Exact Equations (3A)

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Exact First Order ODEs

Differential Form & Equation

A **differential form**

$$P(x,y)dx + Q(x,y)dy$$

A first order **differential equation**

$$P(x,y)dx + Q(x,y)dy = 0$$

this differential form is **exact**
in a region R if there is a function
f(x,y) such that

$$P(x,y)dx + Q(x,y)dy$$

$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = df$$

exact equation
in a region R if there is a function
f(x,y) such that

$$P(x,y)dx + Q(x,y)dy$$

$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \boxed{df = 0}$$

$$P(x,y)dx + Q(x,y)dy$$

is an exact differential in a region R
if it corresponds to
the total differential of
some function f(x,y)

$$df(x,y) = 0$$

$$f(x,y) = c$$

To be exact

$P(x,y)dx + Q(x,y)dy$

to be exact

$P(x,y)dx + Q(x,y)dy = df$

the given equation should be the total differential of a certain function *f*

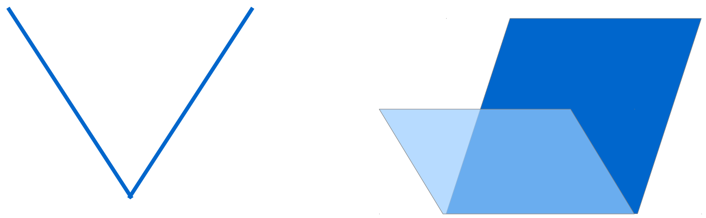
$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

such a function *f* exists

exists all the defined and continuous partial derivatives of

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$$

a smooth surface *f*(*x*, *y*)



continuous but derivatives are undefined

Schwarz' Theorem

In mathematical analysis, *Schwarz' theorem* (or *Clairaut's theorem*[2]) named after Alexis Clairaut and Hermann Schwarz, states that if

$$f : \mathbb{R}^n \to \mathbb{R}$$

has continuous second partial derivatives at any given point in $\mathbb{R}^n$, say, $(a_1, \ldots, a_n)$, then $\forall i, j \in \{1, 2, \ldots, n\}$,

$$\frac{\partial^2 f}{\partial x_i \, \partial x_j}(a_1, \ldots, a_n) = \frac{\partial^2 f}{\partial x_j \, \partial x_i}(a_1, \ldots, a_n).$$

The partial derivations of this function are commutative at that point. One easy way to establish this theorem (in the case where $n = 2$, $i = 1$, and $j = 2$, which readily entails the result in general) is by applying Green's theorem to the gradient of f.

http://en.wikipedia.org/wiki/Symmetry_of_second_derivatives#Schwarz.27_theorem

To be exact

$P(x,y)dx + Q(x,y)dy$ ⟷ to be exact

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ all defined and continuous ⇒ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$P(x,y) = \frac{\partial f}{\partial x}$

$Q(x,y) = \frac{\partial f}{\partial y}$

$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$

$\frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$

$P(x,y)dx + Q(x,y)dy$ is an exact (total) differential ⟺ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Path Independence

A differential *df* of the following form is **exact,**

$$P(x,y)dx + Q(x,y)dy = df$$

if $\int df$ is path independent

the vector field (P, Q) is
a conservative vector field,
with corresponding potential f

path independent $\int df$ $\Leftrightarrow$ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$P(x,y)dx + Q(x,y)dy$
is an exact (total) differential $\Leftrightarrow$ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Exact and Inexact Differential Examples

$$df = 2xy^3\,dx + 3x^2y^2\,dy$$

Is there a function f=f(x,y) such that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\frac{\partial f}{\partial x} = 2xy^3 \qquad \frac{\partial f}{\partial y} = 3x^2y^2$$

$$\frac{\partial^2 f}{\partial y\partial x} = 6xy^2 \qquad \frac{\partial^2 f}{\partial x\partial y} = 6xy^2$$

$f(x,y) = x^2y^3$ exact

$$\int_{(x_1,y_1)}^{(x_2,y_2)} df = f(x_2,y_2) - f(x_1,y_1)$$

Only initial & final points
Path independent

$$df = 2x^2y^3\,dx + 3x^3y^2\,dy$$

Is there a function f=f(x,y) such that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\frac{\partial f}{\partial x} = 2x^2y^3 \qquad \frac{\partial f}{\partial y} = 3x^3y^2$$

$$\frac{\partial^2 f}{\partial y\partial x} = 6x^2y^2 \qquad \frac{\partial^2 f}{\partial x\partial y} = 9x^2y^2$$

no $f(x,y)$ inexact

Integration result depends
on the path also,
in addition to initial & final points

