4')$
$(5')$	$a^{n+1}u(-n-1)$	$a^{-n-1}u(-n-1)$	$(6')$
$(7')$	$a^{n-1}u(n-1)$	$a^{-n+1}u(n-1)$	$(8')$

$(1'')$	$a^{n-1}u(n)$	$a^{-n+1}u(n)$	$(2'')$
$(3'')$	$a^{n+1}u(-n)$	$a^{-n-1}u(-n)$	$(4'')$
$(5'')$	$a^{n-1}u(-n-1)$	$a^{-n+1}u(-n-1)$	$(6'')$
$(7'')$	$a^{n+1}u(n-1)$	$a^{-n-1}u(n-1)$	$(8'')$



permutation over (1) ~ (8)

(x) → (x)

A. Flipping

Base Inverting
Range Flipping

B. Range Shifting = Range Flipping + Range Complementing

Range Flipping
Range Complementing

C. Complementary Inverting

Base Inverting
Range Complementing

permutation over (1') ~ (8')

(x') → (x')

D. Flipping2

Base Inverting
Shifted Range Flipping = Exponent Shifting2 + Range Flipping

E. Shifting2 = Exponent Shifting2 + Range Shifting

Shifted Range Flipping = Exponent Shifting2 + Range Flipping
Range Complementing

F. Complementary Inverting

Base Inverting
Range Complementing

Range Shifting = Range Flipping + Range Complementing

Shifted Range Flipping = Exponent Shifting2 + Range Flipping

Shifting2 = Shifted Range Flipping + Range Complementing

= Exponent Shifting2 + Range (Flipping+Complementing)

= Exponent Shifting2 + Range Shifting

permutation over (1) ~ (8)

A. Flipping	(1) - (4)	(5) - (8)
Base Inverting	(2) - (3)	(6) - (7)
Range Flipping	(3) - (2)	(7) - (6)
	(4) - (1)	(8) - (5)
B. Range Shifting	(1) - (7)	(5) - (3)
Range Flipping	(2) - (8)	(6) - (4)
Range Complementing	(3) - (5)	(7) - (1)
	(4) - (6)	(8) - (2)
C. Complementary Inverting	(1) - (6)	(5) - (2)
Base Inverting	(6) - (1)	(6) - (1)
Range Complementing	(2) - (5)	(7) - (4)
	(5) - (2)	(8) - (3)

permutation over (1') ~ (8')

D. Flipping2	(1') - (4')	(5') - (8')
Base Inverting	(2') - (3')	(6') - (7')
Shifted Range Flipping	(3') - (2')	(7') - (6')
	(4') - (1')	(8') - (5')
E. Shifting2	(1') - (7')	(5') - (3')
Shifted Range Flipping	(2') - (8')	(6') - (4')
Range Complementing	(3') - (5')	(7') - (1')
	(4') - (6')	(8') - (2')
F. Complementary Inverting	(1') - (6')	(5') - (2')
Base Inverting	(6') - (1')	(6') - (1')
Range Complementing	(2') - (5')	(7') - (4')
	(5') - (2')	(8') - (3')



permutation over (1) ~ (8)

	A	B	C
Base Inverting	X		X
Range Flipping	X	X	
Range Complementing		X	X

permutation over (1') ~ (8')

	D	E	F
Base Inverting	X		X
Shifted Range Flipping	X	X	
Range Complementing		X	X

Range Shifting = Range Flipping + Range Complementing
Shifted Range Flipping = Exponent Shifting2 + Range Flipping

Shifting2 = Shifted Range Flipping + Range Complementing
= Exponent Shifting2 + Range (Flipping+Complementing)
= Exponent Shifting2 + Range Shifting



Over (1) ~ (8)

Base Inverting

$$a^n \longleftrightarrow a^{-n}$$

Range Flipping

$$R(n) \longleftrightarrow R(-n)$$

Range Complementing

$$R(n) \longleftrightarrow \overline{R(n)}$$

Over (1') ~ (8')

Base Inverting

$$a^n \longleftrightarrow a^{-n}$$

Shifted Range Flipping

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

Range Complementing

$$R(n) \longleftrightarrow \overline{R(n)}$$

A.I Flipping
Base Inverting
Range Flipping

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow R(-n)$$

$$a^n R(n) \longleftrightarrow a^{-n} R(-n)$$

D.I Flipping2
Base Inverting
Shifted Range Flipping

$$a^n \longleftrightarrow a^{-n}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

$$a^n R(n) \longleftrightarrow a^{-sh2(n)} R(-n)$$

B.I Range Shifting
Range Flipping
Range Complementing

$$R(n) \longleftrightarrow R(-n)$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$R(n) \longleftrightarrow \overline{R(-n)}$$

E.I Shifting2
Shifted Range Flipping
Range Complementing

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

C.I Complementary Inverting
Base Inverting
Range Complementing

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

F.I Complementary Inverting
Base Inverting
Range Complementing

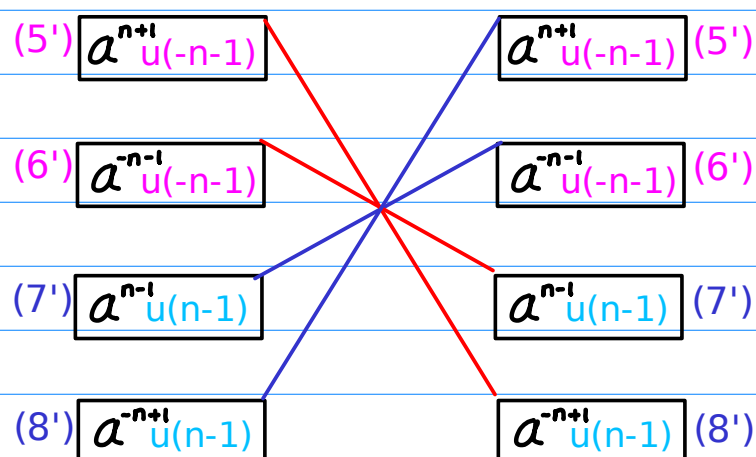
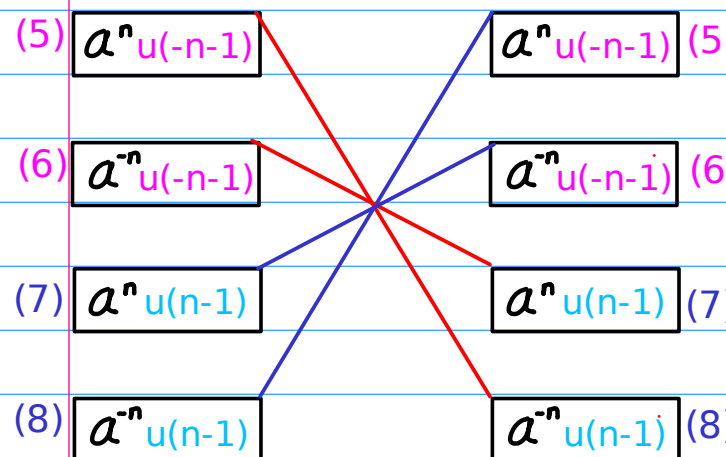
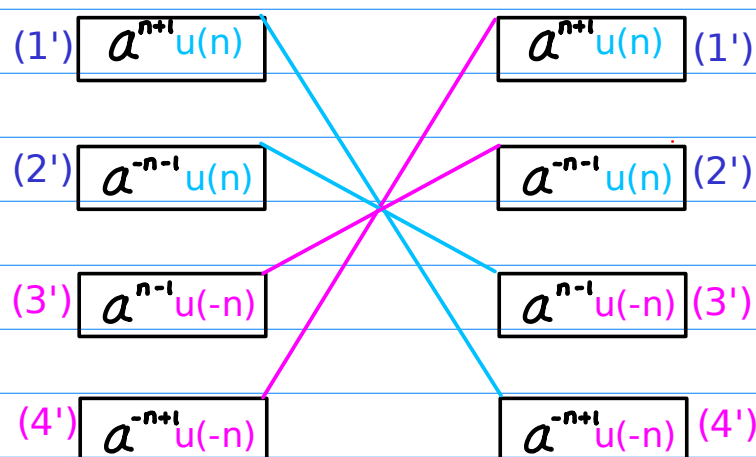
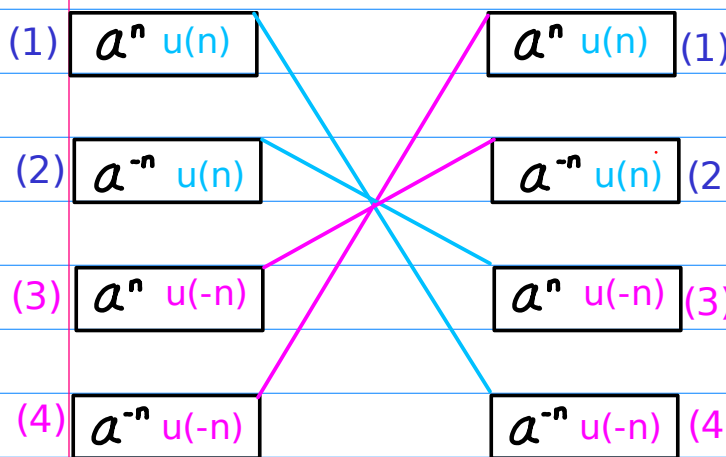
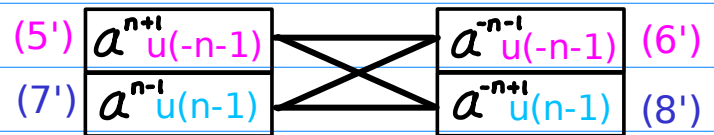
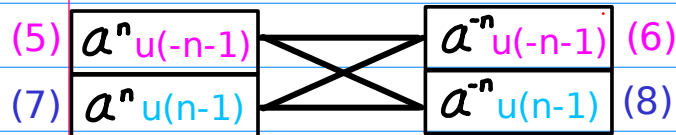
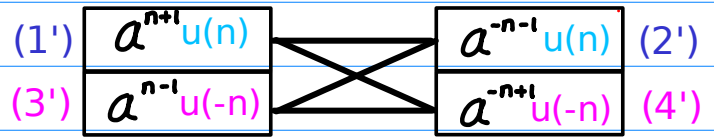
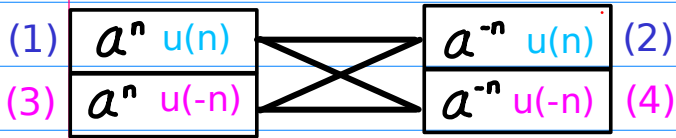
$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

