Z Transform (H.3)
Properties

20161030

Based on
Complex Analysis for Mathematics and Engineering
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Properties of z-transform

		$x_n = x[n]$	$\longleftrightarrow$	$X(z) = \sum_{n=0}^{\infty} x_n z^{-n}$
1	addition	$x_n + y_n$	$\longleftrightarrow$	$X(z) + Y(z)$
2	constant multiple	$c\,x_n$	$\longleftrightarrow$	$c\,X(z)$
3	linearity	$c x_n + d y_n$	$\longleftrightarrow$	$c\,X(z) + d\,Y(z)$
4	delayed unit step	$u[n-m]$	$\longleftrightarrow$	$\dfrac{z^{1-m}}{z-1}$
5	time delay 1 tap	$x_{n-1}\, u[n-1]$	$\longleftrightarrow$	$z^{-1} X(z)$
6	time delayed shift	$x_{n-m}\, u[n-m]$	$\longleftrightarrow$	$z^{-m} X(z)$
7	forward 1 tap	x_{n+1}	$\longleftrightarrow$	$z\,(X(z) - x_0)$
8	forward 2 taps	x_{n+2}	$\longleftrightarrow$	$z^2(X(z) - x_0 - x_1 z^{-1})$
9	time forward	x_{n+m}	$\longleftrightarrow$	$z^m\left(X(z) - \sum_{i=0}^{m-1} x_i z^{-i}\right)$
10	complex translation	$e^{an} x_n$	$\longleftrightarrow$	$X(z e^{-a})$
11	frequency scale	$b^n x_n$	$\longleftrightarrow$	$X\left(\frac{z}{b}\right)$

12	differentiation	$n x_n$	$\longleftrightarrow$ $-zX'(z)$
13	integration	$\frac{1}{n} x_n$	$\longleftrightarrow$ $-\int \frac{X(z)}{z} dz$
14	integration shift	$\frac{1}{n+m} x_n$	$\longleftrightarrow$ $-z^m \int \frac{X(z)}{z^{m+1}} dz$
15	convolution	$x_n * y_n = \sum_{i=0}^{n} x_i y_{n-i}$	$\longleftrightarrow$ $X(z)\, Y(z)$
16	convolution with 1	$x_n * 1 = \sum_{i=0}^{n} x_i$	$\longleftrightarrow$ $\frac{z}{z-1} X(z)$
17	initial value	x_0	$\longleftrightarrow$ $\lim_{z\to\infty} X(z)$
18	final value	$\lim_{n\to\infty} x_n$	$\longleftrightarrow$ $\lim_{z\to 1} (z-1)X(z)$

1 addition $\quad x_n + y_n \Longleftrightarrow X(z) + Y(z)$

2 constant multiple $\quad c\,x_n \Longleftrightarrow c\,X(z)$

3 linearity $\quad c\,x_n + d\,y_n \Longleftrightarrow c\,X(z) + d\,Y(z)$

$$\mathcal{Z}[x_n] = \mathcal{Z}[x[n]] = X(z)$$

$$\mathcal{Z}[y_n] = \mathcal{Z}[y[n]] = Y(z)$$

Linearity

$$\mathcal{Z}[c_1 x_n + c_2 y_n] = \mathcal{Z}[c_1 x[n] + c_2 y[n]] = c_1 X(z) + c_2 Y(z)$$

Delay Shift

$$\mathcal{Z}[x[n-N]\,u[n-N]] = X(z)\,z^{-N}$$

Multiplication by n

$$\mathcal{Z}[n\,x[n]] = -z\frac{d}{dz}X(z)$$

Shifted Sequences and Initial Conditions

4 delayed unit step $\quad u[n-m] \longleftrightarrow \dfrac{z^{1-m}}{z-1}$

5 time delay 1 tap $\quad x_{n-1}\, u[n-1] \longleftrightarrow z^{-1} X(z)$

6 time delayed shift $\quad x_{n-m}\, u[n-m] \longleftrightarrow z^{-m} X(z)$

7 forward 1 tap $\quad x_{n+1} \longleftrightarrow z^{1}(X(z) - x_0)$

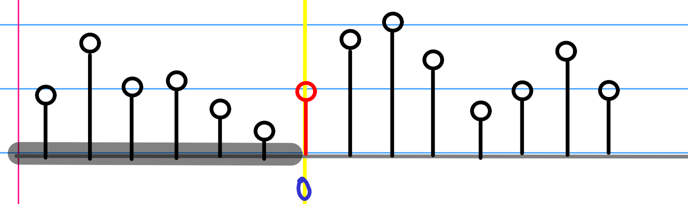
8 forward 2 taps $\quad x_{n+2} \longleftrightarrow z^{2}(X(z) - x_0 - x_1 z^{-1})$

9 time forward $\quad x_{n+m} \longleftrightarrow z^{m}\left(X(z) - \sum_{i=0}^{m-1} x_i z^{-i}\right)$

$x[n-m]$ (shift right $\longrightarrow$) : $\times\, z^{-m}$

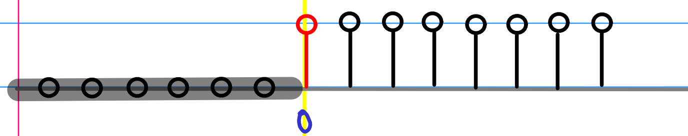
$x[n+m]$ (shift left $\longleftarrow$) : $\times\, z^{+m}$