Exact Equations

$$M(x,y)dx + N(x,y)dy = 0 \qquad \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$$z = f(x,y)$$

$$\frac{\partial f}{\partial x} = M(x,y) \qquad \frac{\partial f}{\partial y} = N(x,y)$$

$$\frac{df}{dx} = M(x) \Rightarrow f(x) = \int M(x)dx + c$$

$$\int \frac{\partial f}{\partial x}dx = \int M(x,y)dx + c \qquad \int \frac{\partial f}{\partial y}dy = \int N(x,y)\,dy + c$$

$$f(x,y) = \int M(x,y)dx + g(y) \qquad f(x,y) = \int N(x,y)dy + h(x)$$

$$\frac{\partial f}{\partial y} = N(x,y) \qquad \frac{\partial f}{\partial x} = M(x,y)$$

$$= \frac{\partial}{\partial y}\int M(x,y)dx + g'(y) \qquad = \frac{\partial}{\partial x}\int N(x,y)dy + h'(x)$$

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx \qquad h'(x) = M(x,y) - \frac{\partial}{\partial x}\int N(x,y)dy$$

Exact Equations (2)

$$f(x,y) = \int M(x,y)\,dx + g(y)$$

$$f(x,y) = \int N(x,y)\,dy + h(x)$$

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx$$

$$h'(x) = M(x,y) - \frac{\partial}{\partial x}\int N(x,y)\,dy$$

$$g(y) = \int g'(y)\,dy$$

$$= \int \left\{ N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx \right\} dy$$

$$h(x) = \int h'(x)\,dx$$

$$= \int \left\{ M(x,y) - \frac{\partial}{\partial x}\int N(x,y)\,dy \right\} dx$$

$$f(x,y) = \int M(x,y)\,dx + \int \left\{ N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx \right\} dy$$

$$f(x,y) = \int N(x,y)\,dy + \int \left\{ M(x,y) - \frac{\partial}{\partial x}\int N(x,y)\,dy \right\} dx$$

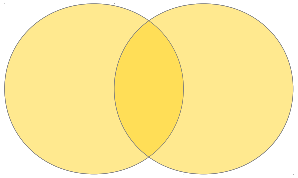
Exact Equations (3)

Exact Equations

$$M(x,y)dx + N(x,y)dy = 0 \qquad \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$$f(x,y) = \int M(x,y)dx + \int \left\{ N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx \right\} dy$$

$$f(x,y) = \int N(x,y)dy + \int \left\{ M(x,y) - \frac{\partial}{\partial x}\int N(x,y)dy \right\} dx$$



$$f(x,y) = \int M(x,y)dx + \int N(x,y)\,dy - \int \frac{\partial}{\partial y}\int M(x,y)dx\,dy \qquad \int \frac{\partial}{\partial y}\int \frac{\partial f}{\partial x}dx\,dy$$

$$f(x,y) = \int N(x,y)dy + \int M(x,y)\,dx - \int \frac{\partial}{\partial x}\int N(x,y)dy\,dx \qquad \int \frac{\partial}{\partial x}\int \frac{\partial f}{\partial y}dy\,dx$$

Exact Equations

$$M(x,y)dx + N(x,y)dy = 0 \qquad \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$$f(x,y) = \int M(x,y)dx + \int \left\{ N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx \right\} dy \qquad \frac{\partial}{\partial y}\frac{\partial}{\partial x}\int M(x,y)dx = \frac{\partial M}{\partial y}$$

$$\frac{\partial f}{\partial x} = M(x,y) + \int \left\{ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\} dy = M(x,y)$$

$$\frac{\partial f}{\partial y} = \int \frac{\partial M}{\partial y}dx + N(x,y) - \int \frac{\partial M}{\partial y}dx = N(x,y)$$

$$f(x,y) = \int N(x,y)dy + \int \left\{ M(x,y) - \frac{\partial}{\partial x}\int N(x,y)dy \right\} dx \qquad \frac{\partial}{\partial x}\frac{\partial}{\partial y}\int N(x,y)dy = \frac{\partial N}{\partial x}$$

$$\frac{\partial f}{\partial x} = \int \frac{\partial N}{\partial x}dy + M(x,y) - \int \frac{\partial N}{\partial x}dy = M(x,y)$$

$$\frac{\partial f}{\partial y} = N(x,y) + \int \left\{ \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right\} dx = N(x,y)$$

NonExact First Order ODEs

NonExact Equations

Exact Equations

$$M(x,y)dx + N(x,y)dy = 0 \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$z = f(x,y)$$

NonExact Equations

$$M(x,y)dx + N(x,y)dy = 0 \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$z = f(x,y)$$

Exact Equations

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0 \qquad \frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

If we can find $\mu(x,y)$

Exact Equations

$$\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0 \qquad \frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

If we can find $\mu(x)$

or $\mu(y)$

$\mu(y)$ $\mu(y)$

NonExact Equations

$$M(x,y)dx + N(x,y)dy = 0 \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$z = f(x,y)$$

Exact Equations

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0 \qquad \frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