A.I Flipping
Base Inverting
Range Flipping

D.I Flipping2
Base Inverting
Shifted Range Flipping

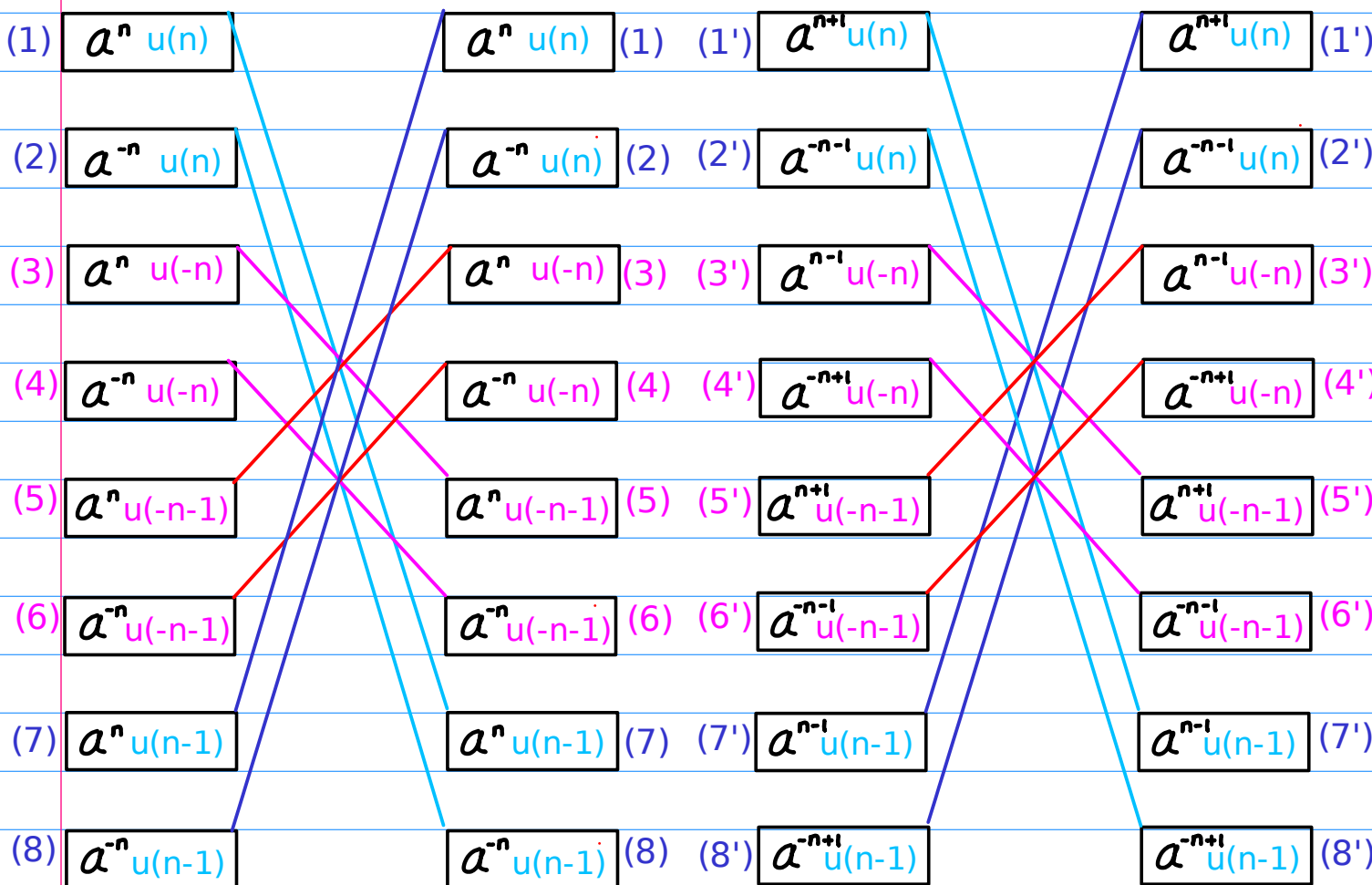
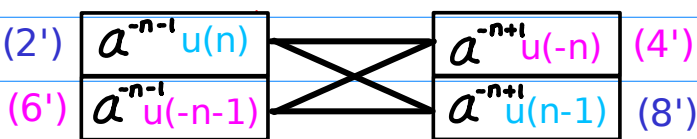
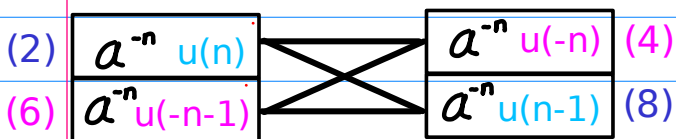
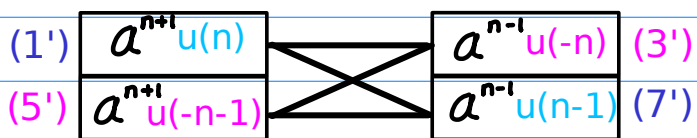
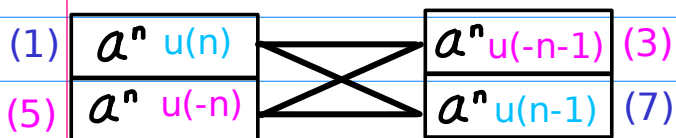


(1) - (4) (5) - (8)
 (2) - (3) (6) - (7)
 (3) - (2) (7) - (6)
 (4) - (1) (8) - (5)