$\boxed{x[n]}$



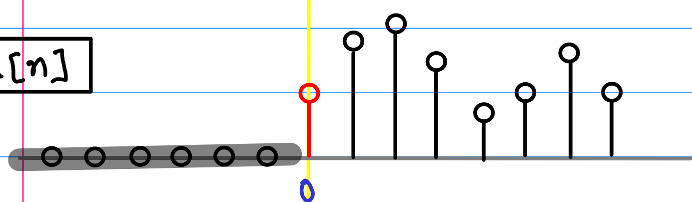
$X(z)$

$\boxed{u[n]}$



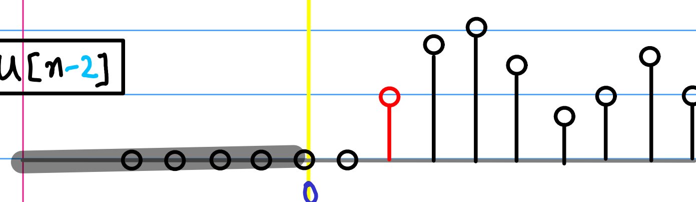
$U(z) = \frac{z}{z-1}$

$\boxed{x[n]\,u[n]}$



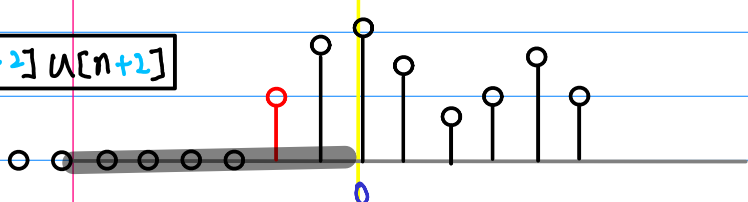
$X(z)$

$\boxed{x[n-2]\,u[n-2]}$



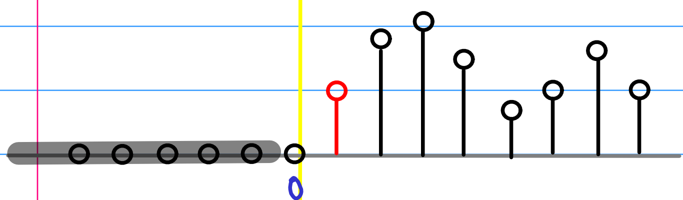
$z^{-2}X(z)$

$\boxed{x[n+2]\,u[n+2]}$

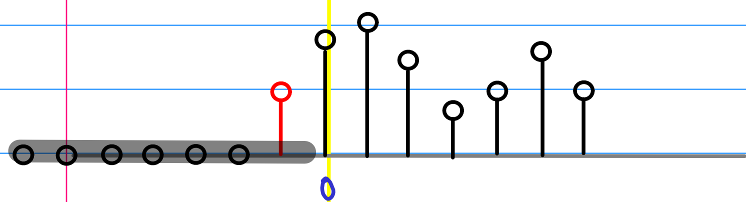


$z^{+2}X(z) - z^2x[0] - z\,x[1]$

* With zero Initial Conditions (all zero $n<0$)



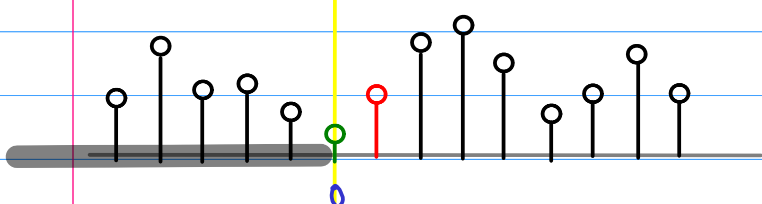
$x[n-1]u[n-1] \longrightarrow z^{-1}X(z)$



$x[n+1]u[n+1] \longleftarrow zX(z) - x[0]z$

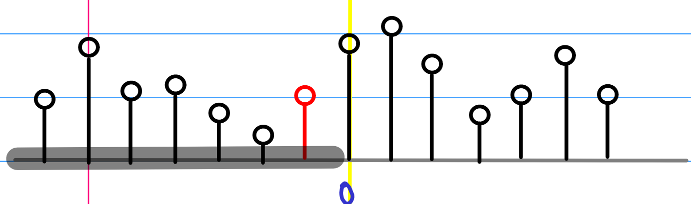
$x[0]$ must vanish

* With non-zero Initial Conditions (not all zero $n<0$)