If we can find $\mu(x,y)$

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \quad \xleftarrow{\mu(x,y)} \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{\partial \mu}{\partial y}M + \mu\frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x}N + \mu\frac{\partial N}{\partial x}$$

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

$$\mu M_y - \mu N_x = \mu_x N - \mu_y M$$

Partial Differential Equation

difficult to find $\mu(x,y)$

Multiplying NonExact Equations by $\mu(x)$

NonExact Equations

$$M(x,y)dx + N(x,y)dy = 0 \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$z = f(x,y)$$

Exact Equations

$$\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0 \qquad \frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

If we can find $\mu(x)$

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \quad \xleftarrow{\mu(x)} \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{\partial \mu}{\partial y}M + \mu\frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x}N + \mu\frac{\partial N}{\partial x}$$

$$\frac{d\mu}{dx}N = \mu\frac{\partial M}{\partial y} - \mu\frac{\partial N}{\partial x}$$

$$\frac{d\mu}{dx}N = \mu M_y - \mu N_x$$

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu$$

generally $P(x,y)$

sometimes $P(x)$

$$\frac{d\mu}{dx} = P(x)\mu(x)$$

$$\frac{d\mu}{dx} - P(x)\mu = 0$$

solve this to find $\mu(x)$

NonExact Equations

$$M(x,y)dx + N(x,y)dy = 0 \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$z = f(x,y)$$

Exact Equations

$$\mu(y)M(x,y)dx + \mu(y)N(x,y)dy = 0 \qquad \frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

If we can find $\mu(y)$

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \quad \longleftarrow \mu(y) \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{\partial \mu}{\partial y}M + \mu\frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x}N + \mu\frac{\partial N}{\partial x}$$

$$M\frac{d\mu}{dy} = \mu\frac{\partial N}{\partial x} - \mu\frac{\partial M}{\partial y}$$

$$M\frac{d\mu}{dy} = \mu N_x - \mu M_y$$

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu$$

generally $P(x,y)$

sometimes $P(y)$

$$\frac{d\mu}{dy} = P(y)\mu(y)$$

$$\frac{d\mu}{dy} - P(y)\mu = 0$$

solve this to find $\mu(y)$

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Solving NonExact Equations

Assumption for $\mu(x)$

$$\frac{\partial}{\partial y}[\mu(x) M(x,y)] = \frac{\partial}{\partial x}[\mu(x) N(x,y)]$$

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu = P(x)\mu$$

$$\frac{d\mu}{dx} - P(x)\mu = 0$$

$$\mu(x) = c e^{\int P(x)\,dx}$$

$$\mu(x) = c e^{\int \left(\frac{M_y - N_x}{N}\right) dx}$$

Solve this exact equation

$$\mu(x) M(x,y)\,dx + \mu(x) N(x,y)\,dy = 0$$

Assumption for $\mu(y)$

$$\frac{\partial}{\partial y}[\mu(y) M(x,y)] = \frac{\partial}{\partial x}[\mu(y) N(x,y)]$$

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu = P(y)\mu$$

$$\frac{d\mu}{dy} - P(y)\mu = 0$$

$$\mu(y) = c e^{\int P(y)\,dy}$$

$$\mu(y) = c e^{\int \left(\frac{N_x - M_y}{M}\right) dy}$$

Solve this exact equation

$$\mu(y) M(x,y)\,dx + \mu(y) N(x,y)\,dy = 0$$

Verifying Exact Equations

$$\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0$$

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu = P(x)\mu$$

$$\mu(x) = c\,e^{\int\left(\frac{M_y - N_x}{N}\right)dx}$$

$$\frac{\partial}{\partial y}[\mu(x)M(x,y)] = \mu_y M + \mu M_y$$

$$\frac{\partial}{\partial x}[\mu(x)N(x,y)] = \mu_x N + \mu N_x$$

$$= \left(\frac{M_y - N_x}{N}\right)\mu N + \mu N_x$$

$$= \mu M_y$$

$$\frac{\partial}{\partial y}[\mu(x)M(x,y)] = \frac{\partial}{\partial x}[\mu(x)N(x,y)]$$

$$\mu(y)M(x,y)dx + \mu(y)N(x,y)dy = 0$$

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu = P(y)\mu$$

$$\mu(y) = c\,e^{\int\left(\frac{N_x - M_y}{M}\right)dy}$$

$$\frac{\partial}{\partial y}[\mu(y)M(x,y)] = \mu_y M + \mu M_y$$

$$= \left(\frac{N_x - M_y}{M}\right)\mu M + \mu M_y$$

$$= \mu N_x$$

$$\frac{\partial}{\partial x}[\mu(y)N(x,y)] = \mu_x N + \mu N_x$$

$$\frac{\partial}{\partial y}[\mu(y)M(x,y)] = \frac{\partial}{\partial x}[\mu(y)N(x,y)]$$

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