(1') - (4') (5') - (8')
 (2') - (3') (6') - (7')
 (3') - (2') (7') - (6')
 (4') - (1') (8') - (5')

**B.I Range Shifting
Range Flipping
Range Complementing**

**E.I Shifting2
Shifted Range Flipping
Range Complementing**

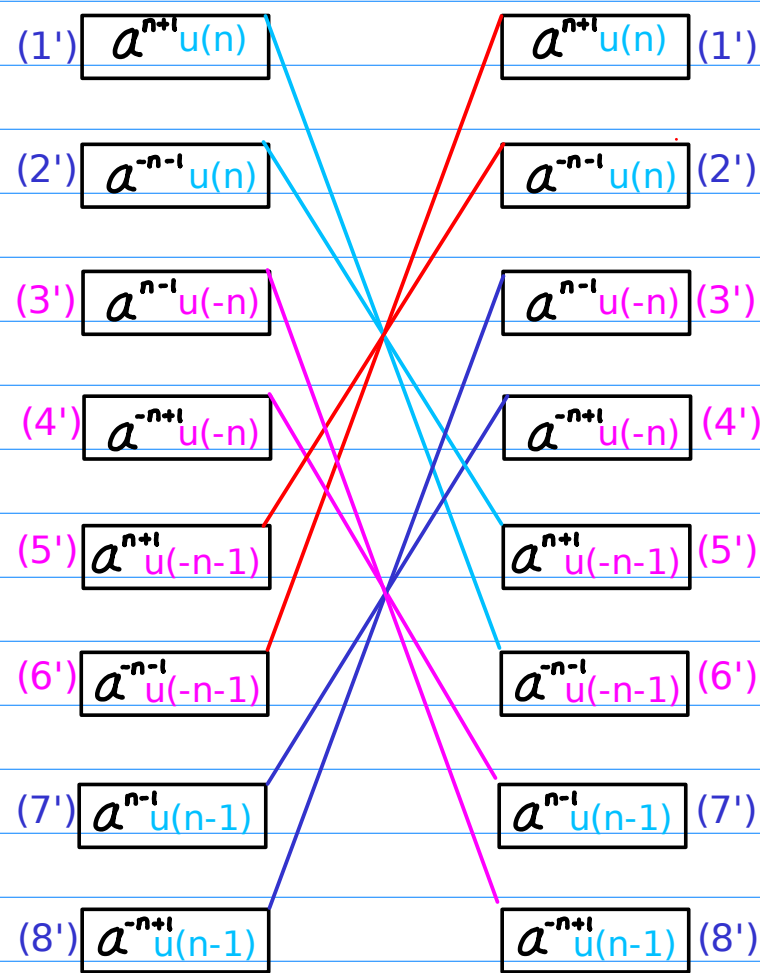
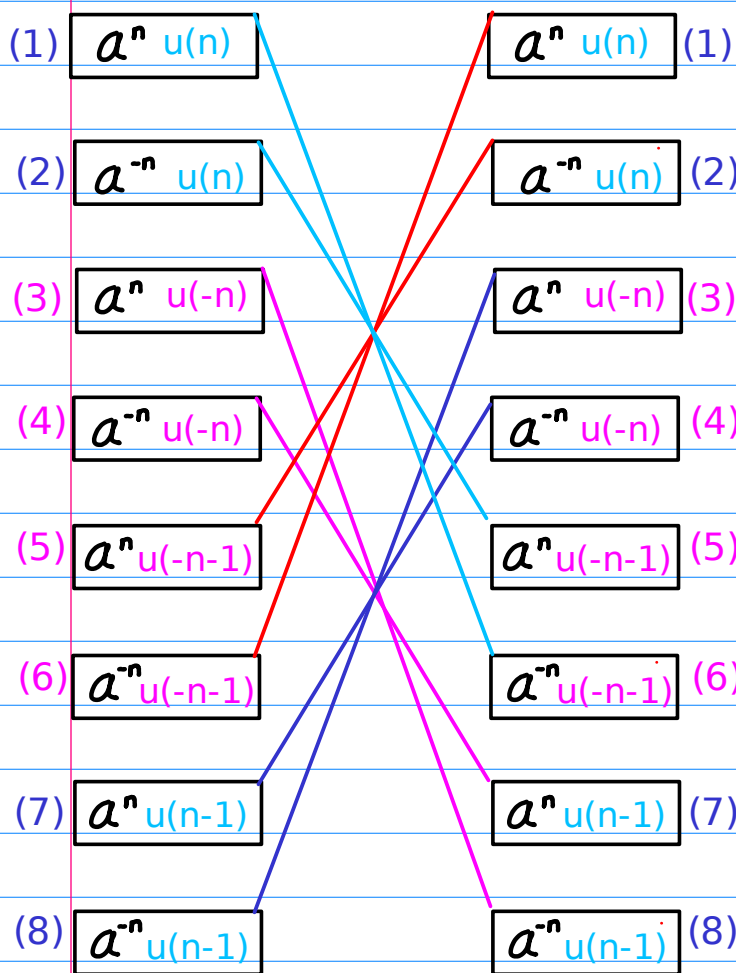
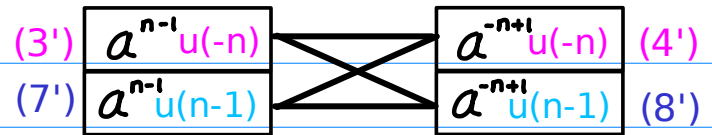
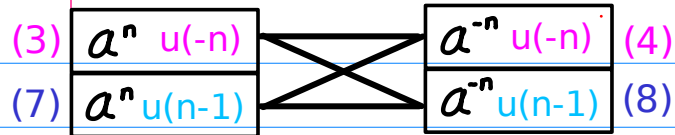
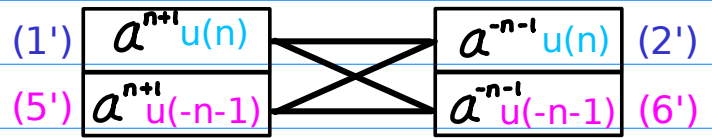
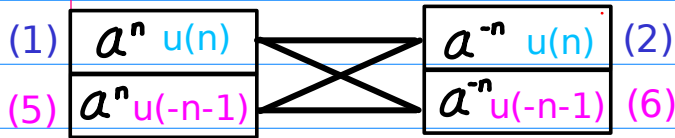


(1) - (7) (5) - (3)
 (2) - (8) (6) - (4)
 (3) - (5) (7) - (1)
 (4) - (6) (8) - (2)

(1') - (7') (5') - (3')
 (2') - (8') (6') - (4')
 (3') - (5') (7') - (1')
 (4') - (6') (8') - (2')

C.I Complementary Inverting Base Inverting Range Complementing

F.I Complementary Inverting Base Inverting Range Complementing



(1) - (6) (5) - (2)
 (6) - (1) (6) - (1)
 (2) - (5) (7) - (4)
 (5) - (2) (8) - (3)

(1') - (6') (5') - (2')
 (6') - (1') (6') - (1')
 (2') - (5') (7') - (4')
 (5') - (2') (8') - (3')

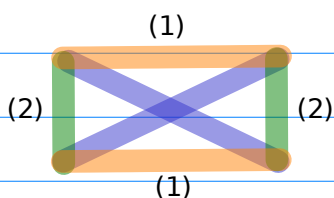


A.I Flipping

- (1) Base Inverting
- (2) Range Flipping

$$\begin{array}{|c|} \hline (1) \ a^n u(n) \\ \hline (3) \ a^n u(-n) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n} u(n) \quad (2) \\ \hline a^{-n} u(-n) \quad (4) \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (5) \ a^n u(-n-1) \\ \hline (7) \ a^n u(n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n} u(-n-1) \quad (6) \\ \hline a^{-n} u(n-1) \quad (8) \\ \hline \end{array}$$



$$\begin{array}{ccc} a^n & \xleftrightarrow{(1)} & a^{-n} \\ R(n) & \xleftrightarrow{(2)} & R(-n) \\ a^n R(n) & \longleftrightarrow & a^{-n} R(-n) \end{array}$$

b^n	b^{-n}
a^n	a^{-n}
a^{-n}	a^n

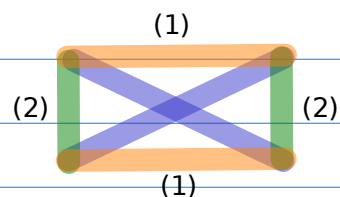
$R(n)$	$R(-n)$
$u(n)$	$u(-n)$
$u(n-1)$	$u(-n-1)$
$u(-n)$	$u(n)$
$u(-n-1)$	$u(n-1)$

