

Complex Inverse Trig & Inverse TrigH (H.1)

20160910

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Inverse Trigonometric & Hyperbolic Functions

$$\sin^{-1}(z)$$

$$\cos^{-1}(z)$$

$$\tan^{-1}(z)$$

$$\cot^{-1}(z)$$

$$\sec^{-1}(z)$$

$$\csc^{-1}(z)$$

$$\sinh^{-1}(z)$$

$$\cosh^{-1}(z)$$

$$\tanh^{-1}(z)$$

$$\coth^{-1}(z)$$

$$\operatorname{sech}^{-1}(z)$$

$$\operatorname{csch}^{-1}(z)$$

$\sin^{-1}(z)$

$$w = \sin^{-1}(z)$$

$$z = \sin(w) = \frac{e^{iw} - e^{-iw}}{2i}$$

$$2iz = e^{iw} - e^{-iw}$$

$$e^{iw} - 2iz - e^{-iw} = 0$$

$$e^{2iw} - 2iz e^{iw} - 1 = 0$$

$$e^{iw}{}^2 - 2iz e^{iw} - 1 = 0$$

$$e^{iw} = iz \pm \sqrt{(iz)^2 + 1}$$
$$= iz + (1 - z^2)^{1/2}$$

$$1 - z^2 = r e^{i\theta}$$
$$(1 - z^2)^{1/2} = r^{1/2} e^{i(\frac{\theta + 2k\pi}{2})} \quad k=0, 1$$

$$e^{iw} = iz + (1 - z^2)^{1/2}$$

$$iw = \ln[iz + (1 - z^2)^{1/2}]$$

$$w = -i \ln[iz + (1 - z^2)^{1/2}]$$

$$= \sin^{-1}(z)$$

$$\sin^{-1}(z) = -i \ln[iz + (1 - z^2)^{1/2}]$$

$\cos^{-1}(z)$

$$w = \cos^{-1}(z)$$

$$z = \cos(w) = \frac{e^{iw} + e^{-iw}}{2}$$

$$2z = e^{iw} + e^{-iw}$$

$$e^{iw} - 2z + e^{-iw} = 0$$

$$e^{2iw} - 2ze^{iw} + 1 = 0$$

$$e^{iw}^2 - 2ze^{iw} + 1 = 0$$

$$e^{iw} = z \pm \sqrt{z^2 - 1} = z \pm \sqrt{i^2(-z^2 + 1)}$$
$$= z + i(1 - z^2)^{1/2}$$

$$1 - z^2 = \rho e^{i\theta}$$
$$(1 - z^2)^{1/2} = \rho^{1/2} e^{i\left(\frac{\theta + 2k\pi}{2}\right)} \quad k=0, 1$$

$$e^{iw} = z + i(1 - z^2)^{1/2}$$

$$iw = \ln[z + i(1 - z^2)^{1/2}]$$

$$w = -i \ln[z + i(1 - z^2)^{1/2}]$$

$$= \cos^{-1}(z)$$

$$\cos^{-1}(z) = -i \ln[z + i(1 - z^2)^{1/2}]$$

$\tan^{-1}(z)$

$$w = \tan^{-1}(z)$$

$$\frac{\sin^{-1}(z)}{\cos^{-1}(z)} = \frac{-i \ln[iz + (1-z^2)^{1/2}]}{-i \ln[z + i(1-z^2)^{1/2}]}$$

$$z = \tan(w) = \frac{e^{iw} - e^{-iw}}{2i} = \frac{(e^{iw} - e^{-iw})}{i(e^{iw} + e^{-iw})}$$

$$z i (e^{iw} + e^{-iw}) = (e^{iw} - e^{-iw})$$
$$(iz - 1) e^{iw} + (iz + 1) e^{-iw} = 0$$

$$(iz - 1) e^{iw} + (iz + 1) = 0$$

$$e^{iw} = \frac{-(iz + 1)}{(iz - 1)} = \frac{(1 + iz)}{(1 - iz)} = \frac{(i - z)}{(i + z)}$$

$$e^{iw} = \left(\frac{i - z}{i + z} \right)^{1/2}$$

$$iw = \frac{1}{2} \ln \left(\frac{i - z}{i + z} \right)$$

$$w = \frac{-i}{2} \ln \left(\frac{i - z}{i + z} \right) = \frac{i}{2} \ln \left(\frac{i - z}{i + z} \right)^{-1}$$