$x[n-1] \longrightarrow z^{-1}X(z) + x[-1]$

$x[-1]$ must come



$x[n+1] \longleftarrow zX(z) - x[0]z$

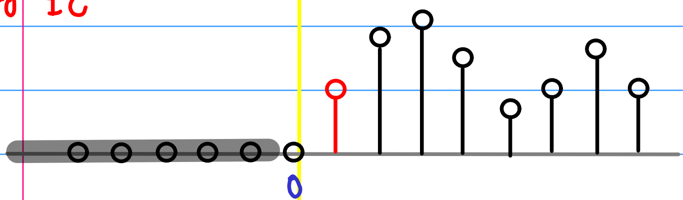
$x[0]$ must vanish

One sided z-transform

disregard $n<0$ portion

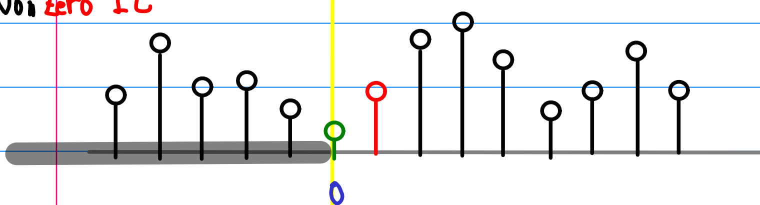
⓪ Shift Right

Zero IC



$x[n-1]\,u[n-1]$ $\quad$ $z^{-1}X(z)$

Non Zero IC

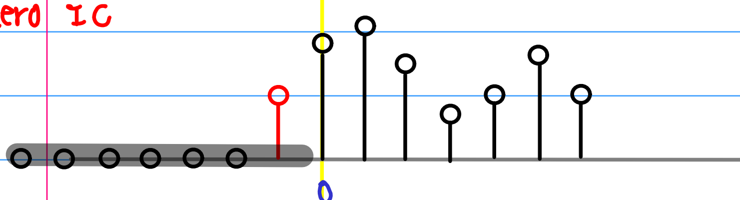


$x[n-1]$ $\quad$ $z^{-1}X(z) + x[-1]$

$x[-1]$ must come

⓪ Shift Left

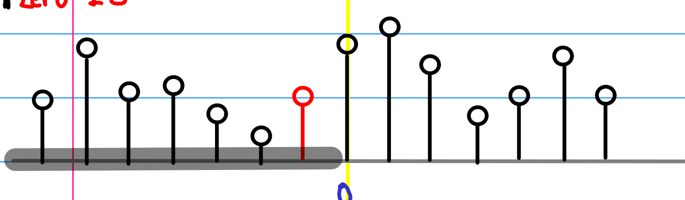
Zero IC



$x[n+1]\,u[n+1]$ $\quad$ $zX(z) - x[0]z$

$x[0]$ must vanish

Non Zero IC



$x[n+1]$ $\quad$ $zX(z) - x[0]z$

$x[0]$ must vanish

One sided z-transform

disregard $n<0$ portion

Z.I.C.	$\rightarrow$	$x[n-1]u[n-1]$	$\longleftrightarrow$	$z^{-1}X(z)$
N.I.C.	$\rightarrow$	$x[n-1]$	$\longleftrightarrow$	$z^{-1}X(z) + x[-1]$
Z.I.C.	$\leftarrow$	$x[n+1]u[n+1]$	$\longleftrightarrow$	$z^{+1}X(z) - x[0]z$
N.I.C.	$\leftarrow$	$x[n+1]$	$\longleftrightarrow$	$z^{+1}X(z) - x[0]z$
Z.I.C.	$\rightarrow$	$x[n-2]u[n-2]$	$\longleftrightarrow$	$z^{-2}X(z)$
N.I.C.	$\rightarrow$	$x[n-2]$	$\longleftrightarrow$	$z^{-2}X(z) + x[-1]z^{-1} + x[-2]$
Z.I.C.	$\leftarrow$	$x[n+2]u[n+2]$	$\longleftrightarrow$	$z^{+2}X(z) - x[0]z^{2} - x[1]z^{1}$
N.I.C.	$\leftarrow$	$x[n+2]$	$\longleftrightarrow$	$z^{+2}X(z) - x[0]z^{2} - x[1]z^{1}$
Z.I.C.	$\rightarrow$	$x[n-m]u[n-m]$	$\longleftrightarrow$	$z^{-m}X(z)$
N.I.C.	$\rightarrow$	$x[n-m]$	$\longleftrightarrow$	$z^{-m}X(z) + x[-1]z^{-m+1} + \cdots + x[-m]$
	$\leftarrow$	$x[n+m]u[n+m]$	$\longleftrightarrow$	$z^{+m}X(z) - x[0]z^{m} - \cdots - x[m-1]z^{1}$
	$\leftarrow$	$x[n+m]$	$\longleftrightarrow$	$z^{+m}X(z) - x[0]z^{m} - \cdots - x[m-1]z^{1}$

		$u[n-m]$	$\longleftrightarrow$	$\dfrac{z^{1-m}}{z-1}$
Z.I.C.	$\rightarrow$	$x_{n-1}\,u[n-1]$	$\longleftrightarrow$	$z^{-1}X(z)$
Z.I.C.	$\rightarrow$	$x_{n-m}\,u[n-m]$	$\longleftrightarrow$	$z^{-m}X(z)$

Z.I.C. N.I.C.	$\leftarrow$	x_{n+1}	$\longleftrightarrow$	$z^{1}(X(z) - x_0)$
Z.I.C. N.I.C.	$\leftarrow$	x_{n+2}	$\longleftrightarrow$	$z^{2}(X(z) - x_0 - x_1 z^{-1})$
Z.I.C. N.I.C.	$\leftarrow$	x_{n+m}	$\longleftrightarrow$	$z^{m}\left(X(z) - \sum_{i=0}^{m-1} x_i z^{-i}\right)$