D.I Flipping2

- (1) Base Inverting
- (2) Shifted Range Flipping

$$\begin{array}{|c|} \hline (1') \ a^{n+1} u(n) \\ \hline (3') \ a^{n-1} u(-n) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n-1} u(n) \quad (2') \\ \hline a^{-n+1} u(-n) \quad (4') \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (5') \ a^{n+1} u(-n-1) \\ \hline (7') \ a^{n-1} u(n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n-1} u(-n-1) \quad (6') \\ \hline a^{-n+1} u(n-1) \quad (8') \\ \hline \end{array}$$



$$\begin{array}{ccc} a^n & \xleftrightarrow{(1)} & a^{-n} \\ a^n R(n) & \xleftrightarrow{(2)} & a^{sh2(n)} R(-n) \\ a^n R(n) & \longleftrightarrow & a^{-sh2(n)} R(-n) \end{array}$$

b^n	$b^{-sh2(n)}$
$a^{(n+1)}$	$a^{-(n-1)}$
$a^{-(n+1)}$	$a^{(n-1)}$
$a^{(n-1)}$	$a^{-(n+1)}$
$a^{-(n-1)}$	$a^{(n+1)}$

$R(n)$	$R(-n)$
$u(n)$	$u(-n)$
$u(n-1)$	$u(-n-1)$
$u(-n)$	$u(n)$
$u(-n-1)$	$u(n-1)$

B.I Range Shifting

(1) Range Complementing
(2) Range Flipping

E.I Shifting2

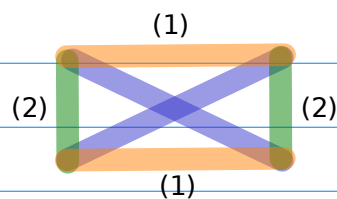
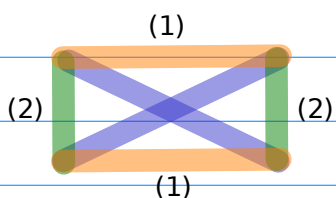
(1) Shifted Range Flipping
(2) Range Complementing

$$\begin{array}{l}
 (1) \quad a^n u(n) \\
 (5) \quad a^n u(-n)
 \end{array}
 \begin{array}{c}
 \diagdown \quad \diagup \\
 \diagup \quad \diagdown
 \end{array}
 \begin{array}{l}
 a^n u(-n-1) \quad (3) \\
 a^n u(n-1) \quad (7)
 \end{array}$$

$$\begin{array}{l}
 (1') \quad a^{n+1} u(n) \\
 (5') \quad a^{n+1} u(-n-1)
 \end{array}
 \begin{array}{c}
 \diagdown \quad \diagup \\
 \diagup \quad \diagdown
 \end{array}
 \begin{array}{l}
 a^{n+1} u(-n) \quad (3') \\
 a^{n+1} u(n-1) \quad (7')
 \end{array}$$

$$\begin{array}{l}
 (2) \quad a^{-n} u(n) \\
 (6) \quad a^{-n} u(-n-1)
 \end{array}
 \begin{array}{c}
 \diagdown \quad \diagup \\
 \diagup \quad \diagdown
 \end{array}
 \begin{array}{l}
 a^{-n} u(-n) \quad (4) \\
 a^{-n} u(n-1) \quad (8)
 \end{array}$$

$$\begin{array}{l}
 (2') \quad a^{-n-1} u(n) \\
 (6') \quad a^{-n-1} u(-n-1)
 \end{array}
 \begin{array}{c}
 \diagdown \quad \diagup \\
 \diagup \quad \diagdown
 \end{array}
 \begin{array}{l}
 a^{-n-1} u(-n) \quad (4') \\
 a^{-n-1} u(n-1) \quad (8')
 \end{array}$$



$$\begin{array}{l}
 R(n) \xleftrightarrow{(1)} \overline{R(-n)} \\
 R(n) \xleftrightarrow{(2)} \overline{R(n)} \\
 R(n) \longleftrightarrow \overline{R(-n)}
 \end{array}$$

$$\begin{array}{l}
 a^n R(n) \xleftrightarrow{(1)} \overline{a^{sh2(n)} R(-n)} \\
 R(n) \xleftrightarrow{(2)} \overline{R(n)} \\
 a^n R(n) \longleftrightarrow \overline{a^{sh2(n)} R(-n)}
 \end{array}$$

b^n	$b^{sh2(n)}$
$a^{(n+1)}$	$a^{(n-1)}$
$a^{-(n+1)}$	$a^{-(n-1)}$
$a^{(n-1)}$	$a^{(n+1)}$
$a^{-(n-1)}$	$a^{-(n+1)}$

$R(n)$	$\overline{R(-n)}$
$u(n)$	$u(n-1)$
$u(n-1)$	$u(n)$
$u(-n)$	$u(-n-1)$
$u(-n-1)$	$u(-n)$

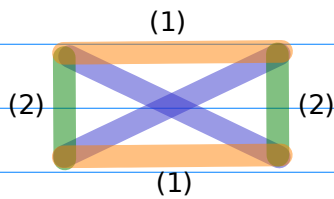
$R(n)$	$\overline{R(-n)}$
$u(n)$	$u(n-1)$
$u(n-1)$	$u(n)$
$u(-n)$	$u(-n-1)$
$u(-n-1)$	$u(-n)$

C.I Complementary Inverting

(1) Base Inverting
(2) Range Complementing

$$\begin{array}{|c|} \hline (1) \quad a^n u(n) \\ \hline (5) \quad a^n u(-n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n} u(n) \quad (2) \\ \hline a^{-n} u(-n-1) \quad (6) \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (3) \quad a^n u(-n) \\ \hline (7) \quad a^n u(n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n} u(-n) \quad (4) \\ \hline a^{-n} u(n-1) \quad (8) \\ \hline \end{array}$$



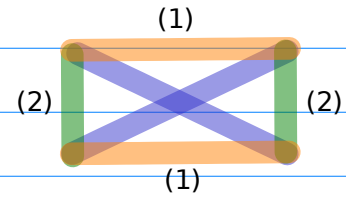
$$\begin{array}{ccc} a^n & \xleftrightarrow{(1)} & a^{-n} \\ R(n) & \xleftrightarrow{(2)} & \overline{R(n)} \\ a^n R(n) & \longleftrightarrow & a^{-n} \overline{R(n)} \end{array}$$

F.I Complementary Inverting

(1) Base Inverting
(2) Range Complementing

$$\begin{array}{|c|} \hline (1') \quad a^{n+1} u(n) \\ \hline (5') \quad a^{n+1} u(-n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n-1} u(n) \quad (2') \\ \hline a^{-n-1} u(-n-1) \quad (6') \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (3') \quad a^{n-1} u(-n) \\ \hline (7') \quad a^{n-1} u(n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n+1} u(-n) \quad (4') \\ \hline a^{-n+1} u(n-1) \quad (8') \\ \hline \end{array}$$



$$\begin{array}{ccc} a^n & \xleftrightarrow{(1)} & a^{-n} \\ R(n) & \xleftrightarrow{(2)} & \overline{R(n)} \\ a^n R(n) & \longleftrightarrow & a^{-n} \overline{R(n)} \end{array}$$

b^n	b^{-n}
a^n	a^{-n}
a^{-n}	a^n

b^n	b^{-n}
$a^{(n+1)}$	$a^{-(n+1)}$
$a^{-(n+1)}$	$a^{(n+1)}$
$a^{(n-1)}$	$a^{-(n-1)}$
$a^{-(n-1)}$	$a^{(n-1)}$