$$w = \frac{i}{2} \ln \left(\frac{i + z}{i - z} \right)$$

$$= \tan^{-1}(z)$$

$$\tan^{-1}(z) = \frac{i}{2} \ln \left(\frac{i + z}{i - z} \right)$$

$$\sin^{-1}(z) = -i \ln[iz + (1-z^2)^{1/2}]$$

$$\cos^{-1}(z) = -i \ln[z + i(1-z^2)^{1/2}]$$

$$\tan^{-1}(z) = \frac{i}{2} \ln\left(\frac{i+z}{i-z}\right)$$

$$\frac{d}{dz} \sin^{-1}(z), \quad \frac{d}{dz} \cos^{-1}(z), \quad \frac{d}{dz} \tan^{-1}(z)$$

$$w = \sin^{-1}(z)$$

$$z = \sin(w)$$

$$\frac{d}{dz} z = \frac{d}{dz} \sin(w) = \cos(w) \frac{dw}{dz}$$

$$\frac{dw}{dz} = \frac{1}{\cos(w)} = \frac{1}{(1 - \sin^2(w))^{\frac{1}{2}}} = \frac{1}{(1 - z^2)^{\frac{1}{2}}}$$

$$w = \cos^{-1}(z)$$

$$z = \cos(w)$$

$$\frac{d}{dz} z = \frac{d}{dz} \cos(w) = -\sin(w) \frac{dw}{dz}$$

$$\frac{dw}{dz} = \frac{-1}{\sin(w)} = \frac{-1}{(1 - \cos^2(w))^{\frac{1}{2}}} = \frac{-1}{(1 - z^2)^{\frac{1}{2}}}$$

$$w = \tan^{-1}(z) \quad \frac{d}{dz} \frac{\sin(z)}{\cos(z)} = \frac{\cos^2(z) + \sin^2(z)}{\cos^2(z)} = 1 + \tan^2(z) = \frac{1}{\cos^2(z)} = \sec^2(z)$$

$$z = \tan(w)$$

$$\frac{d}{dz} z = \frac{d}{dz} \tan(w) = (1 + \tan^2(w)) \frac{dw}{dz}$$

$$\frac{dw}{dz} = \frac{1}{(1 + \tan^2(w))} = \frac{1}{1 + z^2}$$

$$\frac{d}{dz} \sin^{-1}(z) = \frac{1}{(1-z^2)^{1/2}}$$

$$\frac{d}{dz} \cos^{-1}(z) = \frac{-1}{(1-z^2)^{1/2}}$$

$$\frac{d}{dz} \tan^{-1}(z) = \frac{1}{1+z^2}$$

$\sinh^{-1}(z)$

$$w = \sinh^{-1}(z)$$

$$z = \sinh(w) = \frac{e^w - e^{-w}}{2}$$

$$2z = e^w - e^{-w}$$

$$e^w - 2z - e^{-w} = 0$$

$$\boxed{e^{2w}} - 2z \boxed{e^w} - 1 = 0$$

$$\boxed{e^w}^2 - 2z \boxed{e^w} - 1 = 0$$

$$\boxed{e^w} = z \pm \sqrt{(z)^2 + 1}$$
$$= z + (z^2 + 1)^{1/2}$$

$$z^2 + 1 = \rho e^{i\theta}$$
$$(z^2 + 1)^{1/2} = \rho^{1/2} e^{i(\frac{\theta + 2k\pi}{2})} \quad k=0, 1$$

$$\boxed{e^w} = z + (z^2 + 1)^{1/2}$$

$$\boxed{w} = \ln[z + (z^2 + 1)^{1/2}]$$

$$w = \ln[z + (z^2 + 1)^{1/2}]$$
$$= \sinh^{-1}(z)$$

$$\sinh^{-1}(z) = \ln[z + (1 - z^2)^{1/2}]$$

$\cosh^{-1}(z)$

$$w = \cosh^{-1}(z)$$

$$z = \cosh(w) = \frac{e^w + e^{-w}}{2}$$

$$2z = e^w + e^{-w}$$

$$e^w - 2z + e^{-w} = 0$$

$$e^{2w} - 2z e^w + 1 = 0$$

$$e^w{}^2 - 2z e^w + 1 = 0$$

$$e^w = z \pm \sqrt{z^2 - 1}$$
$$= z + (z^2 - 1)^{1/2}$$

$$z^2 - 1 = r e^{i\theta}$$
$$(z^2 - 1)^{1/2} = r^{1/2} e^{i(\frac{\theta + 2k\pi}{2})} \quad k=0, 1$$