Left Shift

$$\{x[n] = x_n\}_{n=0}^{\infty}$$

$$X(z) = \mathcal{Z}[x[n]] = \sum_{n=0}^{\infty} x[n] z^{-n}$$

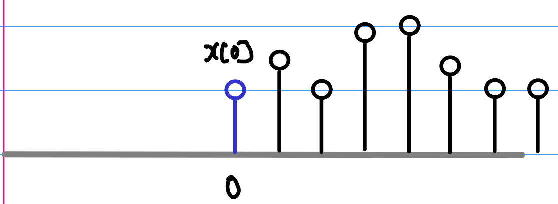
$$\mathcal{Z}[x[n+1]] = z\left(X(z) - x[0]\right)$$

$$\mathcal{Z}[x[n+2]] = z^2\left(X(z) - x[0] - x[1] z^{-1}\right)$$

$$\mathcal{Z}[x[n+3]] = z^3\left(X(z) - x[0] - x[1] z^{-1} - x[2] z^{-2}\right)$$

$$\begin{cases} \text{Z. I.C.} \\ \text{N. I.C.} \end{cases}$$

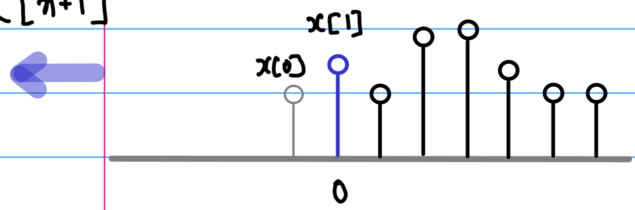
$x[n]$



$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$= x[0] + x[1] z^{-1} + x[2] z^{-2} + \cdots$$

$x[n+1]$

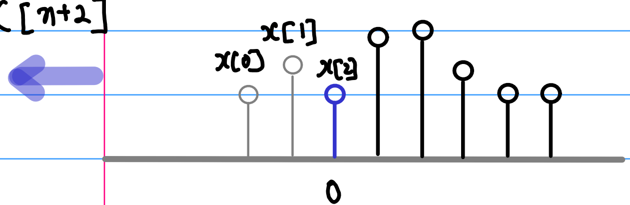


$z\,X(z)$

$$= x[0]z + \left(x[1] + x[2] z^{-1} + x[3] z^{-2} + \cdots\right)$$

$$= x[0]z + \mathcal{Z}[x[n+1]]$$

$x[n+2]$



$$\mathcal{Z}[x[n+1]] = z\,X(z) - x[0]\,z$$

$$\mathcal{Z}[x[n+1]] = z\,\mathcal{Z}[x[n]] - x[0]\,z$$

$$\mathcal{Z}[x[n+2]] = z\,\mathcal{Z}[x[n+1]] - x[1]\,z$$

$$= z\left[z\,\mathcal{Z}[x[n]] - x[0]\,z\right] - x[1]\,z$$

$$= z^2\,\mathcal{Z}[x[n]] - x[0]\,z^2 - x[1]\,z$$

$$= z^2\left(\mathcal{Z}[x[n]] - x[0] - x[1]\,z^{-1}\right)$$

$$\mathcal{Z}[x[n+3]] = z^3\left(\mathcal{Z}[x[n]] - x[0] - x[1]\,z^{-1} - x[2]\,z^{-2}\right)$$

Right Shift with zero initial conditions

$$\left\{ x[n] = x_n \right\}_{n=0}^{\infty}$$

$$X(z) = \mathcal{Z}\left[x[n] \right] = \sum_{n=0}^{\infty} x[n] z^{-n}$$

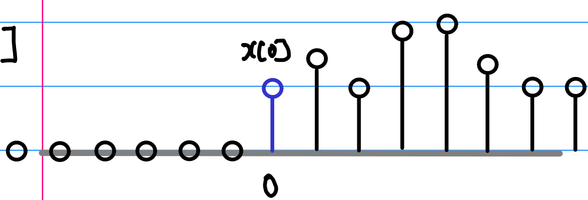
$$\mathcal{Z}\left[x[n-1] \right] = z^{-1} X(z)$$

$$\mathcal{Z}\left[x[n-2] \right] = z^{-2} X(z)$$

$$\mathcal{Z}\left[x[n-3] \right] = z^{-3} X(z)$$

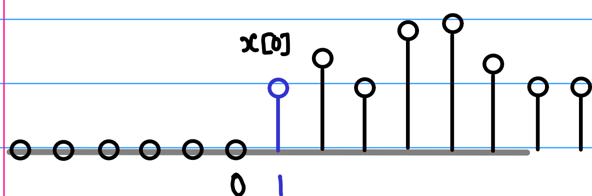
Z.I.C. zero initial conditions

$x[n]$



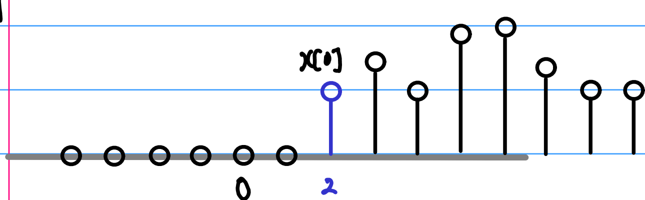
$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$
$$= x[0] + x[1] z^{-1} + x[2] z^{-2} + \cdots$$

$x[n+1]$



$$z^{-1} X(z)$$
$$= x[0] z^{-1} + x[1] z^{-2} + x[2] z^{-3} + x[3] z^{-4} + \cdots$$
$$= z^{-1} \mathcal{Z}\left[x[n+1] \right]$$

$x[n+2]$



$$z^{-2} X(z)$$
$$= x[0] z^{-2} + x[1] z^{-3} + x[2] z^{-4} + x[3] z^{-5} + \cdots$$
$$= z^{-2} \mathcal{Z}\left[x[n+2] \right]$$