$R(n)$	$\overline{R(n)}$
$u(n)$	$u(-n-1)$
$u(n-1)$	$u(-n)$
$u(-n)$	$u(n)$
$u(-n-1)$	$u(n-1)$

$R(n)$	$\overline{R(n)}$
$u(n)$	$u(-n-1)$
$u(n-1)$	$u(-n)$
$u(-n)$	$u(n)$
$u(-n-1)$	$u(n-1)$

A.I Flipping

- (1) Base Inverting
- (2) Range Flipping

$$\begin{array}{ccc}
 a^n & \xleftrightarrow{(1)} & a^{-n} \\
 R(n) & \xleftrightarrow{(2)} & R(-n) \\
 a^n R(n) & \longleftrightarrow & a^{-n} R(-n)
 \end{array}$$

D.I Flipping2

- (1) Base Inverting
- (2) Shifted Range Flipping

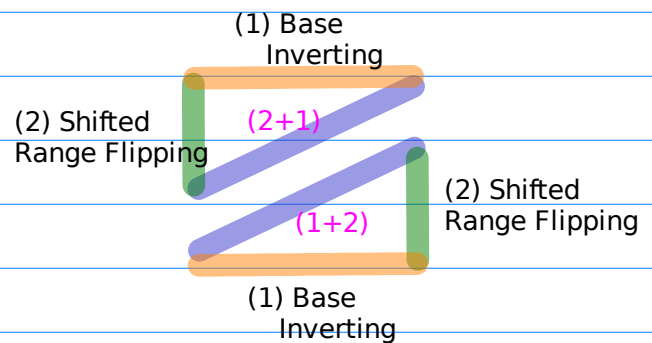
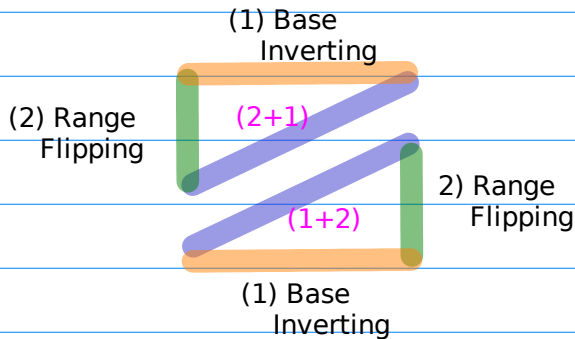
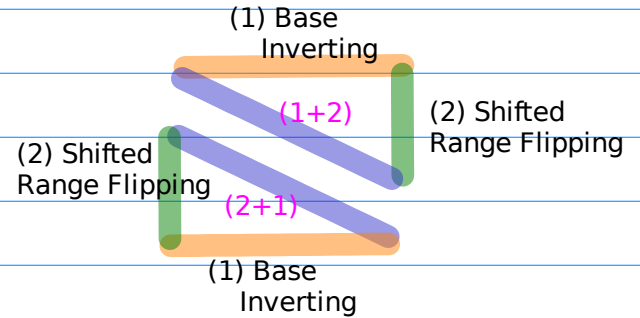
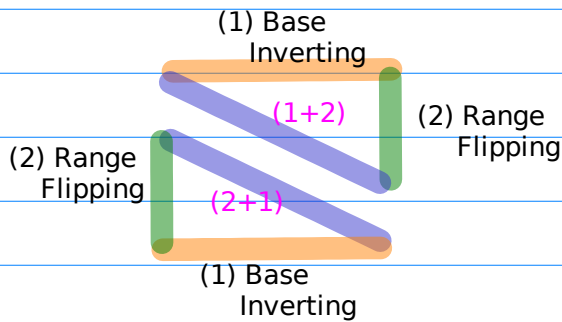
$$\begin{array}{ccc}
 a^n & \xleftrightarrow{(1)} & a^{-n} \\
 a^n R(n) & \xleftrightarrow{(2)} & a^{sh2(n)} R(-n) \\
 a^n R(n) & \longleftrightarrow & a^{-sh2(n)} R(-n)
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(1)} & a^{-n} R(n) \\
 & \xrightarrow{(2)} & a^{-n} R(-n)
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(1)} & a^{-n} R(n) \\
 & \xrightarrow{(2)} & a^{-sh2(n)} R(-n)
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(2)} & a^n R(-n) \\
 & \xrightarrow{(1)} & a^{-n} R(-n)
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(2)} & a^{sh2(n)} R(-n) \\
 & \xrightarrow{(1)} & a^{-sh2(n)} R(-n)
 \end{array}$$



B.I Range Shifting
(1) Range Complementing
(2) Range Flipping

E.I Shifting2
(1) Shifted Range Flipping
(2) Range Complementing

$$R(n) \xleftrightarrow{(1)} R(-n)$$

$$R(n) \xleftrightarrow{(2)} \overline{R(n)}$$

$$R(n) \longleftrightarrow \overline{R(-n)}$$

$$a^n R(n) \xleftrightarrow{(1)} a^{sh2(n)} R(-n)$$

$$R(n) \xleftrightarrow{(2)} \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

$$a^n R(n) \xrightarrow{(1)} a^{-n} R(n)$$

$$\xrightarrow{(2)} a^{-n} \overline{R(n)}$$

$$a^n R(n) \xrightarrow{(1)} a^{-n} R(n)$$

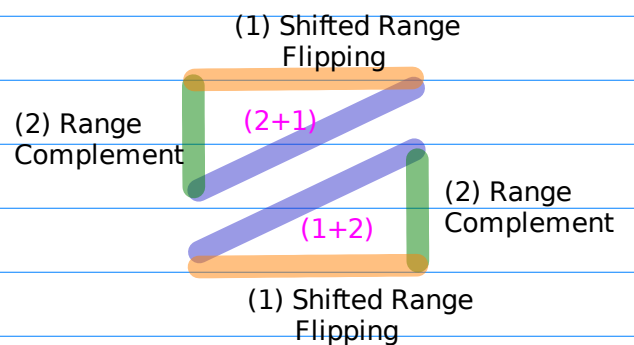
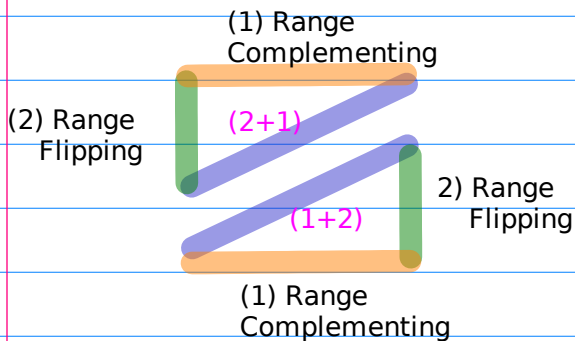
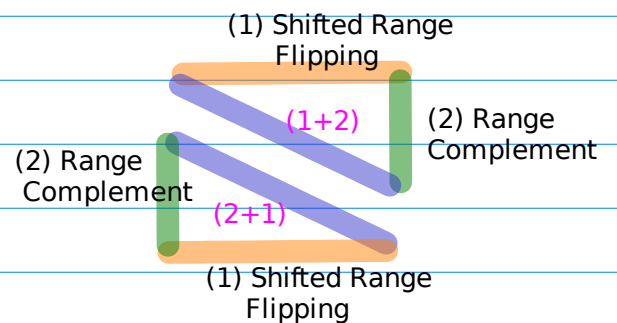
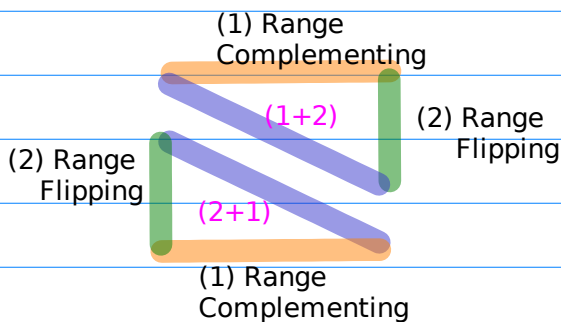
$$\xrightarrow{(2)} a^{-sh2(n)} R(-n)$$

$$a^n rng(n) \xrightarrow{(2)} a^n \overline{R(n)}$$

$$\xrightarrow{(1)} a^{-n} \overline{R(n)}$$

$$a^n R(n) \xrightarrow{(2)} a^{sh2(n)} R(-n)$$

$$\xrightarrow{(1)} a^{-sh2(n)} R(-n)$$



C.I Complementary Inverting

(1) Base Inverting
(2) Range Complementing

F.I Complementary Inverting

(1) Base Inverting
(2) Range Complementing

$$\begin{array}{ccc}
 a^n & \xleftrightarrow{(1)} & a^{-n} \\
 R(n) & \xleftrightarrow{(2)} & \overline{R(n)} \\
 a^n R(n) & \longleftrightarrow & a^{-n} \overline{R(n)}
 \end{array}$$