$$e^w = z + (z^2 - 1)^{1/2}$$

$$w = \ln[z + (z^2 - 1)^{1/2}]$$

$$w = \ln[z + (z^2 - 1)^{1/2}]$$
$$= \cosh^{-1}(z)$$

$$\cosh^{-1}(z) = \ln[z + (z^2 - 1)^{1/2}]$$

$\tanh^{-1}(z)$

$$w = \tanh^{-1}(z)$$

$$\cancel{\frac{\sinh^{-1}(z)}{\cosh^{-1}(z)}} = \frac{\ln[z + (z^2+1)^{\frac{1}{2}}]}{\ln[z + (z^2-1)^{\frac{1}{2}}]}$$

$$z = \tanh(w) = \frac{e^w - e^{-w}}{e^w + e^{-w}} = \frac{(e^w - e^{-w})}{(e^w + e^{-w})}$$

$$z(e^w + e^{-w}) = (e^w - e^{-w})$$
$$(z-1)e^w + (z+1)e^{-w} = 0$$

$$(z-1)e^{2w} + (z+1) = 0$$

$$e^{2w} = \frac{-(z+1)}{(z-1)} = \frac{1+z}{1-z}$$

$$e^{w} = \left(\frac{1+z}{1-z}\right)^{\frac{1}{2}}$$

$$w = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right)$$

$$w = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right)$$

$$= \tanh^{-1}(z)$$

$$\tanh^{-1}(z) = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right)$$

$$\sinh^{-1}(z) = \ln[z + (1 - z^2)^{1/2}]$$

$$\cosh^{-1}(z) = \ln[z + (z^2 - 1)^{1/2}]$$

$$\tanh^{-1}(z) = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right)$$

$$\frac{d}{dz} \sinh^{-1}(z), \quad \frac{d}{dz} \cosh^{-1}(z), \quad \frac{d}{dz} \tanh^{-1}(z)$$

$$w = \sinh^{-1}(z)$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$z = \sinh(w)$$

$$\frac{d}{dz} z = \frac{d}{dz} \sinh(w) = \cosh(w) \frac{dw}{dz}$$

$$\frac{dw}{dz} = \frac{1}{\cosh(w)} = \frac{1}{(1 + \sinh^2(w))^{\frac{1}{2}}} = \frac{1}{(1 + z^2)^{\frac{1}{2}}}$$

$$w = \cosh^{-1}(z)$$

$$z = \cosh(w)$$

$$\frac{d}{dz} z = \frac{d}{dz} \cosh(w) = \sinh(w) \frac{dw}{dz}$$

$$\frac{dw}{dz} = \frac{1}{\sinh(w)} = \frac{1}{(\cosh^2(w) - 1)^{\frac{1}{2}}} = \frac{1}{(z^2 - 1)^{\frac{1}{2}}}$$

$$w = \tanh^{-1}(z)$$

$$\begin{aligned} \frac{d}{dz} \frac{\sinh(z)}{\cosh(z)} &= \frac{\cosh^2(z) - \sinh^2(z)}{\cosh^2(z)} = 1 - \tanh^2(z) \\ &= \frac{1}{\cosh^2(z)} = \operatorname{sech}^2(z) \end{aligned}$$

$$z = \tanh(w)$$

$$\frac{d}{dz} z = \frac{d}{dz} \tanh(w) = (1 - \tanh^2(w)) \frac{dw}{dz}$$

$$\frac{dw}{dz} = \frac{1}{(1 - \tanh^2(w))} = \frac{1}{1 - z^2}$$

$$\frac{d}{dz} \sinh^{-1}(z) = \frac{1}{(z^2 + 1)^{1/2}}$$

$$\frac{d}{dz} \cosh^{-1}(z) = \frac{1}{(z^2 - 1)^{1/2}}$$

$$\frac{d}{dz} \tanh^{-1}(z) = \frac{1}{1 - z^2}$$