Right Shift with non-zero initial conditions

$$\left\{ x[n] = x_n \right\}_{n=0}^{\infty}$$

$$X(z) = \mathcal{Z}\left[x[n] \right] = \sum_{n=0}^{\infty} x[n] z^{-n}$$

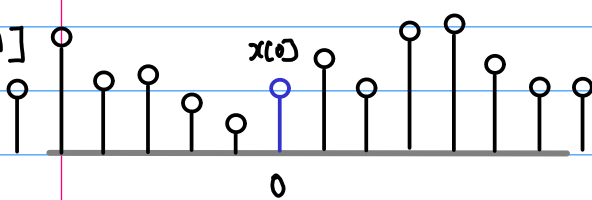
$$\mathcal{Z}\left[x[n-1] \right] = z^{-1} X(z) + x[-1]$$

$$\mathcal{Z}\left[x[n-2] \right] = z^{-2} X(z) + x[-1] z^{-1} + x[-2]$$

$$\mathcal{Z}\left[x[n-3] \right] = z^{-3} X(z) + x[-1] z^{-2} + x[-2] z^{-1} + x[-3]$$

Z.I.C. zero initial conditions

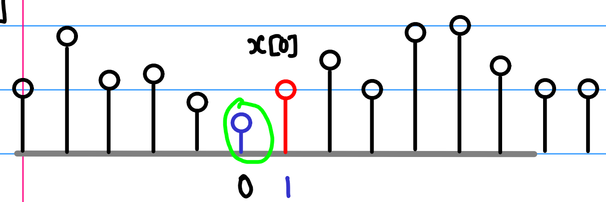
$x[n]$



$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$= x[0] + x[1] z^{-1} + x[2] z^{-2} + \cdots$$

$x[n+1]$

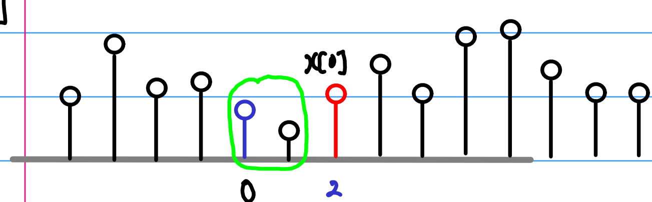


$z^{-1} X(z)$

$$= x[-1] + x[0] z^{-1} + x[1] z^{-2} + x[2] z^{-3} + \cdots$$

$$= z^{-1} \mathcal{Z}\left[x[n+1] \right] + x[-1]$$

$x[n+2]$



$z^{-2} X(z)$

$$= x[-2] + x[-1] z^{-1} + x[0] z^{-2} + x[1] z^{-3} + x[2] z^{-4} + \cdots$$

$$= z^{-2} \mathcal{Z}\left[x[n+2] \right] + z^{-1} x[-1] + x[-2]$$

Left Shift

Z. I.C.
N. I.C.

$$\mathcal{Z}\left[x[n+1]\right] = z\left(X(z) - x[0]\right)$$

$$\mathcal{Z}\left[x[n+2]\right] = z^{2}\left(X(z) - x[0] - x[1]\,z^{-1}\right)$$

$$\mathcal{Z}\left[x[n+3]\right] = z^{3}\left(X(z) - x[0] - x[1]\,z^{-1} - x[2]\,z^{-2}\right)$$

$$\mathcal{Z}\left[x[n+1]\right] = z^{1}X(z) - x[0]\,z^{1}$$

$$\mathcal{Z}\left[x[n+2]\right] = z^{2}X(z) - x[0]\,z^{2} - x[1]\,z^{1}$$

$$\mathcal{Z}\left[x[n+3]\right] = z^{3}X(z) - x[0]\,z^{3} - x[1]\,z^{2} - x[2]\,z^{1}$$

Right Shift

N. I.C.

$$\mathcal{Z}\left[x[n-1]\right] = z^{-1}X(z) + x[-1]$$

$$\mathcal{Z}\left[x[n-2]\right] = z^{-2}X(z) + x[-1]\,z^{-1} + x[-2]$$

$$\mathcal{Z}\left[x[n-3]\right] = z^{-3}X(z) + x[-1]\,z^{-2} + x[-2]\,z^{-1} + x[-3]$$

non-zero $x[-i]$

Z. I.C.

$$\mathcal{Z}\left[x[n-1]\right] = z^{-1}X(z)$$

$$\mathcal{Z}\left[x[n-2]\right] = z^{-2}X(z)$$

$$\mathcal{Z}\left[x[n-3]\right] = z^{-3}X(z)$$

all-zero $x[-i]$

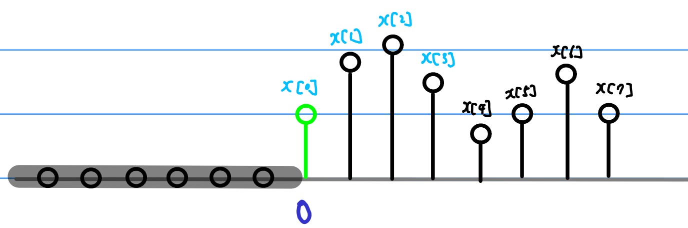
Left Shift = Advance shift

$$\mathcal{Z}\left[x[n+N]\,u[n+N]\right] =$$

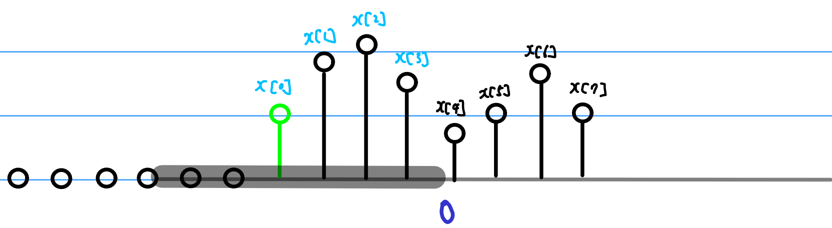
$$\mathcal{Z}\left[x[n+N]\right] = z^{N}\left(X(z) - x[0] - x[1]z^{-1} - x[2]z^{-2} - \cdots - x[N-1]z^{-N+1}\right)$$