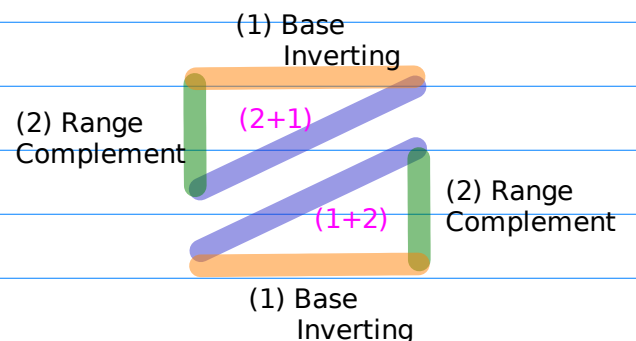
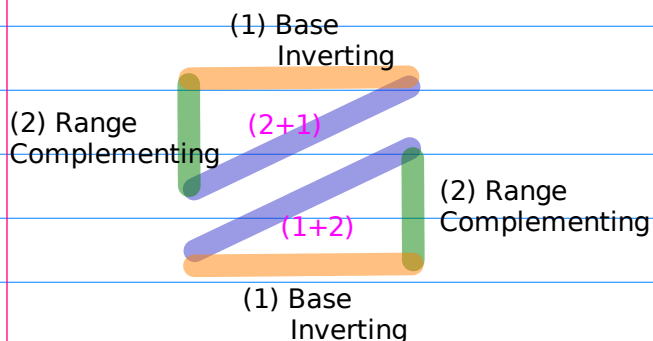
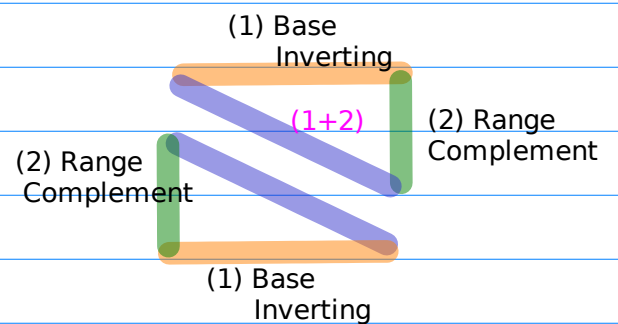
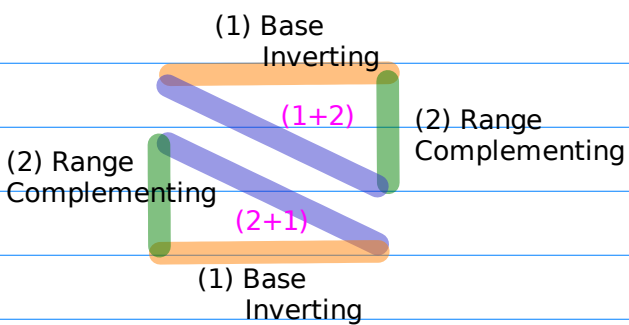
$$\begin{array}{ccc}
 a^n & \xleftrightarrow{(1)} & a^{-n} \\
 R(n) & \xleftrightarrow{(2)} & \overline{R(n)} \\
 a^n R(n) & \longleftrightarrow & a^{-n} \overline{R(n)}
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(1)} & a^{-n} R(n) \\
 & \xrightarrow{(2)} & a^{-n} \overline{R(n)}
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(1)} & a^{-n} R(n) \\
 & \xrightarrow{(2)} & a^{-n} \overline{R(n)}
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(2)} & a^n \overline{R(n)} \\
 & \xrightarrow{(1)} & a^{-n} \overline{R(n)}
 \end{array}$$

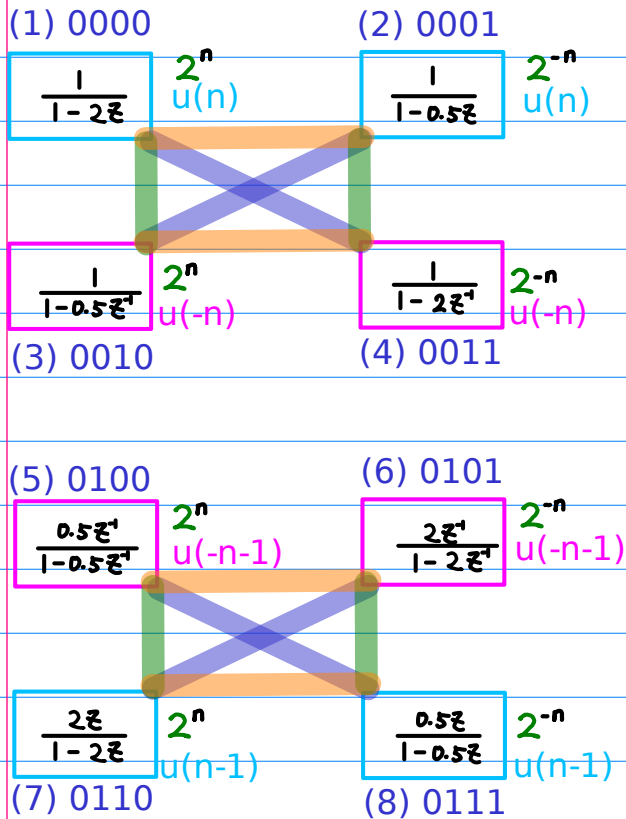
$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(2)} & a^n \overline{R(n)} \\
 & \xrightarrow{(1)} & a^{-n} \overline{R(n)}
 \end{array}$$