$$= z^{N}X(z) - x[0]z^{N} - x[1]z^{N-1} - x[2]z^{N-2} - \cdots - x[N-1]z^{1}$$

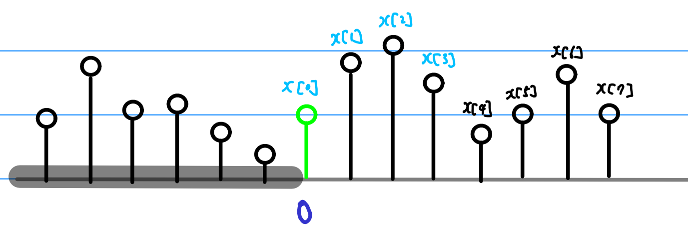
$x[0]$, $x[1]$, $\cdots$, $x[N-1]$ samples must be disappeared



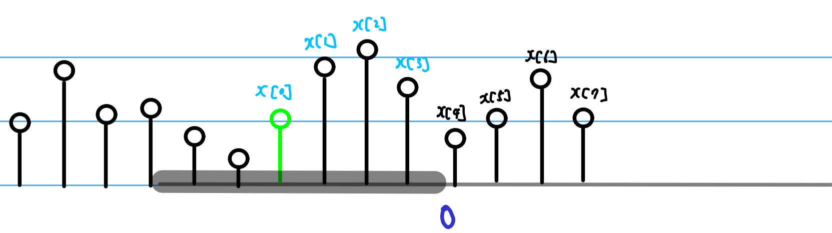
$x[n]\,u[n]$



$x[n+N]\,u[n+N]$



$x[n]$



$x[n+N]$

Left Shift

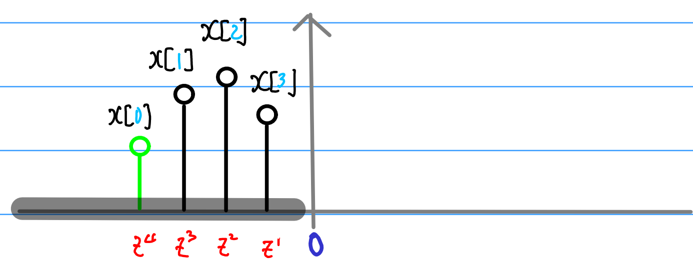
$$\mathcal{Z}[x[n+N]u[n+N]] =$$

$$\mathcal{Z}[x[n+N]] = z^{N}\left(X(z) - x[0] - x[1]z^{-1} - x[2]z^{-2} - \cdots - x[N-1]z^{-N+1}\right)$$

$$= z^{N}X(z) - x[0]z^{N} - x[1]z^{N-1} - x[2]z^{N-2} - \cdots - x[N-1]z^{1}$$

N shifted out samples :

$x[0]$	$x[1]$	$x[2]$	$\cdots$	$x[N-1]$
z^{N}	z^{N-1}	z^{N-2}		z^{1}
$0+N$	$1+N-1$	$2+N-2$		$N-1+1$



must subtract $\left(x[0]z^{N} + x[1]z^{N-1} + x[2]z^{N-2} + \cdots + x[N-1]z^{1}\right)$

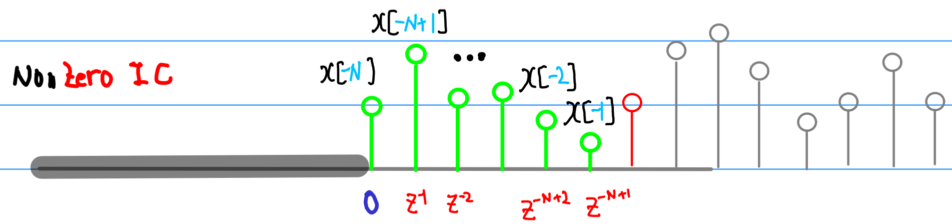
Right Shift

ZIC $\quad \mathcal{Z}\left[x[n-N]\,u[n-N]\right] = z^{-N} X(z)$

NIC $\quad \mathcal{Z}\left[x[n-N]\right] = z^{-N}\left(X(z) + x[-1]z + x[-2]z^{2} + \cdots + x[-N+1]z^{N-1} + x[-N]z^{N}\right)$

$$= z^{-N}X(z) + x[-1]z^{-N+1} + x[-2]z^{-N+2} + \cdots + x[-N+1]z^{-1} + x[-N]$$

Shifted in samples	$x[-N]$	$x[-N+1]$	$\cdots$	$x[-2]$	$x[-1]$
	z^{0}	z^{-1}		z^{-N+2}	z^{-N+1}
	$-N+0$	$-N+1-1$		$-2-N+2$	$-1-N+1$



must add $\quad x[-N] + x[-N+1]z^{-1} + \cdots + x[-2]z^{-N+2} + x[-1]z^{-N+1}$

$$= x[-1]z^{-N+1} + x[-2]z^{-N+2} + \cdots + x[-N+1]z^{-1} + x[-N]$$

ZIC $\mathcal{Z}[x[n+N]u[n+N]] =$

shift out

NIC $\mathcal{Z}[x[n+N]] = \boxed{z^{N}}\left(X(z) - x[0] - x[1]z^{-1} - x[2]z^{-2} - \cdots - x[N-1]z^{-N+1}\right)$

ZIC $\mathcal{Z}[x[n-N]u[n-N]] = \boxed{z^{-N}}\ X(z)$

shift in

NIC $\mathcal{Z}[x[n-N]] = \boxed{z^{-N}}\left(X(z) + x[-1]z + x[-2]z^{2} + \cdots + x[-N+1]z^{N-1} + x[-N]z^{N}\right)$

shift in				shift out			
$x[-N]$	$x[-N+1]$	$x[-2]$	$x[-1]$	$x[0]$	$x[1]$	$x[2]$	$x[N-1]$
z^{N}	z^{N-1}	z^{2}	z	z^{0}	z^{-1}	z^{-2}	z^{-N+1}

shift out				shift in			
$x[0]$	$x[1]$	$x[2]$	$x[N-1]$	$x[-N]$	$x[-N+1]$	$x[-2]$	$x[-1]$
z^{N}	z^{N-1}	z^{N-2}	z	z^{0}	z^{-1}	z^{-N+2}	z^{-N+1}

$\mathcal{Z}[x[n+N]] = z^{N} X(z) - x[0]z^{N} - x[1]z^{N-1} - x[2]z^{N-2} - \cdots - x[N-1]z^{1}$

NIC $\mathcal{Z}[x[n-N]] = z^{-N} X(z) + x[-1]z^{-N+1} + x[-2]z^{-N+2} + \cdots + x[-N+1]z^{-1} + x[-N]$

$$x[n-m]\ (\text{shift right} \longrightarrow):\quad \times z^{-m}$$
$$x[n+m]\ (\text{shift left} \longleftarrow):\quad \times z^{+m}$$

$\longleftarrow$

$x_{n+1}\ u[n+1]$	$\longleftrightarrow$	$z^{+1}X(z)$
$x_{n+2}\ u[n+2]$	$\longleftrightarrow$	$z^{+2}X(z)$
$x_{n+m}\ u[n+m]$	$\longleftrightarrow$	$z^{+m}X(z)$

$\longleftarrow$

x_{n+1}	$\longleftrightarrow$	$z^{+1}X(z) + x[0]z^{+1}$
x_{n+2}	$\longleftrightarrow$	$z^{+2}X(z) + x[0]z^{+2} + x[1]$
x_{n+m}	$\longleftrightarrow$	$z^{+m}X(z) + x[0]z^{+m} + x[1]z^{+m-1} + \cdots + x[m-1]z$

$$\mathcal{Z}\left[x[n+N]\right] = z^{N}X(z) - x[0]z^{N} - x[1]z^{N-1} - x[2]z^{N-2} - \cdots - x[N-1]z^{1}$$

$\longrightarrow$

$x_{n-1}\ u[n-1]$	$\longleftrightarrow$	$z^{-1}X(z)$
$x_{n-2}\ u[n-2]$	$\longleftrightarrow$	$z^{-2}X(z)$
$x_{n-m}\ u[n-m]$	$\longleftrightarrow$	$z^{-m}X(z)$

$\longrightarrow$

x_{n-1}	$\longleftrightarrow$	$z^{-1}X(z) + x[-1]$
x_{n-2}	$\longleftrightarrow$	$z^{-2}X(z) + x[-1]z^{-1} + x[-2]$
x_{n-m}	$\longleftrightarrow$	$z^{-m}X(z) + x[-1]z^{-m+1} + x[-2]z^{-m+2} + \cdots + x[-m]$

NIC $\quad \mathcal{Z}\left[x[n-N]\right] = z^{-N}X(z) + x[-1]z^{-N+1} + x[-2]z^{-N+2} + \cdots + x[-N+1]z^{-1} + x[-N]$

10 complex translation $\quad e^{an} x_n \longleftrightarrow X(ze^{-a})$

11 frequency scale $\quad b^n x_n \longleftrightarrow X\left(\frac{z}{b}\right)$

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$
$$= x[0] + x[1] z^{-1} + x[2] z^{-2} + \cdots$$

$$\mathcal{Z}\left[b^n x[n]\right] = \sum_{n=0}^{\infty} x[n]\, b^n z^{-n}$$
$$= x[0] + x[1]\left(\frac{z}{b}\right)^{-1} + x[2]\left(\frac{z}{b}\right)^{-2} + \cdots$$
$$= X\left(\frac{z}{b}\right)$$

$$\mathcal{Z}\left[e^{an} x[n]\right] = \sum_{n=0}^{\infty} x[n] e^{an} z^{-n}$$
$$= x[0] + x[1]\left(\frac{z}{e^a}\right)^{-1} + x[2]\left(\frac{z}{e^a}\right)^{-2} + \cdots$$
$$= X\left(\frac{z}{e^a}\right)$$
$$= X(ze^{-a})$$

12 differentiation $\quad n x_n \longleftrightarrow -zX'(z)$

13 integration $\quad \frac{1}{n} x_n \longleftrightarrow -\int \frac{X(z)}{z}\, dz$