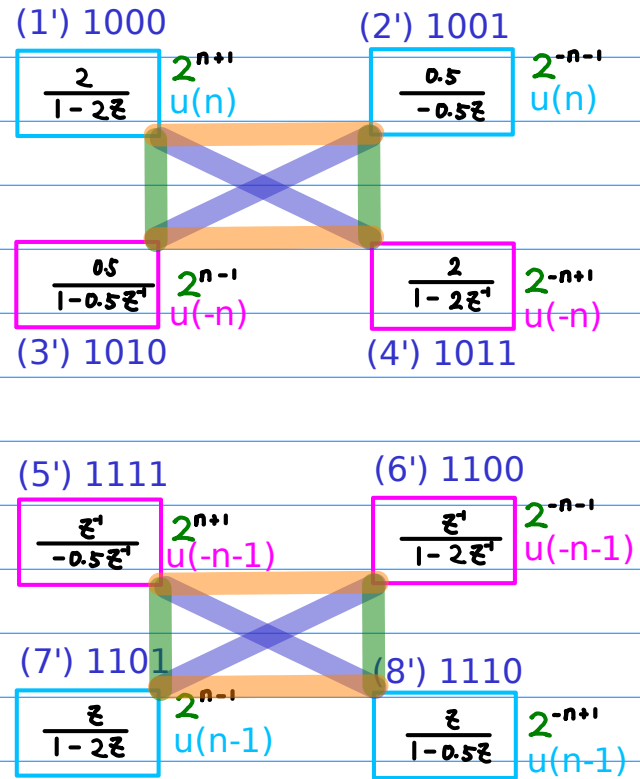




A.I Flipping Base Inverting Range Flipping



D.I Flipping2 Base Inverting Shifted Range Flipping



Shifted Range Flipping
= Exponent Shifting2
+ Range Flipping

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow R(-n)$$

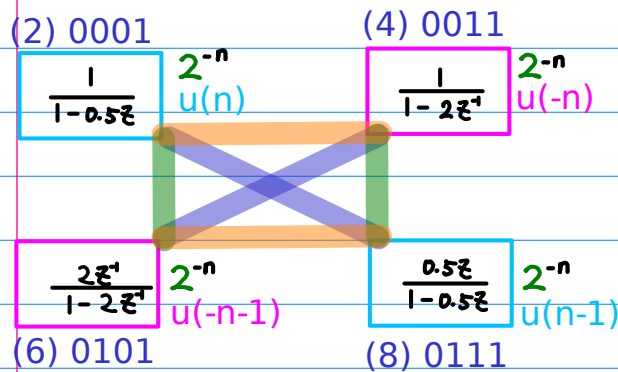
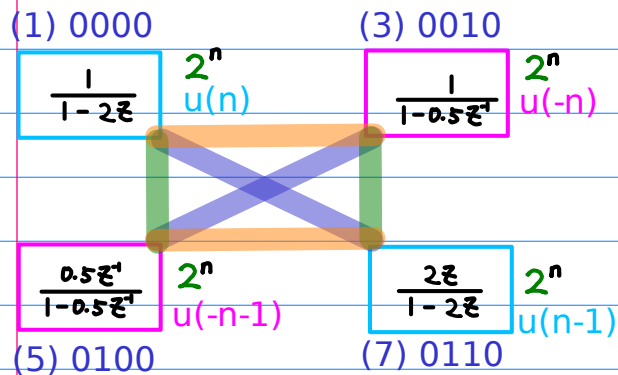
$$a^n R(n) \longleftrightarrow a^{-n} R(-n)$$

$$a^n \longleftrightarrow a^{-n}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

$$a^n R(n) \longleftrightarrow a^{-sh2(n)} R(-n)$$

B.I Range Shifting Range Flipping Range Complementing



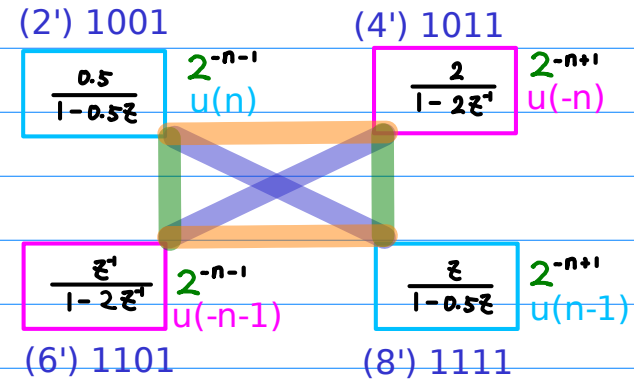
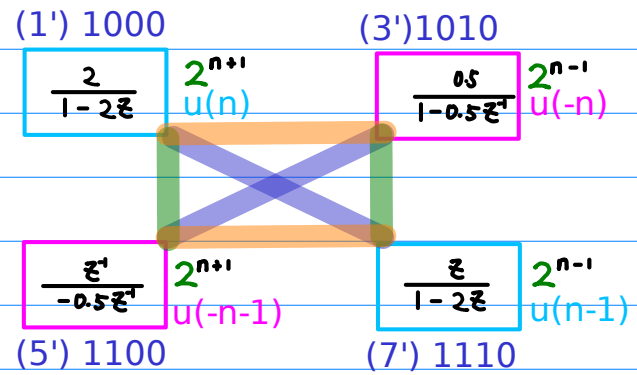
Range Shifting
= Range Flipping
+ Range Complementing

$$R(n) \longleftrightarrow R(-n)$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$R(n) \longleftrightarrow \overline{R(-n)}$$

E.I Shifting2 Shifted Range Flipping Range Complementing



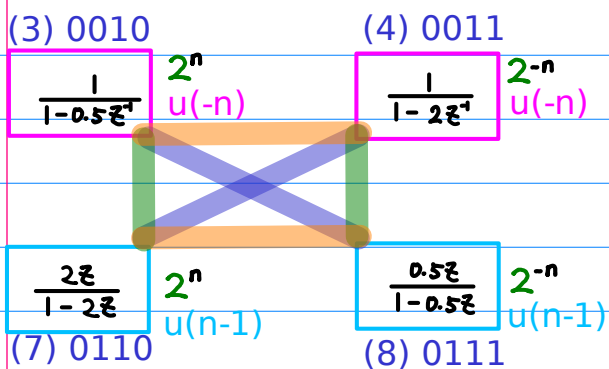
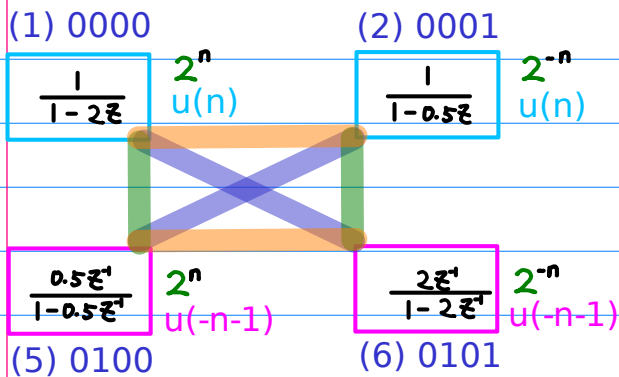
Shifted Range Flipping
= Exponent Shifting2
+ Range Flipping

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

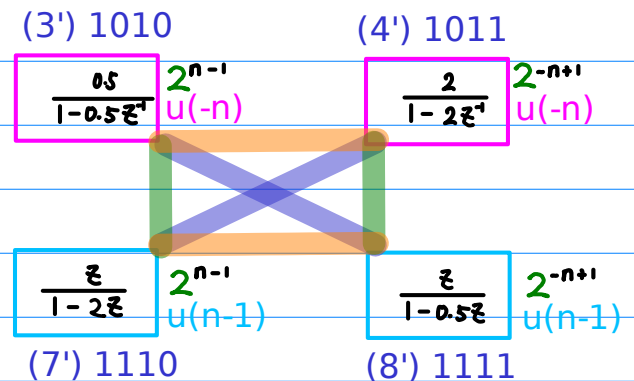
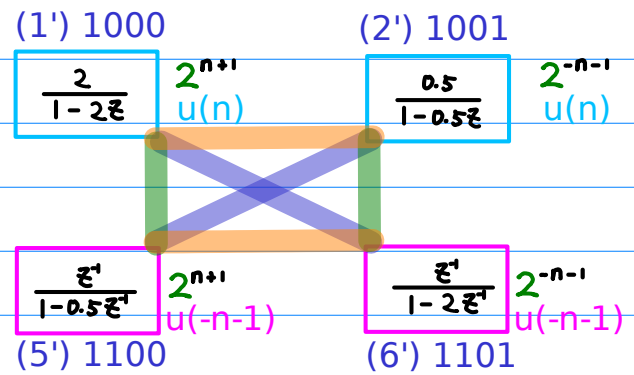
$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

C.I Complementary Inverting Base Inverting Range Complementing



F.I Complementary Inverting Base Inverting Range Complementing



$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

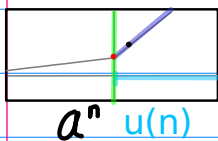
$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

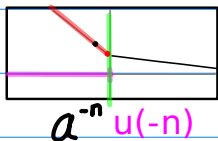
$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