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$
$$= x[0] + x[1] z^{-1} + x[2] z^{-2} + \cdots$$

$$\frac{d}{dz} X(z) = -1 \cdot x[1] z^{-2} - 2\, x[2] z^{-3} - 3\, x[3] z^{-4} - \cdots$$
$$= -z^{-1}\left(1 \cdot x[1] z^{-1} + 2 \cdot x[2] z^{-2} + 3\, x[3] z^{-3} + \cdots\right)$$
$$-zX'(z) = \left(0 \cdot x[0] z^{0} + 1 \cdot x[1] z^{-1} + 2 \cdot x[2] z^{-2} + \cdots\right)$$
$$= \mathcal{Z}\left[n\, x[n]\right]$$

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$
$$= x[0] + x[1] z^{-1} + x[2] z^{-2} + \cdots$$

$$\frac{X(z)}{z} = x[0] z^{-1} + x[1] z^{-2} + x[2] z^{-3} + x[3] z^{-4} + \cdots$$

$$\int \frac{X(z)}{z}\, dz = -x[0] \ln z - x[1] z^{-1} - \frac{1}{2} x[2] z^{-2} - \frac{1}{3} x[3] z^{-3} + \cdots$$

if $x[0] = 0$

$$\int \frac{X(z)}{z}\, dz = - x[1] z^{-1} - \frac{1}{2} x[2] z^{-2} - \frac{1}{3} x[3] z^{-3} + \cdots$$

$$-\int \frac{X(z)}{z}\, dz = \mathcal{Z}\left[\frac{x[n]}{n}\right] \qquad n \geq 1$$

14 integration shift $\frac{1}{n+m} x_n \longleftrightarrow -z^m \int \frac{X(z)}{z^{m+1}} dz$

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$
$$= x[0] + x[1] z^{-1} + x[2] z^{-2} + \cdots$$

$$\frac{X(z)}{z} = x[0] z^{-1} + x[1] z^{-2} + x[2] z^{-3} + x[3] z^{-4} + \cdots$$

$$\frac{X(z)}{z^{m+1}} = x[0] z^{-m-1} + x[1] z^{-m-2} + x[2] X^{-m-3} + \cdots$$

$m \geq -n$

$m > 0$

$m+n > 0$

$$\int \frac{X(z)}{z^{m+1}} dz = -\frac{x[0]}{m} z^{-m} - \frac{x[1]}{m+1} z^{-m-1} - \frac{x[2]}{m+2} z^{-m-2} + \cdots$$
$$= -z^{-m} \left(\frac{x[0]}{m+0} + \frac{x[1]}{m+1} z^{-1} + \frac{x[2]}{m+2} z^{-2} + \cdots \right)$$
$$= -z^{-m} Z\left[\frac{x[n]}{n+m} \right]$$

$$-z^m \int \frac{X(z)}{z^{m+1}} dz = Z\left[\frac{x[n]}{n+m} \right]$$

Convolution

$\{x_n\}_{n=0}^{\infty} \qquad \{y_n\}_{n=0}^{\infty}$

$$x_n * y_n = \sum_{i=0}^{n} x_i \, y_{n-i}$$

$$\mathcal{Z}[x_n * y_n] = X(z)\,Y(z)$$

$z = z^{-1}$

$$X(z)\,Y(z) = \sum_{n=0}^{\infty} x_n z^{-n} \sum_{n=0}^{\infty} y_n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} x_i \, y_{n-i} \right) z^{-n} = \mathcal{Z}\left[\sum_{i=0}^{n} x_i \, y_{n-i} \right] = \mathcal{Z}[x_n * y_n]$$

$$\begin{array}{ll} & (x_0 + x_1 z^{-1} + x_2 z^{-2} + \cdots) \\ \times & (y_0 + y_1 z^{-1} + y_2 z^{-2} + \cdots) \\ \hline \end{array}$$

$n=0 \qquad x_0 y_0$

$n=1 \qquad x_1 y_0 + x_0 y_1$

$n=2 \qquad x_2 y_0 + x_1 y_1 + x_0 y_2$

17 initial value x_0 $\longleftrightarrow$ $\lim\limits_{z\to\infty} X(z)$

18 final value $\lim\limits_{n\to\infty} x_n$ $\longleftrightarrow$ $\lim\limits_{z\to 1} (z-1)X(z)$

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

$$= x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

$$\lim_{z\to\infty} X(z) = x[0] + x[1]\cdot 0 + x[2]\cdot 0 + \cdots$$

$$= X(0)$$

$$zX(z) = \sum_{n=0}^{\infty} x[n]z^{-n+1}$$

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

$$(z-1)X(z) = x[0]z + x[1] + x[2]z^{-1} + \cdots$$

$$x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

$$= \boxed{x[0]z} + x[1] + x[2]z^{-1} + \cdots + x[n]z^{-n+1} + x[n+1]z^{-n} + \cdots$$

$$x[0] + x[1]z^{-1} + \cdots + x[n-1]z^{-n+1} + x[n]z^{-n} + \cdots + \boxed{x[\infty]}$$

$$g(z) = x[0]z + (x[1]-x[0]) + (x[2]-x[1])z^{-1} + \cdots$$

$$\cdots + (x[n+1]-x[n])z^{-n} + \cdots + x[\infty]$$

$$\lim_{z\to 1} g(z) = x[0] + (x[1]-x[0]) + (x[2]-x[1]) \quad + x[\infty]$$

$$\lim_{z\to 1} (z-1)X(z) = \lim_{n\to\infty} x[n]$$