A.II Flipping Base Inverting Range Flipping

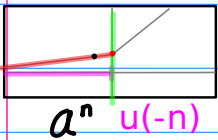
(1) 0000



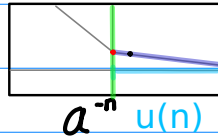
(4) 0011



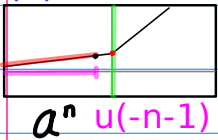
(3) 0010



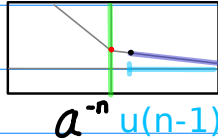
(2) 0001



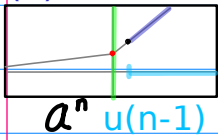
(5) 0100



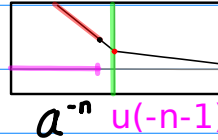
(8) 0111



(7) 0110

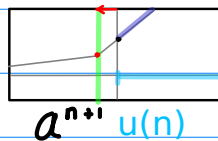


(6) 0101

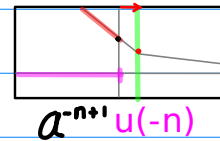


D.II Flipping2 Base Inverting Shifted Range Flipping

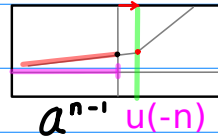
(1') 1000



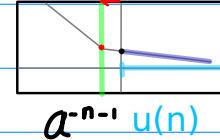
(4') 1011



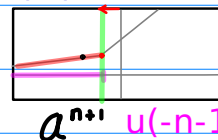
(3') 1010



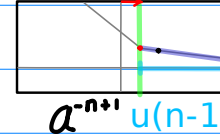
(2') 1001



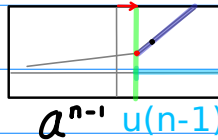
(5') 1100



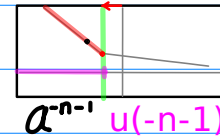
(8') 1111



(7') 1110



(6') 1101



Shifted Range Flipping
= Exponent Shifting2
+ Range Flipping

$$a^n R(n) \longleftrightarrow a^{-n} R(-n)$$

$$a^n R(n) \longleftrightarrow a^{-sh2(n)} R(-n)$$

$$a^n \longleftrightarrow a^{-n}$$

$$a^n \longleftrightarrow a^{-n}$$

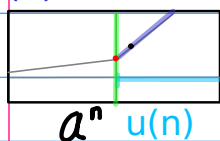
$$R(n) \longleftrightarrow R(-n)$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

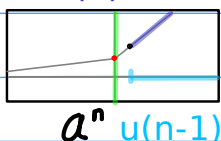
B.II Range Shifting Range Flipping Range Complementing

E.II Shifting2 Shifted Range Flipping Range Complementing

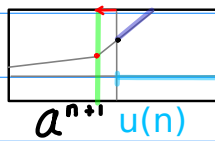
(1) 0000



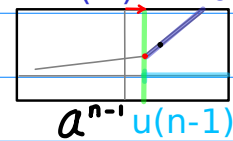
(7) 0110



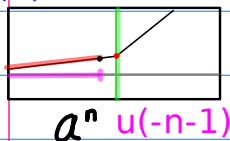
(1') 1000



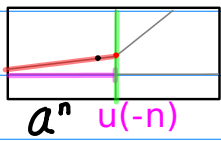
(7') 1110



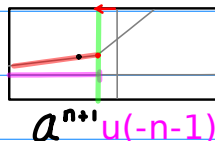
(5) 0100



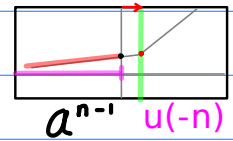
(3) 0010



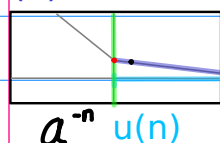
(5') 1100



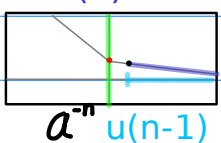
(3') 1010



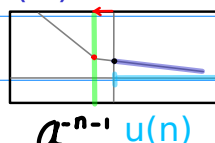
(2) 0001



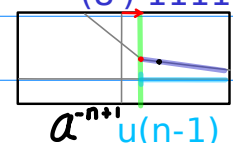
(8) 0111



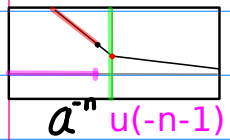
(2') 1001



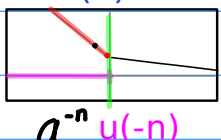
(8') 1111



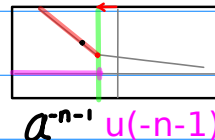
(6) 0101



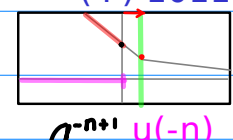
(4) 0011



(6') 1101



(4') 1011



Range Shifting
= Range Flipping
+ Range Complementing

Shifted Range Flipping
= Exponent Shifting2
+ Range Flipping

$$R(n) \longleftrightarrow \overline{R(-n)}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

$$R(n) \longleftrightarrow \overline{R(-n)}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

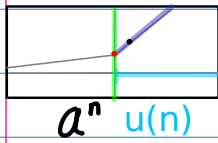
$$R(n) \longleftrightarrow \overline{R(n)}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

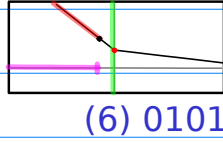
C.II Complementary Inverting Base Inverting Range Complementing

F.II Complementary Inverting Base Inverting Range Complementing

(1) 0000

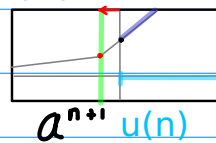


$a^{-n} u(-n-1)$



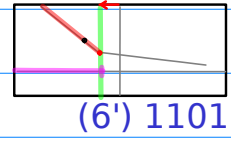
(6) 0101

(1') 1000



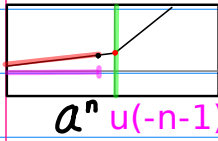
$a^{n+1} u(n)$

$a^{-n-1} u(-n-1)$



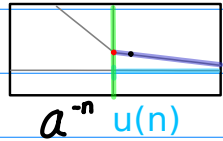
(6') 1101

(5) 0100



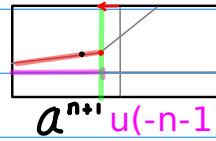
$a^n u(-n-1)$

(2) 0001



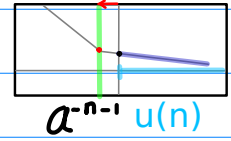
$a^{-n} u(n)$

(5') 1100



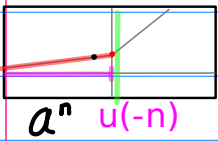
$a^{n+1} u(-n-1)$

(2') 1001



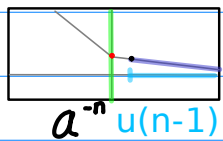
$a^{-n-1} u(n)$

(3) 0010



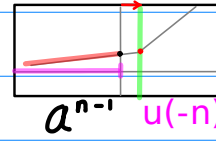
$a^n u(-n)$

(8) 0111



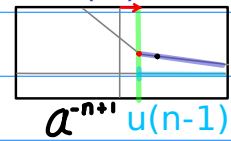
$a^{-n} u(n-1)$

(3') 1010



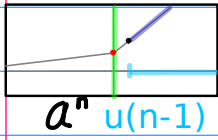
$a^{n-1} u(-n)$

(8') 1111



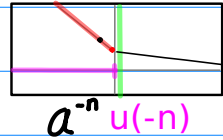
$a^{-n+1} u(n-1)$

(7) 0110



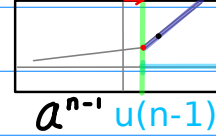
$a^n u(n-1)$

(4) 0011



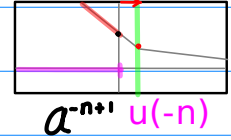
$a^{-n} u(-n)$

(7') 1110



$a^{n-1} u(n-1)$

(4') 1011



$a^{-n+1} u(-n)$

$$a^n R(n) \longleftrightarrow \overline{a^{-n} R(n)}$$

$$a^n R(n) \longleftrightarrow \overline{a^{-n} R(n)}$$

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$



